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Torsion algebraic cycles and a theorem of Roitman

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Let $X$ be a smooth projective algebraic variety defined over an algebraically closed field $k$. Let $CH^*(X)$ denote the chow group of codimension $n$ algebraic cycles on $X$ modulo rational equivalence [8]. In this note I study the $l$-torsion subgroup $CH^*(X)(l) \subseteq CH^*(X)$ for $l$ prime to char $k$. Two sorts of results are proved.

(i) $CH^*(X)(l)$ is related to the $l$-adic étale cohomology via a map

$$\lambda^l : CH^*(X)(l) \longrightarrow H^{2n-1}_{\text{ét}}(X, Q_l/Z_l(n)).$$

(ii) $\lambda^n$ is shown to be an isomorphism for $n = \dim X$, i.e., for $0$-cycles.

The map $\lambda^1$ is also an isomorphism, arising from the identification $CH^1(X) \cong H^1_{\text{ét}}(X, G_m)$ together with the Kummer sequence

$$0 \longrightarrow \mu_{l^n} \longrightarrow G_m \longrightarrow G_m \longrightarrow 0.$$
the full strength of some theorems in [2] (which I like to think are non-trivial) together with the Weil conjectures as proved by Deligne [3] (no question of triviality there.) These prerequisites are discussed in §1. The construction of $\lambda^n$ is given in §2. In §3 the various functoriality properties of $\lambda$ are discussed. §4 contains the proof of Roitman’s theorem, and §5 is a brief discussion of relations between algebraic $K$-theory and torsion in étale cohomology; in particular, relations between $K_1$ of a surface and torsion in $H^3_{\text{ét}}$.

I am indebted to J. Murre for suggesting that $\lambda$ should exist and showing me its value in calculations for chow groups of Fano 3-folds.

1. Prerequisites

We fix once for all a smooth projective variety $X$ defined over an algebraically closed field $k$, and a prime $l \neq \text{char } k$. In this section we recall briefly results from [2] which will be used in the sequel.

Let $H^q(\mu_{1/v}^n)$ denote the Zariski sheaf on $X$ associated to the presheaf

$$U \rightarrow H^q_{\text{ét}}(U, \mu_{1/v}^n)$$

where $U \subset X$ is Zariski open, $q$, $v$, $n$ are given integers, and $\mu_{1/v}^n$ denotes the étale sheaf of $l$-th roots of 1 on $X$, tensored with itself $n$ times. The Leray spectral sequence associated to the morphisms of sites $X_{\text{ét}} \rightarrow X_{\text{zar}} \rightarrow \text{point}$ is

$$E_2^{p,q} = H^p(X, H^q(\mu_{1/v}^n)) \Rightarrow H^{p+q}_{\text{ét}}(X, \mu_{1/v}^n).$$

Let $X^r$ denote the set of points of $X$ of codimension $r$. The filtration by codimension of support on $R\Gamma(X_{\text{ét}}, \mu_{1/v}^n)$ gives rise to another spectral sequence

$$E_1^{p,q} = \prod_{x \in X^p} H^q_{\text{gal}}(k(x), \mu_{1/v}^{n-p}).$$

The basic fact is that the spectral sequences (1.1) and (1.2) coincide from $E_2$ onward. In fact, we can localize (1.2) for the Zariski topology to obtain a complex of sheaves

$$0 \rightarrow H^q(\mu_{1/v}^n) \rightarrow \prod_{x \in X^0} i_x H^q(k(x), \mu_{1/v}^n) \rightarrow \cdots \rightarrow \prod_{x \in X^{q-1}} i_x H^1(k(x), \mu_{1/v}^{n-q+1}) \rightarrow \prod_{x \in X^q} i_x \mu_{1/v}^{n-q}.$$
where $i_x(A)$ for an abelian group $A$ denotes the constant sheaf $A$ supported on the Zariski closure of the point $x$. The main theorem of [2] says that (1.3) is an acyclic resolution of $H^q$. Taking global sections, the cohomology of $E_i^q$ is seen to be $H(X, H^q)$.

Two corollaries will be particularly useful.

**COROLLARY 1.4:**

\[(1.4) \quad H^p(X, H^q(\mu_{l^r}^{\otimes n})) = (0) \quad \text{for} \quad p > q.\]

In particular, for $n \geq 1$, $E_2^{n+1,n-1} = (0)$, $i \geq 1$, so there is a boundary map from (1.1)

\[
H^{n-1}(X, H^n(\mu_{l^r}^{\otimes n})) \rightarrow H^{2n-1}_{\text{ét}}(X, \mu_{l^r}^{\otimes n}).
\]

**COROLLARY 1.5:** $H^{n-1}(X, H^n(\mu_{l^r}^{\otimes n}))$ is the cohomology of the complex

\[
(1.5) \coprod_{x \in X^{n-2}} H^2(k(x), \mu_{l^r}^{\otimes 2}) \rightarrow \coprod_{x \in X^{n-1}} k(x)^* / k(x)^{l^r} \rightarrow \coprod_{x \in X^n} Z / l^r Z.
\]

One further fact of this sort which will be used is the existence of a resolution (cf. [6]).

\[
(1.6) \coprod_{x \in X^{n-1}} k(x)^* \xrightarrow{\partial} \coprod_{x \in X^n} Z \rightarrow CH^*(X) \rightarrow 0.
\]

The map $\partial_{l^r}$ in (1.5) is obtained by reduction modulo $l^r$ from $\partial$ in (1.6).

**2. Construction of $\lambda$**

We consider the diagram which exact rows

\[
0 \rightarrow \coprod_{x \in X^{n-1}} k(x)^* / k^* \xrightarrow{l^r} \coprod_{x \in X^n} k(x)^* / k^* \rightarrow \coprod_{x \in X^n} k(x)^* / k(x)^{l^r} \rightarrow 0
\]

\[(2.1) \quad \begin{array}{ccc}
0 & \rightarrow & \prod_{x \in X^n} Z \\
\downarrow \partial & & \downarrow \partial \\
\prod_{x \in X^n} Z & \rightarrow & \prod_{x \in X^n} Z / l^r Z \\
\downarrow l^r & & \downarrow l^r \\
0 & \rightarrow & 0
\end{array}
\]

The serpent lemma applied to (2.1) says the horizontal row in (2.2) below is exact.
Passing to the limit over $\nu$ we get

$$
\rho : \text{Ker} \partial \longrightarrow H^{2n-1}_d(X, \mathbb{Z}_l(n))
$$

The key fact necessary to construct $\lambda$ is

**Lemma 2.4:** The image of $\rho$ is torsion.

**Proof:** Suppose for a moment that $k = \tilde{F}_p$ is the algebraic closure of a finite field. Fix $x \in \text{Ker} \partial$ and assume $x$ and $X$ are defined over $\mathbb{F}_p^s$, $X = X_0 \times_{\text{Sp} \tilde{F}_p} \text{Sp} \tilde{F}_p$. Take

$$
f = (p^a \text{-th power frobenius on } X_0) \times (\text{Identity on } \tilde{F}_p).
$$

Then $f$ acts compatibly with $\rho$

$$
f \rho(x) = \rho(f(x)) = p^{an}\rho(x).
$$

By the Weil conjectures (proved in [3]), $p^{an}$ is not a proper value of $f$ on $H^{2n-1}_d(X, \mathbb{Q}_l(n))$, so $\rho(x)$ is torsion.

In general, we may spread $X$ out to a smooth scheme $\mathcal{X}$ over $\text{Sp} R$, where $R$ is a valuation ring with quotient field $k$ and residue field $k_0 = \tilde{F}_p$. We have a specialization isomorphism

$$
s : H^{2n-1}_d(X, \mathbb{Z}_l(n)) \sim H^{2n-1}_d(X_0, \mathbb{Z}_l(n)).
$$

In fact, specialization (actually cospecialization, cf. [4], Arcata exposé, section V) induces a map of complexes of $E_1$ terms (1.2)
To see this, let $Z_k \subset X_k$ be closed, and let $\bar{Z} \subset \bar{X}$ denote the closure of $Z_k$ in $\bar{X}$. Let $U_k = X_k - Z_k$, $\mathcal{U} = \bar{X} - \bar{Z}$. Consider the long exact sequences of local cohomology

$$\cdots \to H^r_{\mathcal{Z}}(\bar{X}, Z/l^rZ) \to H^r(\bar{X}, Z/l^rZ) \to H^r(\mathcal{U}, Z/l^rZ) \to \cdots$$

$$H^r_{\mathcal{Z}}(X_k, Z/l^rZ) \to H^r(\bar{X}_k, Z/l^rZ) \to H^r(U_k, Z/l^rZ).$$

$\mathcal{X}$ is smooth over $\text{Sp} R$ and hence locally acyclic (op. cit.) so $\alpha$ and $\beta$ are isomorphisms. Hence $\gamma$ is an isomorphism. We obtain a specialization map

$$H^r_{\mathcal{Z}}(X_k, Z/l^rZ) \xrightarrow{\gamma^{-1}} H^r_{\mathcal{Z}}(\bar{X}, Z/l^rZ) \to H^r_{\mathcal{Z}}(X_0, Z/l^rZ).$$

Such a $\tau$ can be thought of as a specialization map for étale homology (cf. [2]). Examining the construction of the spectral sequence (1.2) in [2], one easily constructs the arrow in (2.5).

Notice now that the subgroups $\coprod_{x \in X_{k}^{X_k}} k^* \subset \coprod_{x \in X_{X_k}^{X_k}} k(x)^*$ (resp. $\coprod_{x \in X_{X_k}^{X_k}} k_{0}(x)^* \subset \coprod_{x \in X_{X_k}^{X_k}} k_{0}(x)^*$) are $l$-divisible and map to zero in $\coprod_{x \in X_{X_k}^{X_k}} Z$ (resp. $\coprod_{x \in X_{X_k}^{X_k}} Z$). A standard specialization argument yields homomorphisms $t$ fitting into a commutative diagram with exact rows.

$$0 \to \text{Ker} \tilde{\delta}_k \to \coprod_{x \in X_{k}^{X_k}} k(x)^*/k^* \to \coprod_{x \in X_{X_k}^{X_k}} Z \to \text{Ch}^n(X_k) \to 0$$

$$0 \to \text{Ker} \tilde{\delta}_0 \to \coprod_{x \in X_{X_k}^{X_k}} k_{0}(x)^*/k_{0}^* \to \coprod_{x \in X_{X_k}^{X_k}} Z \to \text{CH}^n(X_0) \to 0$$

Using divisibility we see easily that $\tilde{\rho}$ in (2.3) factors through a $\tilde{\rho}_k$ with domain $\text{Ker} \tilde{\delta}_k$ (resp. $\tilde{\rho}_0$ with domain $\text{Ker} \tilde{\delta}_0$). Moreover $\tilde{\rho}_0 \circ t = s \circ \tilde{\rho}_k$. Since Image $\tilde{\rho}_0$ is torsion and $s$ is an isomorphism, Image $\tilde{\rho}_k = \text{Image} \rho_k$ is necessarily torsion as well. Q.E.D.
Construction of $\lambda$: We take the direct limit of exact sequences (2.2)

$$0 \to \text{Ker } \partial/I' \to \text{Ker } \partial \to \text{Ker } \partial/I' \to CH^n(X)_I' \to 0$$

and maps

$$\begin{align*}
\text{Ker } \partial/I' & \longrightarrow H^{2n-1}_\text{et}(X, \mu_{I'}^n) \\
\text{Ker } \partial/I' & \longrightarrow H^{2n-1}_\text{et}(X, \mu_{I'}^n)
\end{align*}$$

The image in $H^{2n-1}_\text{et}(X, \mu_{I'}^n)$ of the torsion in $H^{2n-1}_\text{et}(X, Z_l(n))$ dies in the limit group $H^{2n-1}_\text{et}(X, Q_l/Z_l(n))$, so we obtain a map

$$(2.7) \quad CH^n(X)(I) \longrightarrow H^{2n-1}_\text{et}(X, Q_l/Z_l(n)).$$

We define $\lambda^n_I$ to be the negative of (2.7). (The reason for the change of sign is to get (3.6).)

3. Functoriality of $\lambda^n_I$

**Lemma 3.1:** The presentation

$$\coprod_{x \in X^{n-1}} k(x)^* \xrightarrow{\partial} \coprod_{x \in X^n} Z \longrightarrow CH^n(X) \longrightarrow 0$$

is functorial under pullback by a flat morphism $f: W \to X$ and under direct image by a proper morphism $g: X \to Y$.

**Proof:** The maps

$$\coprod_{w \in W^n} Z \leftarrow f^* \coprod_{x \in X^n} Z \xrightarrow{g^*} \coprod_{y \in Y^{n-m}} Z \quad (m = \text{fibre dim } X/Y)$$

are defined as usual in cycle theory [8]. Note that flatness insures that $f^{-1}([x])$ has codimension $n$ for any $x \in X^n$, so $f^*$ is everywhere defined. Let $x \in X^{n-1}$, $y \in W^{n-1}$ and suppose the cycle $f^{-1}([x])$ contains $\{\bar{y}\}$ with multiplicity $m$. Then $f^*: k(x)^* \to k(y)^*$ is the $m$-th power.
of the map induced by the morphism of schemes \{y\} \to \{x\}. Similarly, 
\(g*: k(x)^* \to k(g(x))^*\) is the norm map if \([k(g(x)): k(x)] < \infty\) and zero otherwise. The fact that these maps are compatible is classical cycle theory, and is left for the reader. Q.E.D.

**Lemma 3.2:** The spectral sequence (1.2) is functorial under pull-back by flat morphisms and direct image by proper morphisms.

**Proof:** Let \(w \xrightarrow{f} X \xrightarrow{g} Y\) be as above. For \(q \geq 0\) let 
\[H^q_{\text{ét}}(X, Z|l^rZ) = \lim_{\text{codim } Z = q} H^q_{\text{ét}}(X, Z|l^rZ).\]
Since \(f\) is flat, we get 
\[f^*: H^q_{\text{ét}}(X, Z|l^rZ) \to H^q_{\text{ét}}(W, Z|l^rZ).\]
This suffices to show \(f^*\)-functoriality for (1.2) (cf. [2]). The existence of compatible maps 
\[g^*: H^q_{\text{ét}}(X, \mu_{l^r}^{\times n}) \to H^q_{\text{ét}}(Y, Z|l^rZ)\]
is the duality theory for étale cohomology (It can be interpreted as saying that étale homology is covariant for proper maps, cf. [2].) and suffices to establish covariant \(g^*\) functoriality for the spectral sequence (1.2). Notice, however, that \(g^*\) shifts indices 
\[g^*: E^{p,q}_1 = \bigoplus_{x \in X} H^{q-p}(k(x), \mu_{l^r}^{\times m}) \to \bigoplus_{y \in Y} H^{q-p}(k(y), Z|l^rZ) = E^{p-m,q-m}_1.\]

Combining these two lemmas (again compatibilities will not be pursued in detail) we obtain

**Proposition 3.3:** With notation as above, there is a commutative diagram 
\[\begin{array}{ccc}
CH^n(W)(l) & \xrightarrow{f^*} & CH^n(X)(l) \\
\downarrow{\lambda^n_l} & & \downarrow{\lambda^n_{l^{-m}}} \\
H^{2n-1}_{\text{ét}}(W, Q/lZ_l(n)) & \xrightarrow{f^*} & H^{2n-1}_{\text{ét}}(X, Q/lZ_l(n)) \\
\downarrow{\lambda^{2n-1}_{l^{-m}}} & & \downarrow{\lambda^{2n-2m-1}_{l^{-m}}} \\
H^{2n-2m-1}_{\text{ét}}(Y, Q/lZ_l(n - m)). & & \\
\end{array}\]

Now let \(z \in CH^q(X)\) be a cycle class, and let \([z] \in H^{2q}_{\text{ét}}(X, Z_l(q))\) be the class of \(z\) in cohomology.
**Proposition 3.4:** The diagram

\[
\begin{array}{ccc}
CH^n(x)(l) & \xrightarrow{\cdot z} & CH^{n+q}(X)(l) \\
\downarrow{\lambda^q} & & \downarrow{\lambda^{q+q}} \\
H^{2n-1}(X, Q_l/Z_l(n)) & \xrightarrow{[z]} & H^{2n+2q-1}(X, Q_l/Z_l(n+q))
\end{array}
\]

commutes, where \(\cdot z\) denotes multiplication in the Chow ring and \(\cdot [\cdot]\) the cup product induced by the obvious bilinear map

\[
Q_l/Z_l(n) \times Z_l(q) \longrightarrow Q_l/Z_l(n + q).
\]

**Proof:** We may assume \(z\) is the class of an irreducible subvariety \(Z \subset X\). Let \(S\) denote the set of all pairs \((x, R^*(x))\) where \(x \in X^{n-1}\) and \(R(x) \subset k(x)\) is a finitely generated \(k\)-subalgebra. For \(A \subset S\) a subset, let \(A' \subset X^n\) denote the set of all generic points of \(\{x\} - \text{Sp } R(x)\) as \(x\) runs through \(A\). Given \(A \subset S\) a finite subset, we can arrange, by moving \(Z\), for \(Z\) to “meet \(A\) properly”, i.e., for the cycle intersection \(Z \cdot \{x\}\) to be defined for all \(x\) in \(A\) or \(A'\). We obtain in this case a diagram

\[
\begin{array}{ccc}
\coprod_{x \in X^{n+q-1}} R(x)^* / R(x)^{l_r} \xrightarrow{\cdot z} \coprod_{y \in X^{n+q-1}} k(y)^* / k(y)^{l_r'} \xrightarrow{\cdot z} \coprod_{x \in X^{n+q-1}} Z / l_r' Z \\
\downarrow{\lambda^{n-1}} & & \downarrow{\lambda^{n+q-1}} \\
\coprod_{x \in X^{n+q-1}} Z / l_r' Z \\
\end{array}
\]

and hence a map \(\cdot Z : \text{Ker } \partial^{n-1}_{l_r, A} \rightarrow \text{Ker } \partial^{n+q-1}_{l_r'}\).

The diagrams (cf. (2.2))

\[
\begin{array}{ccc}
\text{Ker } \partial^{n-1}_{l_r, A} & \xrightarrow{\cdot z} & \text{Ker } \partial^{n+q-1}_{l_r'} \\
\downarrow{\cdot z} & & \downarrow{\cdot z} \\
CH^n(X)_{l_r} & \xrightarrow{\cdot z} & CH^{n+q}(X)_{l_r} \\
\downarrow{\lambda^{n-1}} & & \downarrow{\lambda^{n+q-1}} \\
H^{2n-1}(X, Q_l/Z_l(n)) & \xrightarrow{[z]} & H^{2n+2q-1}(X, Q_l/Z_l(n+q))
\end{array}
\]

Commutes. Given \(u \in CH^n(X)_{l_r}\) we can choose \(A\) finite but sufficiently large so \(u \in \text{Image}(\text{Ker } \partial^{n+q-1}_{l_r') \rightarrow CH^n(X)_{l_r})\). It is now straightforward to verify \([Z] \cdot \lambda_{l_r}(x) = \lambda_{l_r}(u \cdot z)\). Q.E.D.
Proposition 3.5: Let $X$, $Y$ be smooth projective varieties, $m$ and $n$ integers, and $\Gamma$ a cycle on $Y \times X$ of dimension $= \dim X + m - n$. Then $\Gamma$ induces correspondences

$$\Gamma_*: CH^m(Y) \longrightarrow CH^n(X)$$
$$\Gamma_*: H^{2m-1}(Y, Q_l/Z_l(m)) \rightarrow H^{2n-1}(X, Q_l/Z_l(n))$$

and the diagram

$$\begin{array}{ccc}
CH^m(Y)(l) & \xrightarrow{\Gamma_*} & CH^n(X)(l) \\
\downarrow{\lambda} & & \downarrow{\lambda} \\
H^{2m-1}(Y, Q_l/Z_l(m)) & \xrightarrow{\Gamma_*} & H^{2n-1}(X, Q_l/Z_l(n))
\end{array}$$

commutes.

Proof: $\Gamma_* = p_2^* \circ (\cdot \Gamma) \circ p_1^*$ where $p_1, p_2$ are the two projections on $Y \times X$. The assertions follow from (3.3)–(3.4). Q.E.D.

Proposition 3.6: The map $\lambda^1: CH^1(X)(l) \rightarrow H^1(X, Q_l/Z_l(1))$ is the natural isomorphism arising from the Kummer sequence

$$0 \longrightarrow \mu_{l^\nu} \longrightarrow G_m \xrightarrow{\iota^*} G_m \longrightarrow 0$$

and the identification

$$CH^1(X) \equiv H^1(X, G_m).$$

Proof: We have in the Zariski topology an exact sequence

$$0 \longrightarrow O_X^* \longrightarrow \bigoplus_{x \in X} \bigoplus_{i_{x^*}} i_{x^*}Z/l^\nu Z \longrightarrow 0$$

and a commutative diagram of cohomology

$$\begin{array}{ccc}
\Ker \partial_{l^\nu} \cong H^1_{et}(X, \mu_{l^\nu}) & \longrightarrow & H^1_{et}(X, G_m) \\
\cong & & \cong \\
\Gamma(X, O^*/O^{l^\nu}) & \longrightarrow & H^1(X, O^*)
\end{array}$$

For $\tilde{f} \in \Ker \partial_{l^\nu} \subset k(X)^*/k(X)^{l^\nu}$, we lift to $f \in k(X)^*$ and write
for some open cover $U_\alpha$ of $X$. The image of $f$ in $H^1(X, O_\alpha)$ is represented by the cocycle $g_\alpha f_\alpha = (f_\alpha f_\beta)^{-1}$. This cocycle is associated to the divisor $-l^{-1}(f)$. The assertion of the proposition is now straightforward. (Note that we changed the sign to define $\lambda$ in (2.7).) Q.E.D.

Suppose now that the ground field is $\mathbb{C}$, the complex numbers. Let $CH^n(X)(l)_0 \subset CH^n(X)(l)$ denote the subgroup of cycles homologous to zero (in $H^{2n}(X, Z)$). Griffiths has defined [5] a complex structure on the torus

$$J^n(X) = H^{2n-1}(X, \mathbb{R})/H^{2n-1}(X, Z)$$

and a cycle map $CH^n(X)_0 \to J^n(X)$. He obtains in this way a map

$$\psi^n_1 : CH^n(X)(l)_0 \to H^{2n-1}(X, \mathbb{Q}/\mathbb{Z}).$$

**Proposition 3.7:** Identify $\mathbb{Q}/\mathbb{Z} \cong \mathbb{Q}/\mathbb{Z}(n)$ by taking $e^{2\pi i/n}$ as the generator of the $l'$-th roots of 1. Then $\psi^n_1 = \lambda^n_1$.

**Proof:** For $x \in X^{n-1}$ and $f \in k(x)^*$, let $|f|$ denote the support of the divisor $(f)$. Such an $f$ gives a map $f : \{x\} - |f| \to \mathbb{P}^1 - \{0, \infty\}$. We fix a simple path $l$ on $\mathbb{P}^1$ with $\partial l = (0) - (\infty)$. $f^{-1}(l)$ is a chain on $\{x\}$ representing a class $f^{-1}(l) \in H_{2e-1}(\{x\}, |f|; \mathbb{Z})$, where $e = \dim(x)$.

Suppose now that $F = (\ldots, f_i, \ldots) \in \ker \partial$. Since $|f_i|$ has complex dimension $e - 1$, we get an exact sequence

$$0 \to H_{2e-1}(X; Z) \to H_{2e-1}(X, \bigcup |f_i|; Z) \to H_{2e-2}(\bigcup |f_i|; Z)$$

and the assignment $F \mapsto F^{-1}(l) = \sum f_i^{-1}(l)$ clearly defines a map $-\lambda' : \ker \partial \to H_{2e-1}(X, Z)$. Similarly, one defines $-\lambda'' : \ker \partial'' \to H_{2e-1}(X, \mathbb{Z}/l'^* \mathbb{Z})$. In fact, the image of $\ker \partial$ is torsion and $\lambda''$ coincides with $\lambda''$ (2.2) up to the identification

$$H_{2e-1}(X, \mathbb{Z}/l'^* \mathbb{Z}) \cong H^{2n-1}(X, \mathbb{Z}/l'^* \mathbb{Z}), \; n = \dim X - e + 1.$$
The cycle map $CH_{e-1}(X)_0 \xrightarrow{\psi} J_{e-1}(X)$ is defined (roughly speaking) by $\psi(\gamma) = \int_\gamma$, where $\partial \Gamma = \gamma$. If we take $\gamma = f^{-1}(\alpha) - f^{-1}(\beta)$ for $\alpha, \beta \in \mathbb{P}^1$, we can take $\Gamma = f^{-1}(l_{\alpha\beta})$ for a path $l_{\alpha\beta}$ from $\alpha$ to $\beta$. Since $\gamma$ is trivial in $CH_{e-1}(X)$, we conclude that $\int_\gamma f^{-1}(l_{\alpha\beta}) \omega$ is constant independent of $\alpha, \beta$ for $\omega \in H^{2e-1,0} + \cdots + H^{e,e-1}$. Allowing $\alpha \to \beta$, we see that this integral is trivial. Since any cycle in $\lambda'(\text{Ker } \partial)$ can be represented as a sum of chains $f^{-1}(l)$, we conclude

$$\int_{\lambda'(\text{Ker } \partial)} (H^{2e-1,0} + \cdots + H^{e,e-1}) = (0),$$

so $\lambda'(\text{Ker } \partial)$ is torsion.

Suppose now $\gamma \in CH_{e-1}(X)_0$ and $l^* \gamma = 0$. We compare the two prescriptions for obtaining a torsion homology class:

Griffiths prescription: Write $\gamma = \partial \Gamma$ and note $\int_\Gamma = l^{-v}. \text{ period.}$

$\lambda'$-prescription: Write $l^* \gamma = \partial F$, $F = (\ldots f_1, \ldots)$. Then $F^{-1}(l)$ is a cycle mod $l^*$ representing $-\lambda' \partial (\gamma)$. Since $l^* \Gamma - \partial F = \text{ period and } f_{\partial F}$ is trivial on $H^{2e-1,0} + \cdots + H^{e,e-1}$, it follows that these two prescriptions coincide. Q.E.D.

**Proposition 3.8:** $\lambda$ is compatible with specialization. Given $X$ smooth and projective over $\text{Sp } R$ ($R$ local) with general fibre $X$ and special fibre $X_0$, there exist for $l$ prime to char $X_0$ specialization maps

$$\sigma : CH^n(X)(l) \longrightarrow CH^n(X_0)(l)$$

$$\tau : H^{2n-1}(X, Q_l/Z_l(n)) \xrightarrow{\cong} H^{2n-1}(X_0, Q_0/Z_l(n)),$$

and $\lambda \sigma = \tau \lambda$.

**Proof:** The existence of $\sigma$ is classical [8]. For $\tau$, cf. the discussion following (2.6) and also [4]. To verify $\lambda \sigma = \tau \lambda$, we may assume after base change that $R$ is a valuation ring with algebraically closed quotient and residue fields. For $x \in X^{*n-1}$ ($X \subset \mathcal{X}$, is the generic fibre) let $R(x) \subset k(x)$ denote the ring of all functions on the normalization of $\{x\}$ (closure in $\mathcal{X}$) which have no pole along components of the special fibre. Let $\mathcal{Y}_x$ denote the same open set of smooth points on this normalization, and let $Y_x \subset \mathcal{Y}_x$ be the general fibre. The morphism $Y_x \subset \mathcal{Y}_x$ is locally acyclic, so $H^1(Y_x, \mu_{l^r}) \cong H^1(\mathcal{Y}_x, \mu_{l^r})$ [4]. Removing all horizontal divisors on $\mathcal{Y}_x$ (i.e. divisors whose support contains no
component of the special fibre) we obtain in the limit
\[ k(x)^* / k(x)^{*l^r} \cong R(x)^* / R(x)^{*l^r}. \]

There is a commutative diagram

\[
\begin{array}{ccc}
\bigcup_{x \in X^{n-1}} k(x)^* / k(x)^{*l^r} & \xrightarrow{\partial_{l^r}(X)} & \bigcup_{x \in X^n} Z/l^r Z \\
\bigcup R(x)^* / R(x)^{*l^r} & \parallel & \bigcup Z/l^r Z \\
\bigcup_{x \in X_0^{n-1}} k_0(x_0)^* / k_0(x_0)^{*l^r} & \xrightarrow{\partial_{l^r}(X_0)} & \bigcup_{x_0 \in X_0^n} Z/l^r Z
\end{array}
\]

The two squares

\[
\begin{array}{ccc}
H^{2n-1}_l(X, Q_l/Z_l(n)) & \xleftarrow{\tau} & \text{Ker } \partial_{l^r}(X) \xrightarrow{\sigma} CH^n_l(X)_{l^r} \\
\tau & & \sigma
\end{array}
\]

also commute, so one deduces \( \lambda \sigma = \tau \lambda \). Q.E.D.

One final compatibility. Suppose \( n = \dim X \) and let \( \text{Alb}(X) \) denote the albanese of \( X \) (the dual of the Picard variety). The \( e_m \)-pairing gives an isomorphism

\[ \text{Alb}(X)(l) \cong H^{2n-1}_l(X, Q_l/Z_l(n)). \]

There is also a map \( CH^n_l(X)(l) \to \text{Alb}(X)(l) \) obtained by mapping a zero cycle on \( X \) to the sum of the corresponding points on the albanese.

**Proposition 3.9**: With notation as above, the diagram

\[
\begin{array}{ccc}
CH^n_l(X)(l) & \longrightarrow & \text{Alb}(X)(l) \\
\bigcup & & \bigcup
\end{array}
\]

\[ H^{2n-1}_l(X, Q_l/Z_l(n)) \]

commutes.
PROOF: We will see in the next section that $CH^n(X)(l) \cong \text{Alb}(X)(l)$. Let $C \subseteq X$ be a general linear space section of dimension 1. Then it is known that $CH^1(C)(l) \rightarrow \text{Alb}(X)(l)$. Using this and (3.3) we reduce to verifying commutativity for

$$
\begin{array}{ccc}
CH^1(C)(l) & \longrightarrow & \text{Alb}(X)(l) \\
\downarrow & & \downarrow \\
H^1(C, \mathbb{Q}/\mathbb{Z}(1)) & \longrightarrow & H^{2n-1}(X, \mathbb{Q}/\mathbb{Z}(n))
\end{array}
$$

This compatibility is known. Q.E.D.

4. On a theorem of Roitman

Roitman has announced a proof of the following important

**Theorem 4.1:** Let $X$ be a smooth projective variety over an algebraically closed field $k$. Let $CH_0(X)_{\text{tors}}$ denote the torsion subgroup of the Chow group $CH_0(X)$ of zero cycles on $X$ modulo rational equivalence, and let $\text{Alb}(X)_{\text{tors}}$ be the torsion subgroup of the Albanese of $X$. Then the natural map

$$
\psi : CH_0(X)_{\text{tors}} \longrightarrow \text{Alb}(X)_{\text{tors}}
$$

is an isomorphism.

I will present in detail the proof of a slightly weaker result.

**Theorem 4.2:** The above map $\psi$ is surjective and is an isomorphism prime to the characteristic of $k$.

**Proof:** Surjectivity. The subgroup $CH_0(X)_{\text{deg}0}$ of zero cycles of degree 0 is known to be divisible, so it suffices to show $CH_0(X)_{l} \rightarrow \text{Alb}(X)_l$, when the subscript indicates the kernel of multiplication by a prime $l$. We use induction on dimension $X$.

When $\dim X = 1$, $CH_0(X)$ is the Picard group, and the assertion is well-known. Assume $\dim X > 1$ and let $Y \subseteq X$ be a smooth hyperplane section. The top horizontal arrow in the diagram

$$
\begin{array}{ccc}
CH_0(Y)_l & \longrightarrow & \text{Alb}(Y)_l \\
\downarrow & & \downarrow \\
CH_0(X)_l & \longrightarrow & \text{Alb}(X)_l
\end{array}
$$
may be assumed surjective, so it suffices to show $\text{Alb}(Y) \to \text{Alb}(X)_t$. Ignoring twists by roots of 1 we have

$$\text{Hom}(H^1_{\text{ét}}(X, Z/lZ), Z/lZ) \to \text{Alb}(X)_t.$$  

(This is true even for $l = \text{char } k$) so it suffices to show $H^1_{\text{ét}}(X, Z/lZ) \to H^1_{\text{ét}}(Y, Z/lZ)$. This follows from Zariski’s connectedness theorem.

**Injectivity.** We use a presentation of the Chow group

$$\prod_{x \in X_1} k(x)^* \to \prod_{x \in X_0} Z \to \text{CH}_0(X) \to 0$$

where $X_i$ denotes the set of points on $X$ whose Zariski closure has dimension $i$. Fix a cycle $z$ on $X$ and an integer $l \geq 1$ such that $lz \sim 0$. Assume further that $\psi(z) = 0$. We must show $z \sim 0$. We may assume $l$ prime.

Let $x_0, \ldots, x_n \in X_1, f_i \in k(x_i), C_i = \{x_i\}$ so that $lz = \partial(f_1, \ldots, f_n)$.

**First reduction:** We may assume the $C_i$ are smooth, and no more than two $C_i$ pass through any point.

Indeed, blowing up a point on $X$ changes neither $\text{CH}_0(X)$ nor $\text{Alb}(X)$. Blow up a succession of points so that the strict transform $\tilde{C}$ of $C$ becomes a disjoint union of smooth curves

$$\tilde{C} = \bigsqcup \tilde{C}_i.$$  

Let $\pi : \tilde{X} \to X$ denote the blowing down map. Let $\tilde{z}$ be a cycle on $\tilde{C}$ with $\pi \tilde{z} = z$, and let

$$w = l\tilde{z} - \tilde{\partial}(f_0, \ldots, f_n)$$

where $f_i$ is viewed as a function on $\tilde{C}_i$, and

$$\tilde{\partial} : \bigsqcup_{\tilde{x} \in \tilde{X}_1} k(\tilde{x})^* \to \bigsqcup_{\tilde{x} \in \tilde{X}_0} Z.$$  

Note $\pi(w) = 0$. Let $E_1, \ldots, E_r$ denote the connected components of the exceptional locus of $\pi$. $w$ can be written $w = w_1 + \cdots + w_r$ where each $w_i$ has degree 0 and is supported on $E_i$. Since $E_i$ is a union of projective spaces we can find lines $l_{ij} \subset E_i$ with $w_i$ supported on $\bigcup l_{ij}$ and functions $g_{ij}$ on $l_{ij}$ such that $\partial(g_{ij}, \ldots, g_{ij}, \ldots) = w_i$. We can further
arrange that $C' = \bar{C} \cup \bigcup_{i,j} l_{ij}$ has at most two components through any point as desired.

**Second reduction:** We may assume $X$ is a surface.

Indeed, let notation be as above, and let $Y$ be a general linear space section of large degree and dimension 2 containing $C$. Our hypothesis about $C$ will guarantee that $Y$ is smooth. Moreover, $\text{Alb}(X) \cong \text{Alb}(Y)$ and $l_z \sim 0$ on $Y$. Hence we may replace $X$ by $Y$.

**Third reduction:** We may assume $|z|$ consists of smooth points of $C = \bigcup C_i$.

It suffices to write $z = z_0 + \cdots + z_n$ with $|z_i| \subset C_i$, and then move $z_i$ by a rational equivalence on $C_i$ so that $|z_i| \cap C_j = \emptyset$, all $i \neq j$.

**Fourth reduction:** We may assume $z$ represents an $l$-torsion point in the (generalized) jacobian $J(C)$.

This is trickier. We proceed by induction on $N =$ number of pairs of irreducible components $C_i, C_j$ such that some point of $C_i \cap C_j$ is a zero or pole of $f_i$ or $f_j$. If $N = 0$, then the class of $l_z$ lies in $\text{Ker}(J(C) \rightarrow J(\bar{C})) = K$. Since $K$ is divisible and lies in $\text{Ker}(J(C) \rightarrow \text{CH}_0(X))$ we can replace $z$ by $z + k$ for some $k \in K$ and suppose $l_z = 0$ in $J(C)$.

Suppose now $N > 0$. Write $C_\infty = \bigcup_{i=1}^n C_i$. Renumbering if necessary, we may assume some point of $C_0 \cap C_\infty$ is a zero or pole of $f_0$. Then adding irreducible curves in general position to $C_0$ and $C_\infty$, we may assume $C_0$ linearly equivalent to $C_\infty$ and very ample. ($C_0$ may no longer be irreducible.) We may further suppose that a general element of the corresponding pencil $\{C_i\}$ is smooth.

We next write

$$z = z_0 - z_\infty$$

with $|z_0| \subset C_0$, $|z_\infty| \subset C_\infty$ and degree $z_0 = \text{degree } z_\infty = 0$. This can be done in such a way that $l_{z_0} = \delta$ in $J(C_0)$, with $|\delta| \subset C_0 \cap C_\infty$. Let $\mathcal{C} \rightarrow \mathbb{P}^1$ be obtained by blowing up the base locus of the pencil $\{C_i\}$. $\delta$ gives rise to a divisor $\Delta$ on $\mathcal{C}$ with $\delta = \Delta \cdot C_0$.

Write $\Delta$ also for the corresponding section of the relative Picard scheme $\Delta : \mathbb{P}^1 \rightarrow \text{Pic}(\mathcal{C}/\mathbb{P}^1)$. Multiplication by $l$ induces a map $l_* : \text{Pic} \rightarrow \text{Pic}$ and we consider the scheme $l_*(\Delta) \subset \text{Pic}$. This is a principal homogeneous space under $\text{Pic}_l = l_*(0)$ and is non-empty. Let $Z$ be the completion of the normalization of an irreducible component of $l_*(\Delta)$ containing the point of $J(C_0) \subset \text{Pic}(\mathcal{C}/\mathbb{P}^1)$ corresponding to $z_0$. It follows from Lemma 1 (below) that $Z$ maps onto $\mathbb{P}^1$. Let $w_0, w_\infty \in Z$ be points of $Z$ lying over 0 and $\infty$, with $w_0$ coinciding in $\text{Pic}$ with the class of $z_0$. 

\[\text{arrange that } C' = \bar{C} \cup \bigcup_{i,j} l_{ij} \text{ has at most two components through any point as desired.}\]

\[\text{Second reduction: We may assume } X \text{ is a surface.}\]

\[\text{Indeed, let notation be as above, and let } Y \text{ be a general linear space section of large degree and dimension 2 containing } C. \text{ Our hypothesis about } C \text{ will guarantee that } Y \text{ is smooth. Moreover, } \text{Alb}(X) \cong \text{Alb}(Y) \text{ and } l_z \sim 0 \text{ on } Y. \text{ Hence we may replace } X \text{ by } Y.\]

\[\text{Third reduction: We may assume } |z| \text{ consists of smooth points of } C = \bigcup C_i.\]

\[\text{It suffices to write } z = z_0 + \cdots + z_n \text{ with } |z_i| \subset C_i, \text{ and then move } z_i \text{ by a rational equivalence on } C_i \text{ so that } |z_i| \cap C_j = \emptyset, \text{ all } i \neq j.\]

\[\text{Fourth reduction: We may assume } z \text{ represents an } l\text{-torsion point in the (generalized) jacobian } J(C).\]

\[\text{This is trickier. We proceed by induction on } N = \text{number of pairs of irreducible components } C_i, C_j \text{ such that some point of } C_i \cap C_j \text{ is a zero or pole of } f_i \text{ or } f_j. \text{ If } N = 0, \text{ then the class of } l_z \text{ lies in } \text{Ker}(J(C) \rightarrow J(\bar{C})) = K. \text{ Since } K \text{ is divisible and lies in } \text{Ker}(J(C) \rightarrow \text{CH}_0(X)) \text{ we can replace } z \text{ by } z + k \text{ for some } k \in K \text{ and suppose } l_z = 0 \text{ in } J(C).\]

\[\text{Suppose now } N > 0. \text{ Write } C_\infty = \bigcup_{i=1}^n C_i. \text{ Renumbering if necessary, we may assume some point of } C_0 \cap C_\infty \text{ is a zero or pole of } f_0. \text{ Then adding irreducible curves in general position to } C_0 \text{ and } C_\infty, \text{ we may assume } C_0 \text{ linearly equivalent to } C_\infty \text{ and very ample. (}C_0 \text{ may no longer be irreducible.) We may further suppose that a general element of the corresponding pencil } \{C_i\} \text{ is smooth.}\]

\[\text{We next write } z = z_0 - z_\infty\]

\[\text{with } |z_0| \subset C_0, |z_\infty| \subset C_\infty \text{ and degree } z_0 = \text{degree } z_\infty = 0. \text{ This can be done in such a way that } l_{z_0} = \delta \text{ in } J(C_0), \text{ with } |\delta| \subset C_0 \cap C_\infty. \text{ Let } \mathcal{C} \rightarrow \mathbb{P}^1 \text{ be obtained by blowing up the base locus of the pencil } \{C_i\}. \text{ }\]

\[\text{Write } \Delta \text{ also for the corresponding section of the relative Picard scheme } \Delta : \mathbb{P}^1 \rightarrow \text{Pic}(\mathcal{C}/\mathbb{P}^1). \text{ Multiplication by } l \text{ induces a map } l_* : \text{Pic} \rightarrow \text{Pic} \text{ and we consider the scheme } l_*(\Delta) \subset \text{Pic}. \text{ This is a principal homogeneous space under } \text{Pic}_l = l_*(0) \text{ and is non-empty. Let } Z \text{ be the completion of the normalization of an irreducible component of } l_*(\Delta) \text{ containing the point of } J(C_0) \subset \text{Pic}(\mathcal{C}/\mathbb{P}^1) \text{ corresponding to } z_0. \text{ It follows from Lemma 1 (below) that } Z \text{ maps onto } \mathbb{P}^1. \text{ Let } w_0, w_\infty \in Z \text{ be points of } Z \text{ lying over } 0 \text{ and } \infty, \text{ with } w_0 \text{ coinciding in } \text{Pic} \text{ with the class of } z_0.\]
With reference to the diagram

\[ \begin{array}{ccc}
X & \xrightarrow{\pi} & Y \\
\downarrow{\sigma} & & \downarrow{\sigma'} \\
Z \times \mathbb{P}^1 & \xrightarrow{p} & \mathbb{P}^1 \\
\downarrow{p'} & & \downarrow{f} \\
Z & \xrightarrow{q} & \mathbb{P}^1
\end{array} \]

there exists a divisor \( D \) on \( Z \times \mathbb{P}^1 \) such that

\[ lD \sim \pi^*\Delta + p^*(t) \]

for some divisor \( t \) on \( Z \). If we fix a divisor \( w \) on \( Z \) with \( lw \sim (w_0) - (w_\infty) \), we find

\[
\pi_*(p^*((w_0) - (w_\infty)) \cdot D) = \pi_*(p^*w \cdot \pi^*\Delta) = \sigma_*(\pi_*(p^*w) \cdot \Delta) \\
= \sigma_*(f^*q^*w \cdot \Delta) \sim 0
\]

because \( q_*w = 0 \).

Let \( z'_o = \pi_*(p^*w_\infty \cdot D) \). We have \( z'_o \subset C_\infty, z'_o \sim z_0 \), and

\[ l_\infty z'_o \sim \pi_*(p^*w_\infty \cdot \pi^*\Delta) \sim \sigma_*(f^*(z) \cdot \Delta) = \delta. \]

In fact, \( [lz'_o] = [\delta] \) in \( J(C_\infty) \). In particular, there exist functions \( g_i \) on \( C_i, i = 1, \ldots, n \) such that the \( g_i \) have no zeros or poles on \( C_i \cap C_j \) (\( i, j = 1, \ldots, n \)) and such that

\[ \sum (g_i) = lz'_o - \delta. \]

Then taking \( z' = z'_o - z_\infty \) we find \( z' \sim z \) and

\[ \delta(f_1g_1, f_2g_2, \ldots, f_ng_n) = lz'_o - \delta - (lz_\infty - \delta) = lz'. \]

Since \( N(z, (f_0, \ldots, f_n)) > N(z', (f_1g_1, \ldots, f_ng_n)) \) the induction is complete.

For simplicity, I assume henceforth \( l \) prime to char \( k \).

**Lemma 1**: With notation as above, all irreducible components of the scheme \( \text{Pic}(\mathbb{P}/\mathbb{P}^1) \), map onto non-empty open sets of \( \mathbb{P}^1 \).
PROOF: The fibre of Pic, over a point 0 ∈ P^1 is H^1(C_0, Z/lZ) (up to twisting). Since an étale cover of C_0 lifts to an étale cover of C × P^1 Sp(Ω^1_{P^1,0}) we see that a closed point in the fibre of Pic, over 0 necessarily spreads out to cover some open set in P^1. Q.E.D.

Returning now to the proof that

\[ \psi: CH_0(X)_{\text{tors}} \longrightarrow \text{Alb}(X)_{\text{tors}} \]

is injective prime to the characteristic, we have a cycle z, an integer l prime to char k, and a very ample curve C such that |z| ⊂ smooth points of C, [lz] = 0 in J(C), and ψ(z) = 0. We must show z ∼ 0 in CH_0(X). For this we have

Fifth reduction: We may assume C smooth.

Indeed, we fix a general pencil ℂ as before with C = C_0 and denote by Z the completion of the normalization of an irreducible component of Pic_0 containing a point w_0 lying over 0 ∈ P^1 and mapping to [z] ∈ J(C_0). We pick some general, smooth fibre C_1 ⊂ ℂ and a point w_1 ∈ Z lying over the same point 1 ∈ P^1. As before there is a divisor D on Z × P^1 ℂ well-defined up to rational equivalence and vertical fibres such that (cf. diagram (1))

\[ \pi_*(p^*w_0 \cdot D) \sim z \]

\[ lD \sim p^*(t) \] some divisor t on Z.

Let z' = π_*(p^*w_1 · D), and let w be a divisor on Z such that lw ∼ (w_1) − (w_0). Then

\[ z' \sim z + \pi_*(p^*(lw) \cdot D) \sim z \]

and |z'| ⊂ C_1. Also [lz'] = 0 in J(C_1), completing the reduction.

And now (the moment you’ve all been waiting for) comes the point.

Key Lemma: Let X be a smooth projective surface, C ⊂ X a smooth hyperplane section. Let l be an integer prime to the characteristic. Then the two maps

\[ \begin{array}{ccc}
J(C)_l & \xrightarrow{\text{Alb}(X)_l} & CH_0(X)_l \\
\end{array} \]

have the same kernel.
Notice that this lemma will prove the theorem.

**Proof of Lemma:** As before we take a general (lefschetz) pencil $\mathcal{C}$ with $C = \mathcal{C}_0$. We have a morphism of schemes

$$\text{Pic}(\mathcal{C}/\mathbb{P}^1) \xrightarrow{h} \text{Alb}(X) \times_{\text{Sp}_k} \mathbb{P}^1.$$ 

Let $V = \text{Ker } h$, $V_{l^r} \subset V$ the $l^r$-torsion. There are diagrams

$$V_{l^r} = \text{Ker}(H^1_{\text{et}}(C, \mathbb{Z}/l^r) \to H^3_{\text{et}}(X, \mathbb{Z}/l^r)) \subseteq V_{l^r,0}$$

which induce an isomorphism in the limit

$$\text{Ker}(H^1_{\text{et}}(C, \mathbb{Q}/\mathbb{Z}) \to H^3_{\text{et}}(X, \mathbb{Q}/\mathbb{Z})) \xrightarrow{\sim} V_{l^r,0}.$$ 

Thus, given $\gamma \in J(C) \cong H^1_{\text{et}}(C, \mathbb{Z}/l\mathbb{Z})$ with $\gamma \mapsto 0$ in $\text{Alb}(X)$, $\gamma' = "l^r"\gamma \in V_{l^r,0}$ for $\nu \gg 0$, i.e., $\gamma'$ is a vanishing cycle. I claim $V_{l^r}$ contains an irreducible component $Z$ such that any $\gamma' \in V_{l^r,0}$ can be written

$$\gamma' = \sum n_i \gamma_i$$

with $\gamma_i \in Z_0$ and $\sum n_i = 0$. Assume this claim for a moment. Given $\gamma \in V_{l^r,0}$, write $\gamma' = "l^r"\gamma = \sum n_i \gamma_i$. Just as before, there is a divisor $D$ on $Z \times_{\mathbb{P}^1} \mathcal{C}$ and (notation as in diagram 1)

$$l^rD \sim p^*(t)$$

$$\gamma' \sim \pi_* \left( p^* \sum n_i(\gamma_i) \cdot D \right) \text{ in } CH_0(X).$$

Since $V_{l^r}$ is étale over 0 the $\gamma_i$ are smooth points of $Z$, so $\sum n_i(\gamma_i) \in J(Z)$, a divisible group. It follows that $\gamma' \sim 0$ in $CH_0(X)$ whence $\gamma \sim 0$ in $CH_0(X)$ as desired. (Note $\gamma' = \gamma$ viewed as a point of order $l^r$.)

**Proof of Claim:** The group $V_{l^r,0}$ of vanishing cycles is known to be generated by certain cycles $\delta_i (\text{mod } l^r)$ (the vanishing cycles) which are all conjugate under the monodromy group of the pencil.
Let \( Z \) be the component of \( V_r \) containing one (and hence all) the \( \delta_i \). Associated with each \( \delta_i \) is a \( \sigma_i \in \pi_i(\mathbb{P}^1 - \Sigma, 0) \) where \( \Sigma \) is the finite set of points where the fibres of \( f \) are singular. The \( \sigma_i \) generate \( \pi_1 \), and for \( \chi \in V_{r,0} \), the Picard-Lefschetz formula says

\[
\sigma_i(\chi) = \chi \pm \langle \chi \cdot \delta_i \rangle \delta_i
\]

where \( \langle \cdot \rangle \) denotes the intersection pairing. Since the \( \delta_i \) generate \( V'_{r,0} \) we can write

\[
\gamma' = \sum m_i \delta_i.
\]

The trick is to get \( \Sigma m_i = 0 \pmod{l^r} \). Suppose \( \Sigma m_i = m \) and that \( m_i|m_i \), all \( i \). Let \( q \) be such that \( \langle \delta_i \cdot \delta_i \rangle \) is invertible in \( \mathbb{Z}/l^r \mathbb{Z} \). Such a \( q \) must exist, otherwise by Picard-Lefschetz, writing \( V' = \text{Ker}(H^1(C, \mathbb{Z}_l) \to H^2(X, \mathbb{Z}_l)) \), we would have \( \sigma_q(\delta_i) - \delta_i \in IV' \). Since the \( \sigma_q(\delta_i) \) generate \( V' \) and rank \( V' \) is even, this is not possible. Let \( r \) be such that

\[
\sigma'_q(\delta_i) = \delta_i + r < \delta_i \cdot \delta_q \delta_q = \delta_i + \frac{m}{m_i} \delta_q.
\]

Then \( \sigma'_q(\delta_i) \in \mathbb{Z}_0 \), and

\[
\gamma' = m_i \sigma'_q(\delta_i) + \sum_{i>1} m_i \delta_i - m \delta_q.
\]

This verifies the claim and completes the proof of the theorem.

5. Relations with algebraic K-theory

In this final section I want to reconsider the map \( \rho \) in (2.3) from the point of view of algebraic K-theory. \( K_i \) will denote the Zariski sheaf on \( X \) associated to the \( i \)-th Quillen K-group, [6]. We will take \( n = \dim X \). \( K_i(X) \) will denote the \( i \)-th global K-group of (the category of vector bundles on) the variety, and \( SK_1(X) = \text{Ker}(K_1(X) \to k^* = \Gamma(X, O_X^*)) \). Finally \( T(l) \subset H^{2n-1}_\text{et}(X, \mathbb{Z}_l(n)) \) will denote the torsion subgroup (Pontryagin dual to the torsion subgroup of the Neron-Severi group of \( X \)).

**Theorem 5.1:** There is a surjective map \( H^{n-1}(X, K_n) \to T(l) \). When \( \dim X = 2 \), we obtain \( SK_1(K) \to T(l) \).
PROOF: A construction of Tate gives a map

$$K_2(F) \to H^2(F, \mu_{l^2}) \quad l \text{ prime to char } F$$

for any field $F$. We obtain a commutative diagram

$$
\begin{array}{ccc}
\coprod_{x \in X^{n-2}} K_2(k(x)) & \to & \coprod_{x \in X^{n-1}} k(x)^* \\
\downarrow & & \downarrow \\
\coprod_{x \in X^{n-2}} H^2(k(x), \mu_{l^2}^{\otimes 2}) & \to & \coprod_{x \in X^{n-1}} k(x)^*/k(x)^{*l} \to \coprod_{x \in X^{n}} \mathbb{Z}/l^n\mathbb{Z}
\end{array}
$$

where the top line comes via a spectral sequence

$$
\coprod_{x \in X^n} K_{q-p}(k(x)) \Rightarrow K_{q-p}(X)
$$

due to Quillen. Quillen also shows that the $E_1$ complexes compute the cohomology of the sheaves $K_p$ just as in (1.3). We deduce from the above diagram (cf. (2.3)) a map

$$\tilde{\rho}: H^{n-1}(X, K_n) \to T(l).$$

For $\nu \gg 0$ we have $T(l) \subset H^{2n-1}(X, Z(n))/l^n \subset H^{2n-1}(X, Z/l^n\mathbb{Z}(n))$ and a diagram

$$
\begin{array}{ccc}
0 & \to & H^{n-1}(X, K_n)/l^n \\
\downarrow \tilde{\rho} & & \downarrow \alpha \\
0 & \to & H^{n-1}(X, Z/l^n\mathbb{Z}(n)) \to H^{2n-1}(X, Z/l^n\mathbb{Z}(n))/T(l) \to 0
\end{array}
$$

where $\alpha$ comes as in (2.2) and is surjective because for $n = \dim X$, $H^{n-1}(X, H^n) = H^{2n-1}_d(X)$, and $\lambda$ is an isomorphism by §4. We conclude that $\tilde{\rho}$ is surjective. When $n = 2$, an examination of (5.2) shows $SK_1(X) \to H^1(X, K_2)$, so we obtain $SK_1(X) \to T(l)$. Q.E.D.
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