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## TANGENT CONES AT GORENSTEIN SINGULARITIES

Judith D. Sally\*

### Introduction

If  $x$  is a Cohen–Macaulay point of a  $d$ -dimensional affine variety  $V$ , then the embedding dimension of  $V$  at  $x$  is bounded above by  $e_x + d - 1$ , where  $e_x$  is the multiplicity of  $V$  at  $x$ , [1]. In this paper the following result is proved.

**THEOREM:** *If  $x$  is a Gorenstein point of an affine variety  $V$ , the tangent cone at  $x$  is Gorenstein if the embedding dimension of  $V$  at  $x$  is  $d$ ,  $d + 1$ ,  $e_x + d - 2$  or  $e_x + d - 1$ . Also, for any  $d \geq 0$ , there is an affine variety  $V$  with Gorenstein point  $x$  such that the embedding dimension of  $V$  at  $x$  is  $d + 2 = e_x + d - 3$  and the tangent cone at  $x$  is not Gorenstein.*

An immediate application of this result is that the Hilbert function of a Gorenstein singularity  $x$ , in the case where the embedding dimension at  $x$  is  $e_x + d - 2$  with  $d \geq 1$ , is a polynomial for  $n \geq 2$  and is completely determined by the multiplicity at  $x$ . In fact,

$$H_x(n) = e_x \binom{n+d-2}{d-1} + \binom{n+d-3}{n}, \quad n \geq 2.$$

Other applications will be presented in a subsequent paper.

The author proved in [12] an analogous result for Cohen–Macaulay points. Namely that if  $x$  is a Cohen–Macaulay point then the tangent cone at  $x$  is Cohen–Macaulay if the embedding dimension at  $x$  is  $d$ ,  $d + 1$  or  $e_x + d - 1$ . D. Eisenbud and J. Wahl informed the author of the application of [12] to rational surface singularities which have, by a result of M. Artin, [2], embedding dimension  $e_x + 1$ . In [14], Wahl proves directly that the tangent cone at a rational surface singularity

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is Cohen–Macaulay. He also proves that the minimally elliptic surface singularities of Laufer [7] have Gorenstein tangent cones. This is a special case of the result to be proved in this paper as minimally elliptic singularities are Gorenstein surface singularities  $x$  with embedding dimension  $e_x$ . The result answers a question posed to the author by D. Eisenbud.

## 1. Some preliminaries

The techniques which we will use are purely algebraic so we rephrase the problem in terms of local rings and associated graded rings. Henceforth,  $(R, \mathfrak{m})$  is a local ring of dimension  $d \geq 0$ . (The term “local” includes “Noetherian.”)  $G(R) = R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \dots$  is the associated graded ring with maximal homogeneous ideal  $\mathcal{M} = \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \dots$ . We denote the map  $R \rightarrow G(R)$  which takes each element  $x$  of  $R$  to its initial form in  $G(R)$  by “ $-$ ”, i.e.,  $x \rightarrow \bar{x}$  in  $\mathfrak{m}^t/\mathfrak{m}^{t+1}$ , where  $x \in \mathfrak{m}^t \setminus \mathfrak{m}^{t+1}$ , some  $t \geq 0$ .  $v_R = \dim_{R/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$  is the embedding dimension of  $R$ . If  $I$  is an  $\mathfrak{m}$ -primary ideal of  $(R, \mathfrak{m})$ ,  $e(I)$  will denote the multiplicity of  $I$ .  $e(\mathfrak{m})$  is the multiplicity of  $R$  and will be denoted by  $e_R$ .

We will need some facts about reductions of ideals, cf. [11]. A minimal reduction for  $\mathfrak{m}$  is a sequence  $x_1, \dots, x_d$  of elements of  $\mathfrak{m}$  such that  $\mathfrak{m}^{r+1} = (x_1, \dots, x_d)\mathfrak{m}^r$ , for some non-negative integer  $r$ . Viewed in  $G(R)$ , a minimal reduction for  $\mathfrak{m}$  is just a sequence  $x_1, \dots, x_d$  of elements of  $\mathfrak{m}$  with initial forms  $\bar{x}_1, \dots, \bar{x}_d$  forming a system of homogeneous parameters of degree 1 for  $G(R)$ , i.e.,  $\bar{x}_1, \dots, \bar{x}_d$  are elements of  $\mathfrak{m}/\mathfrak{m}^2$  and  $\mathcal{M}^{r+1} \subset (\bar{x}_1, \dots, \bar{x}_d)G(R)$ , for some non-negative integer  $r$ . From this point of view, it is well known that, if  $R/\mathfrak{m}$  is infinite – a hypothesis which will never cause us any problem, then there is a system  $\bar{x}_1, \dots, \bar{x}_d$  of homogeneous parameters of degree 1 for  $G(R)$ ; in other words, if  $R/\mathfrak{m}$  is infinite, minimal reductions for  $\mathfrak{m}$  exist, cf. [11]. The integer  $r$  that appears above is important, so we make the following definition.

**1.1. DEFINITION:** Assume that  $R/\mathfrak{m}$  is infinite. The *reduction exponent*  $r(R)$  of  $R$  is the least integer  $r$  such that there is a system of parameters  $x_1, \dots, x_d$  of  $R$  with  $\mathfrak{m}^{r+1} = (x_1, \dots, x_d)\mathfrak{m}^r$ .

It is not hard to see that if  $x_1, \dots, x_d$  is a minimal reduction for  $\mathfrak{m}$ , then  $e((x_1, \dots, x_d)R) = e_R$ . Thus, if  $(R, \mathfrak{m})$  is a Cohen–Macaulay ring and  $x_1, \dots, x_d$  is a minimal reduction for  $\mathfrak{m}$ , then  $e_R =$

$\lambda(R/(x_1, \dots, x_d)R)$ , where  $\lambda = \lambda_A$  denotes the length of an  $A$ -module over an Artinian local ring  $A$ .

It is proved in [11] that if  $x_1, \dots, x_d$  is a minimal reduction for  $\mathfrak{m}$ , then the elements  $x_1, \dots, x_d$  are analytically independent in  $\mathfrak{m}$ . This means that if  $f(Y_1, \dots, Y_d)$  is any form of (arbitrary) degree  $s$  with coefficients in  $R$  such that  $f(x_1, \dots, x_d) \in \mathfrak{m}^{s+1}$ , then all the coefficients of  $f$  are in  $\mathfrak{m}$ .

The proof of the result stated in the introduction has two steps. First, we show that for any Cohen–Macaulay local ring  $(R, \mathfrak{m})$ ,  $G(R)$  is Cohen–Macaulay if  $r(R) \leq 2$ . Then we show that the given hypotheses on  $(R, \mathfrak{m})$  force  $r(R) = 2$ .

## 2. The reduction exponent

The following theorem generalizes Theorem 2 in [12].

**2.1. THEOREM:** *Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional Cohen–Macaulay local ring with  $R/\mathfrak{m}$  infinite. If  $r(R) \leq 2$ , then  $G(R)$  is Cohen–Macaulay.*

**PROOF:** By hypothesis, there is a regular sequence  $x_1, \dots, x_d$  in  $R$  with  $\mathfrak{m}^3 = (x_1, \dots, x_d)\mathfrak{m}^2$ . By [4] or [9], it is sufficient to prove that  $\bar{x}_1, \dots, \bar{x}_d$  is a regular sequence in  $G(R)$ . We do this by induction on  $d$ . There is no problem if  $d = 0$ , so we let  $d = 1$ . Suppose  $yx_1 \in \mathfrak{m}^t$ , with  $t \geq 3$ . Then  $yx_1 \in x_1\mathfrak{m}^{t-1}$  and, since  $x_1$  is a nonzero divisor in  $R$ ,  $y \in \mathfrak{m}^{t-1}$ . This shows that  $\bar{x}_1$  is a nonzero divisor in  $G(R)$ .

Assume that  $d > 1$ . We first check that  $\bar{x}_1$  is a nonzero divisor in  $G(R)$ . Suppose  $yx_1 \in \mathfrak{m}^t$  with  $t \geq 3$ . Then  $yx_1 \in (x_1, \dots, x_d)^{t-2}\mathfrak{m}^2$ . We have  $yx_1 = x_1g(x_1, \dots, x_d) + h(x_2, \dots, x_d)$ , where  $g$  is a homogeneous polynomial of degree  $t-3$  in  $x_1, \dots, x_d$  with coefficients in  $\mathfrak{m}^2$  and  $h$  is a homogeneous polynomial of degree  $t-2$  in  $x_2, \dots, x_d$  with coefficients in  $\mathfrak{m}^2$ . Thus,  $(y-g)x_1 = h$ . Since the associated graded ring of  $R$  with respect to the ideal  $\mathfrak{x}R = (x_1, \dots, x_d)R$ , namely the ring  $R/\mathfrak{x}R \oplus \mathfrak{x}R/(\mathfrak{x}R)^2 \oplus (\mathfrak{x}R)^2/(\mathfrak{x}R)^3 \oplus \dots$ , is a polynomial ring over  $R/\mathfrak{x}R$ , it follows that  $y-g \in (x_1, \dots, x_d)^{t-2}R$ . (Actually, by [5],  $y-g \in (x_2, \dots, x_d)^{t-2}R$ .) Thus  $y = g + \ell$ , where  $\ell$  is a homogeneous polynomial of degree  $t-2$  in  $x_1, \dots, x_d$ . But  $yx_1 = gx_1 + \ell x_1$  is in  $\mathfrak{m}^t$  and, since  $gx_1 \in \mathfrak{m}^t$ , it follows that  $\ell x_1 \in \mathfrak{m}^t$ . By the analytic independence in  $\mathfrak{m}$  of  $x_1, \dots, x_d$ , all the coefficients of  $\ell$  are in  $\mathfrak{m}$ . Hence  $y \in \mathfrak{m}^{t-1}$ . Thus the image of  $x_1$  in  $G(R)$  is a nonzero divisor in  $G(R)$  and  $G(R/x_1R) \cong G(R)/\bar{x}_1G(R)$ .  $R/x_1R$  is a  $(d-1)$ -dimensional Cohen–

Macaulay local ring which satisfies the hypotheses of the theorem, so the theorem follows by induction.

2.2. EXAMPLE: If  $(R, \mathfrak{m})$  is Cohen–Macaulay and  $r(R) = 3$ , then  $G(R)$  need not be Cohen–Macaulay. Let  $R = k[[t^4, t^5, t^{11}]]$ ,  $k$  any (infinite) field and  $t$  an indeterminate. Then  $\mathfrak{m}^4 = t^4 \mathfrak{m}^3$ , where  $\mathfrak{m} = (t^4, t^5, t^{11})R$ .  $R \cong k[[X, Y, Z]]/(XZ - Y^3, YZ - X^4, Z^2 - X^3 Y^2)$  and  $G(R) \cong k[X, Y, Z]/(XZ, YZ, Z^2, Y^4)$ .  $G(R)$  is not Cohen–Macaulay.

With regard to this example and similar ones, it is interesting to recall, cf. [10], that if  $(R, \mathfrak{m})$  is any 1-dimensional complete local domain with  $e_R > 1$ , then  $G(R)$  has nonzero prime nilradical. However, the analysis in [10] does not give information about whether  $\mathcal{M}$  belongs to zero in  $G(R)$ .

### 3. Gorenstein local rings

If  $(R, \mathfrak{m})$  is any  $d$ -dimensional local Cohen–Macaulay ring, then, by [1],  $v_R \leq e_R + d - 1$ . If  $v_R = d$  or  $d + 1$ , it is well-known that  $G(R)$  is Cohen–Macaulay. In [12], it is proved that for  $v_R = e_R + d - 1$ , it is also true that  $G(R)$  is Cohen–Macaulay. In fact, it is proved that  $v_R = e_R + d - 1$  implies that  $r(R) = 1$ . The following proposition shows that  $v_R = e_R + d - 1$  is *not* an interesting case if  $R$  is Gorenstein.

3.1. PROPOSITION: *Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional local Cohen–Macaulay ring of embedding dimension  $e_R + d - 1$ , with  $e_R > 1$ . Then,*

$$\dim_{R/\mathfrak{m}} \text{Ext}_R^d(R/\mathfrak{m}, R) = e_R - 1.$$

PROOF: We may assume that  $R/\mathfrak{m}$  is infinite. By [12],  $v_R = e_R + d - 1$  implies that there is a regular sequence  $x = x_1, \dots, x_d$  in  $R$  with  $\mathfrak{m}^2 = x\mathfrak{m}$ . Let  $R^* = R/xR$  and  $\mathfrak{m}^* = \mathfrak{m}/xR$ . Then  $\mathfrak{m}^* = (0 : \mathfrak{m}^*) = \{a \in R^* \mid a\mathfrak{m}^* = 0\}$ . But  $\text{Ext}_R^d(R/\mathfrak{m}, R) \cong \text{Hom}_R(R/\mathfrak{m}, R/xR) \cong (0 : \mathfrak{m}^*)$ . Now,  $\lambda_{R^*}(R^*) = e_R$ . Hence,  $\dim_{R/\mathfrak{m}} \text{Ext}_R^d(R/\mathfrak{m}, R) = \dim_{R/\mathfrak{m}}(0 : \mathfrak{m}^*) = \dim_{R^*/\mathfrak{m}^*} \mathfrak{m}^* = e_R - 1$ .

3.2. COROLLARY: *If  $(R, \mathfrak{m})$  is a Gorenstein local ring of embedding dimension  $e_R + d - 1$  with  $e_R > 1$ , then  $e_R = 2$ .*

PROOF:  $R$  Gorenstein implies that  $\dim_{R/\mathfrak{m}} \text{Ext}_R^d(R/\mathfrak{m}, R) = 1$ .

We turn to Cohen–Macaulay rings of embedding dimension  $e_R + d - 2$ . Unlike the case of embedding dimension  $e_R + d - 1$ , there do

exist Gorenstein local rings of embedding dimension  $e_R + d - 2$  with  $e_R$  any positive integer  $> 2$ . For example, let  $e$  be any positive integer  $> 2$  and set  $R = k[[t^e, t^{e+1}, \dots, t^{2e-2}]]$ , where  $k$  is a field.  $R$  is a 1-dimensional complete local domain with  $v_R = e - 1$  and  $e_R = e$  so  $v_R = e_R + d - 2$ . The numerical semigroup  $S$  generated by  $e, e + 1, \dots, 2e - 2$  is symmetric because the conductor of  $S$  is  $2e$  and the number of elements in  $S$  which are less than  $2e$  is  $e$ . It follows from [6] that  $R$  is Gorenstein. Let  $d > 1$ . Then  $T = R[[X_1, \dots, X_{d-1}]]$  is a Gorenstein local ring of dimension  $d$ .  $e_T = e_R = e$  and  $v_T = e_T + d - 2$ .

3.3. PROPOSITION: *Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional local Cohen-Macaulay ring. Let  $\mathbf{x} = x_1, \dots, x_d$  be a minimal reduction for  $\mathfrak{m}$ . Then  $\lambda_{R/xR}(\mathfrak{m}^2/x\mathfrak{m}) = 1$  and  $\mathfrak{m}^3 \subset x\mathfrak{m}$  if and only if  $v_R = e_R + d - 2$ .*

PROOF: We have the exact sequence

$$0 \rightarrow \text{Tor}_1^R(R/xR, R/\mathfrak{m}) \rightarrow \mathfrak{m}/x\mathfrak{m} \rightarrow R/xR \rightarrow R/\mathfrak{m} \rightarrow 0$$

from which it follows that  $\lambda_{R/xR}(\mathfrak{m}/x\mathfrak{m}) = e_R + d - 1$ . Since

$$0 \rightarrow \mathfrak{m}^2/x\mathfrak{m} \rightarrow \mathfrak{m}/x\mathfrak{m} \rightarrow \mathfrak{m}/\mathfrak{m}^2 \rightarrow 0$$

is exact,  $\lambda_{R/xR}(\mathfrak{m}^2/x\mathfrak{m}) = 1$  if and only if  $v_R = e_R + d - 2$ . We have  $\mathfrak{m}^2 \supseteq (\mathfrak{m}^3, x\mathfrak{m}) \supseteq x\mathfrak{m}$ . If  $v_R = e_R + d - 2$ , then  $\mathfrak{m}^2 \neq x\mathfrak{m}$ , so  $\mathfrak{m}^3 \subset x\mathfrak{m}$ .

In general,  $v_R = e_R + d - 2$  does not imply that  $\mathfrak{m}^3 = x\mathfrak{m}^2$ , as Example 2.2 shows. However, we will now see that if  $(R, \mathfrak{m})$  is Gorenstein of embedding dimension  $v_R = e_R + d - 2$ , then  $\mathfrak{m}^3 = x\mathfrak{m}^2$ . In the proof of (3.4) below, we will use the fact that if  $(R, \mathfrak{m})$  is a 0-dimensional local Gorenstein ring and  $I$  is an ideal, then  $\lambda_R(R/I) = \lambda_R((0 : I))$ , cf., for example [3].

3.4. THEOREM: *Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional local Gorenstein ring with embedding dimension  $v_R = e_R + d - 2$ . Then  $G(R)$  is Gorenstein.*

The following lemma will be needed for the proof of (3.4).

3.5. LEMMA: *Let  $k$  be any field and  $V$  a  $n$ -dimensional vector space over  $k$ . Let  $(\ , \ )$  be a symmetric bilinear form on  $V$ . There is a basis  $v_1, \dots, v_n$  for  $V$  such that for each  $i, 1 \leq i \leq n$ , either  $(v_i, v_i) = 0$  or  $(v_i, v_j) = 0$  for  $j \neq i$ .*

PROOF: Let  $v_1, \dots, v_n$  be a basis for  $V$ . If  $(v_i, v_i) = 0$  for all  $i$ , the

proof is finished, so we assume that  $(v_1, v_1) \neq 0$ . Let  $V_1 = \{x \in V \mid (x, v_1) = 0\}$ .  $\dim V_1 < \dim V$  so there is, by induction, a basis  $w_1, \dots, w_t$  for  $V_1$  with the required properties. In addition we have  $(v_1, w_j) = (w_j, v_1) = 0$  for  $1 \leq j \leq t$ . To see that  $v_1, w_1, \dots, w_t$  span  $V$ , note that if  $x \in V$ , then  $x' = x - [(x, v_1)/(v_1, v_1)]v_1 \in V_1$ .

**PROOF OF (3.4):** We may assume that  $R/m$  is infinite and let  $x = x_1, \dots, x_d$  be a minimal reduction for  $m$ . By (3.3),  $m^3 \subset xm$  and  $\lambda_{R/xR}(m^2/xm) = 1$ . We will first prove that  $G(R)$  is Cohen–Macaulay. For this it is sufficient, by (2.1), to show that  $m^3 = xm^2$ . We prove this by induction on  $d$ . Since  $d = 0$  is no problem, we begin with  $d = 1$ . Let  $x = x_1$ . We may assume that  $v_R > 2$ . For if  $v_R = 2$  then, since  $\lambda_{R/m}(m^2/m^3) > 2$ , we have  $\lambda_{R/m}(m^2/m^3) = 3$  and  $m^3 = xm^2$ .

The first step is to show that  $m^4 \subset x^2m$ . Since  $m^3 \subset xm$  and  $\lambda_{R/xR}(m^2/xm) = 1$ , it is sufficient to show that there is some element in  $m^2 \setminus xm$  whose square is in  $x^2m$ . Pass to the 0-dimensional Gorenstein ring  $\tilde{R} = R/xR$  with  $\tilde{m} = m/xR$  and let  $k = \tilde{R}/\tilde{m}$ . Let  $V = \tilde{m}/\tilde{m}^2$ .  $V$  is a  $k$ -vector space of dimension  $e_R - 2 > 1$ .  $V$  has a non-degenerate symmetric bilinear form given by multiplication in  $\tilde{R}$ . For  $m^2 = (xm, g)$ , where  $g$  is any element of  $m^2 \setminus xm$ .  $\tilde{m}^2 = \tilde{g}\tilde{R}$  and we define for  $\alpha, \beta \in \tilde{m}/\tilde{m}^2$ ,  $(\alpha, \beta) = \bar{u}_{\alpha\beta}$ , where for  $a \in \tilde{m}$  mapping to  $\alpha$  and  $b \in \tilde{m}$  mapping to  $\beta$ ,  $ab = u_{\alpha\beta}\tilde{g}$  and  $\bar{u}_{\alpha\beta}$  is the image of  $u_{\alpha\beta}$  in  $k$ .  $\tilde{m}^3 = 0$ , so this is well defined. Since  $R$  is Gorenstein, this form is also non-degenerate. By (3.5), we can find a minimal basis  $x, z_1, \dots, z_{e_R-2}$  for  $m$  with the property that, for  $1 \leq i \leq e_R - 2$ , either  $z_i^2 \in xm$  or  $z_iz_j \in xm$  for  $j \neq i$ . If there is an  $i$  with  $z_i^2 \notin xm$ , then  $z_i^4 \in x^2m$ . For if  $j \neq i$ , there is an  $\ell \neq i$ , such that  $z_iz_\ell \notin xm$ . Thus,  $z_i^2 = uz_jz_\ell + x\mu$  with  $u \in R \setminus m$  and  $\mu \in m$ . Then,  $z_i^4 = uz_jz_\ell z_i^2 + x\mu z_i^2 \in x^2m$ . If  $z_i^2 \in xm$  for all  $i$ , then there is a  $j \neq 1$  such that  $z_1z_j \notin xm$ . Then  $(z_1z_j)^2 = z_1^2z_j^2 \in x^2m$ .

Thus we have that  $m^4 \subset x^2m$ . Now we pass to the 0-dimensional Gorenstein ring  $R^* = R/x^2R$  with  $m^* = m/x^2R$ . We have  $\lambda_{R^*}(R^*) = 2e_R$ ,  $\lambda_{R^*}(m^*/m^{*2}) = e_R - 1$  and  $\lambda_{R^*}(m^{*3}) = 1$ . Thus,  $\lambda_{R^*}(m^{*2}/m^{*3}) = 2e_R - e_R - 1 = e_R - 1$ . Consequently,  $\lambda_{R/m}(m^2/m^3) = e_R$  and, since  $e_R = \lambda_{R/xR}(R/xR) = \lambda_{R/xR}(m^2/xm^2)$ , it follows that  $m^3 = xm^2$ . This concludes the case  $d = 1$ .

Assume that  $d > 1$ . Suppose that  $m^3 \not\subset xm^2$ . We will show that there is an  $i$ ,  $1 \leq i \leq d$ , such that  $(m/x_iR)^3 \not\subset (x_1, \dots, \hat{x}_i, \dots, x_d)(m/x_iR)^2$ . Since  $R/x_iR$  is a local Gorenstein ring of dimension  $d - 1$  and embedding dimension  $v_{R/x_iR} = e_{R/x_iR} + (d - 1) - 2$ , for  $e_{R/x_iR} = e_R$ , this will contradict the induction hypothesis.<sup>1</sup> Since  $m^3 \neq xm^2$ , there is an element  $z$  in  $m^3$

<sup>1</sup> The author is grateful to B. Singh for his simplification of this part of the proof and for several incisive comments on the paper.

with  $z \notin \mathfrak{m}^2$ . Write  $z = ax_1 + bx_2 + \dots + cx_d$  with  $a, b, \dots, c \in \mathfrak{m}$ . By the analytic independence of  $x$  in  $\mathfrak{m}$  we have  $x^2R \cap \mathfrak{m}^3 = x^2\mathfrak{m} \subseteq \mathfrak{m}^2$ . Therefore,  $z \notin \mathfrak{m}^2 + x^2R$ . Now, if  $a, b, \dots, c \in \mathfrak{m}^2 + xR$  then  $z \in \mathfrak{m}^2 + x^2R$ , a contradiction. We may therefore assume that  $a \notin \mathfrak{m}^2 + xR$ . We then claim that any  $i \geq 2$  meets the requirement. To see this, say for  $i = d$ , suppose  $z \in (x_1, \dots, x_{d-1})\mathfrak{m}^2 + x_dR$ . Then there exists  $y \in \mathfrak{m}^2$  such that  $(a - y)x_1 \in (x_2, \dots, x_d)R$ . Therefore  $a - y \in (x_2, \dots, x_d)R$ , which shows that  $a \in \mathfrak{m}^2 + xR$ , a contradiction. This completes the proof that  $\mathfrak{m}^3 = \mathfrak{m}^2$ .

It remains to show that  $G(R)$  is Gorenstein.  $\bar{x}_1, \dots, \bar{x}_d$  is a regular sequence in  $G(R)$  and  $G(R/xR) \cong G(R)/(\bar{x}_1, \dots, \bar{x}_d)G(R)$ . Since  $G(R)$  is Gorenstein if and only if  $G(R)_{\mathcal{M}}$  is Gorenstein, [8], it suffices to show that  $G(R/xR)$  is Gorenstein. Let  $R_0 = R/xR$  and  $\mathfrak{m}_0 = \mathfrak{m}/xR$ . Then  $G(R_0) = R_0/\mathfrak{m}_0 \oplus \mathfrak{m}_0/\mathfrak{m}_0^2 \oplus \mathfrak{m}_0^2$  and  $\mathcal{M}_0 = \mathfrak{m}_0/\mathfrak{m}_0^2 \oplus \mathfrak{m}_0^2$ . We must show that  $\dim_{R/m} (0 : \mathcal{M}_0) = 1$ . But  $\bar{y}(\mathfrak{m}_0/\mathfrak{m}_0^2) = 0$  means  $y\mathfrak{m}_0 \subseteq \mathfrak{m}_0^3 = 0$ . Hence  $(0 : \mathcal{M}_0) = \mathfrak{m}_0^2$ . Since  $R_0$  is Gorenstein,  $\dim_{R/m}(\mathfrak{m}_0^2) = \dim_{R/m}(0 : \mathfrak{m}_0) = 1$ . This concludes the proof of the theorem.

(3.6) EXAMPLE: We show that for any  $d \geq 0$  there is a local Gorenstein ring  $(R, \mathfrak{m})$  of embedding dimension  $v_R = d + 2 = e_R + d - 3$  with  $G(R)$  not Gorenstein. We begin with  $d = 1$ . Let  $k$  be a field and  $t$  an indeterminate. Let  $R = k[[t^5, t^6, t^9]]$ . The numerical semigroup generated by 5, 6 and 9 is symmetric so, by [6],  $R$  is a 1-dimensional local Gorenstein ring.  $R \cong k[[X, Y, Z]]/(YZ - X^3, Z^2 - Y^3)$  and  $G(R) = k[X, Y, Z]/(YZ, Z^2, Y^4 - ZX^3)$ .  $3 = v_R = e_R + d - 3 = d + 2$  and  $G(R)$  is not Gorenstein.  $R/t^5R$  is a similar example for  $d = 0$ . Examples for  $d > 1$  are obtained by adjoining analytic indeterminates  $W_1, \dots, W_{d-1}$  to  $k[[t^5, t^6, t^9]]$ .

In summary, then, we have the following. If  $(R, \mathfrak{m})$  is a  $d$ -dimensional Gorenstein local ring of multiplicity  $e_R$  and embedding dimension  $v_R$ , then  $G(R)$  is Gorenstein if  $v_R = d, d + 1$  or  $e_R + d - 2$ . These are the only embedding dimensions which will always give  $G(R)$  Gorenstein.

3.7. COROLLARY: *Let  $(R, \mathfrak{m})$  be a local Gorenstein ring of multiplicity at most 4. Then  $G(R)$  is Gorenstein.*

It follows from (3.4) that the Hilbert function for a  $d$ -dimensional local Gorenstein ring of embedding dimension  $e_R + d - 2$  is completely determined by  $e_R$ . To see this, we recall that the Hilbert sum transforms for any local ring  $(R, \mathfrak{m})$  are defined inductively as follows. Let  $n$  be any non-negative integer.



$$H_R^0(n) = \dim_{R/m}(\mathfrak{m}^n/\mathfrak{m}^{n+1})$$

and

$$H_R^i(n) = \sum_{j=0}^n H_R^{i-1}(j).$$

3.8. REMARK: Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional local ring. It is known, cf. [13], that if  $x_1, \dots, x_t$  are elements of  $\mathfrak{m}/\mathfrak{m}^2$  with images  $\bar{x}_1, \dots, \bar{x}_t$  in  $G(R)$  forming a regular sequence, then

$$H_R^0(n) = H_{R/(x_1, \dots, x_t)R}^i(n),$$

for all  $n \geq 0$ . Thus, if  $G(R)$  is Cohen–Macaulay, there is an Artin local ring  $R^*$  of embedding dimension  $v_R - d$  and length  $e_R$  such that

$$H_R^0(n) = H_{R^*}^d(n),$$

for all  $n \geq 0$ . (If  $R/\mathfrak{m}$  is infinite, take  $R^* = R/(x_1, \dots, x_d)R$ , where  $x_1, \dots, x_d$  is a minimal reduction for  $\mathfrak{m}$ . If  $R/\mathfrak{m}$  is finite, it may be necessary first to pass to the ring  $R(u) = R[u]_{\mathfrak{m}R[u]}$ , where  $u$  is an indeterminate.)

3.9. COROLLARY: Let  $(R, \mathfrak{m})$  be a  $d$ -dimensional local Gorenstein ring of embedding dimension  $e_R + d - 2$ . Then there exists a 0-dimensional local Gorenstein ring  $(R^*, \mathfrak{m}^*)$  with  $H_{R^*}^0(1) = e_R - 2$ ,  $H_{R^*}^0(2) = 1$  and  $H_{R^*}^0(n) = 0$ , for  $n > 2$ , such that

$$H_R^0(n) = H_{R^*}^d(n),$$

for all  $n \geq 0$ . In fact, for  $d \geq 1$ ,

$$H_R^0(n) = e_R \binom{n+d-2}{d-1} + \binom{n+d-3}{n}, \quad n \geq 2.$$

PROOF: We may assume that  $R/\mathfrak{m}$  is infinite. In the first part of the proof of (3.4) we saw that  $(R, \mathfrak{m})$  Gorenstein with  $v_R = e_R + d - 2$  implies that there is a regular sequence  $x_1, \dots, x_d$  in  $R$  such that  $\mathfrak{m}^3 = (x_1, \dots, x_d)\mathfrak{m}^2$  and the images  $\bar{x}_1, \dots, \bar{x}_d$  form a regular sequence in  $G(R)$ . With  $R^* = R/(x_1, \dots, x_d)R$ , the first statement of the corollary follows from (3.8). The second statement follows from the first by double induction on  $n$  and  $d$ , using the fact that for any local ring  $(R, \mathfrak{m})$ ,  $H_R^d(n) = H_R^d(n-1) + H_R^{d-1}(n)$ .

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