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TWO THEOREMS ON DE RHAM COHOMOLOGY

William E. Lang*

In this paper, we prove two theorems on the de Rham cohomology of a nonsingular variety over a perfect field of characteristic $p > 0$. The first theorem relates the differential in the conjugate spectral sequence of de Rham cohomology to the Bockstein operation of Serre's Witt vector cohomology. Applications of this theorem to quasi-elliptic surfaces will be published later. The second theorem gives the de Rham cohomology of an Enriques surface, and, as a corollary, gives the number of vector fields on a non-classical Enriques surface.

The two theorems are logically independent, but Theorem 1 was discovered in the course of my work on Theorem 2, and both theorems are closely related to fundamental work of Oda [6]. Therefore, it seems best to publish them together. After the first version of this paper was written, I learned that L. Illusie has obtained the results of Part B independently, using the deep theory of the de Rham–Witt complex [4]. The proof given here is more elementary, but Illusie's gives additional information on the crystalline cohomology of an Enriques surface. Nevertheless, I believe that this proof is still of some interest.

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A. De Rham cohomology and the first Bockstein operation

The algebraic de Rham cohomology was introduced by Grothendieck in [2]. The facts which we need concerning this theory are proved in Katz [5] and summarized in [3, III, 7–8]. We give a brief summary below to fix notation.

Recall that if X is a non-singular variety of dimension n over a perfect field k , the de Rham cohomology $H_{DR}^*(X)$ is defined as the hypercohomology of the de Rham complex

$$\Omega_X^0 = 0 \rightarrow 0_X \rightarrow \Omega_X^1 \rightarrow \Omega_X^2 \rightarrow \cdots \rightarrow \Omega_X^n \rightarrow 0.$$

The first spectral sequence of hypercohomology

$$E_1^{pq} = H^q(X, \Omega_X^p) \Rightarrow H_{DR}^n(X)$$

is called the Hodge–de Rham spectral sequence. It degenerates in characteristic zero, but does not always degenerate in characteristic p .

There is another spectral sequence which is useful in computing de Rham cohomology in characteristic p . We note that the Frobenius morphism F induces a p -linear isomorphism between the hypercohomology of Ω_X and $F_*\Omega_X$. Let Z^i denote the kernel of $F_*d: F_*\Omega_X^i \rightarrow F_*\Omega_X^{i+1}$ and let B^i denote the image of $F_*d: F_*\Omega_X^{i-1} \rightarrow F_*\Omega_X^i$. The Cartier operator C induces an isomorphism $Z^i/B^i \cong \Omega^i$. Then the second spectral sequence of hypercohomology gives the conjugate spectral sequence

$$E_2^{pq} = H^p(X, \Omega^q) \Rightarrow H_{DR}^*(X)$$

of de Rham cohomology.

THEOREM 1: *Let X be a non-singular variety over a perfect field k of characteristic $p > 0$. Then the following diagram is commutative:*

$$\begin{array}{ccc} H^0(X, B^1) & \longrightarrow & H^0(X, \Omega_X^1) \\ \downarrow \delta & & \downarrow d_2 \\ H^1(X, 0_X) & \xrightarrow{-\beta_1} & H^2(X, 0_X) \end{array}$$

*The horizontal arrow is the obvious inclusion, β_1 is the first Bockstein operation of Witt vector cohomology, the left vertical arrow comes from the exact sequence $0 \rightarrow 0_X \xrightarrow{F} F_*0_X \xrightarrow{d} B^1 \rightarrow 0$, and the right vertical arrow is the differential in the conjugate spectral sequence of de Rham cohomology.*

PROOF: This will be a straightforward computation in Čech cohomology. Pick a cover of X by open affines $\{U_i\}$. Then on U_i , an element of $H^0(X, B^1)$ can be written as dg_i . Let $h = \delta(dg_i)$. Then h can be represented by an alternating 1-cochain such that $h_{ij}^p = g_j - g_i$, $h_{jk}^p = g_k - g_j$, $h_{ik}^p = g_k - g_i$.

Recall that β_1 is the connecting homomorphism in the cohomology sequence of the following exact sequence:

$$0 \rightarrow 0_X \rightarrow W_2 \rightarrow 0_X \rightarrow 0$$

where W_2 is the sheaf of Witt vectors of length 2. Recall also that Witt vectors of length 2 add as follows: If $a = (a_0, a_1)$ and $b = (b_0, b_1)$ then

$$a + b = \left(a_0 + b_0, a_1 + b_1 - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} a_0^m b_0^{p-m} \right),$$

where $p^{-1} \binom{p}{m}$ is defined as an element of k by the formula $((p-1)!)/(m!(p-m)!)$. (See Serre [7].)

Assume $p \neq 2$. To find a representative cocycle for $d = \beta_1(h)$, we compute

$$\begin{aligned} & (h_{jk}, 0) - (h_{ik}, 0) + (h_{ij}, 0) \\ &= \left(h_{jk} + h_{ij}, - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} h_{jk}^m h_{ij}^{p-m} \right) + (-h_{ik}, 0) \\ &= \left(h_{jk} + h_{ij} - h_{ik}, - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} h_{jk}^m h_{ij}^{p-m} \right. \\ & \quad \left. - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} (-1)^m h_{ik}^m (h_{jk} + h_{ij})^{p-m} \right). \end{aligned}$$

Since $h_{jk} + h_{ij} = h_{ik}$, this reduces to

$$\left(0, - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} h_{jk}^m h_{ij}^{p-m} - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} (-1)^m h_{ik}^m \right).$$

Since $p \neq 2$, $\sum_{m=1}^{p-1} p^{-1} \binom{p}{m} (-1)^m = 0$. So $\beta_1(h)$ is represented by the 2-cocycle

$$(1) \quad d_{ijk} = - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} h_{jk}^m h_{ij}^{p-m}.$$

Now we go around the other way. Since the conjugate spectral sequence is a second spectral sequence of hypercohomology, d_2 is the

composition of connecting homomorphisms obtained from the cohomology of the following exact sequences.

$$\begin{aligned} 0 \longrightarrow B^1 \longrightarrow Z^1 \xrightarrow{C} \Omega^1 \longrightarrow 0 & \quad H^0(\Omega^1) \xrightarrow{\delta_1} H^1(B^1) \\ 0 \longrightarrow 0_X \xrightarrow{F} F_*0_X \xrightarrow{d} B^1 \longrightarrow 0 & \quad H^1(B^1) \xrightarrow{\delta_2} H^2(0_X). \end{aligned}$$

Let $e = \delta_1(dg_i)$. To find a representative cocycle for e , observe that $dg_i = C(g_i^{p-1}dg_i)$. Then

$$\begin{aligned} e_{ij} &= g_j^{p-1}dg_i - g_i^{p-1}dg_j = (g_j^{p-1} - g_i^{p-1})dg_i = ((g_i + h_{ij}^p)^{p-1} - g_i^{p-1})dg_i \\ &= \sum_{m=1}^{p-1} \binom{p-1}{m} g_i^{p-1-m} h_{ij}^{pm} dg_i = d \left(\sum_{m=1}^{p-1} p^{-1} \binom{p}{m} g_i^{p-m} h_{ij}^{pm} \right). \end{aligned}$$

Similarly,

$$e_{ik} = d \left(\sum_{m=1}^{p-1} p^{-1} \binom{p}{m} g_i^{p-m} h_{ik}^{pm} \right), \quad e_{jk} = d \left(\sum_{m=1}^{p-1} p^{-1} \binom{p}{m} g_j^{p-m} h_{jk}^{pm} \right).$$

Let $f = \delta_2(e)$, then f can be represented by a 2-cocycle f_{ijk} such that

$$\begin{aligned} (2) \quad f_{ijk}^p &= \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} g_j^{p-m} h_{jk}^{pm} - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} g_i^{p-m} h_{ik}^{pm} \\ &\quad + \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} g_i^{p-m} h_{ij}^{pm} \\ &= \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} (g_i + h_{ij}^p)^{p-m} h_{jk}^{pm} \\ &\quad - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} (h_{ij}^p + h_{jk}^p)^m g_i^{p-m} + \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} g_i^{p-m} h_{ij}^{pm} \\ &= \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} \sum_{q=0}^{p-m} \binom{p-m}{q} g_i^q h_{ij}^{p(m-q)} h_{jk}^{pm} \\ &\quad - \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} \sum_{q=0}^m \binom{m}{q} h_{ij}^{p,q} h_{jk}^{p(m-q)} g_i^{p-m} \\ &\quad + \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} g_i^{p-m} h_{ij}^{pm} \\ &= \sum_{m=1}^{p-1} p^{-1} \binom{p}{m} h_{ij}^{p(p-m)} h_{jk}^{pm} + (\text{terms involving } g_i). \end{aligned}$$

Comparing (2) with (1), we see that it is enough to show that the sum of the terms involving g_i is 0. To see this, choose ℓ , $1 \leq \ell \leq p-1$. Then the contribution of g_i^ℓ is

$$\begin{aligned}
 &g_i^\ell \left(\sum_{m=1}^{p-\ell} p^{-1} \binom{p}{m} \binom{p-m}{\ell} h_{ij}^{p(m-\ell)} h_{jk}^m - p^{-1} \binom{p}{p-\ell} \right. \\
 &\quad \times \sum_{q=0}^{p-\ell} \binom{p-\ell}{q} h_{ij}^{pq} h_{jk}^{p(\ell-q)} + p^{-1} \binom{p}{p-\ell} h_{ij}^{p(p-\ell)} \Big) \\
 &= g_i^\ell \left(\sum_{m=1}^{p-\ell} p^{-1} \binom{p}{m} \binom{p-m}{\ell} h_{ij}^{p(m-\ell)} h_{jk}^m - p^{-1} \binom{p}{p-\ell} \right. \\
 &\quad \times \sum_{q=0}^{p-\ell-1} \binom{p-\ell}{q} h_{ij}^{pq} h_{jk}^{p(\ell-q)} \Big).
 \end{aligned}$$

Now match up terms by putting $m = p - \ell - q$. We need only show that

$$\binom{p}{p-\ell-q} \binom{\ell+q}{\ell} = \binom{p}{p-\ell} \binom{p-\ell}{q}.$$

This is left to the reader.

We must also check the case $p = 2$. We know that $\beta_1(x) = x \cup x$ for all $x \in H^1(0_X)$ [7]. Therefore $d = \beta_1(h)$ is represented by $d_{ijk} = h_{ij}h_{jk}$. However, the computation going around the other way does not use $p \neq 2$, hence the result is

$$\left(\sum_{m=1}^1 \frac{1}{2} \binom{2}{m} h_{ij}^2 h_{jk}^2 \right)^{1/2} = h_{ij}h_{jk}.$$

B. De Rham cohomology and Enriques surfaces

For the results on Enriques surfaces that we use in this section, see [1].

THEOREM 2: *The first de Rham cohomology group of an Enriques surface is 0, if $\text{char } k \neq 2$. If $\text{char } k = 2$, it is 1-dimensional. Furthermore, if $\text{char } k = 2$,*

$$h^0(\Omega_X^1) = \begin{cases} 1 & \text{if } X \text{ is classical} \\ 0 & \text{if } X \text{ is singular} \\ 1 & \text{if } X \text{ is supersingular.} \end{cases}$$

In the supersingular case, the injection $H^0(\Omega_X^1) \rightarrow H_{\text{DR}}^1(X)$ induced by the Hodge–de Rham spectral sequence is an isomorphism.

PROOF: We use a result of Oda [6] relating the first de Rham

cohomology group to the Picard scheme. The conjugate spectral sequence gives an exact sequence

$$0 \longrightarrow H^1(0_X) \longrightarrow H^1_{DR}(X) \xrightarrow{V} H^0(\Omega^1_X).$$

Oda defines a map V^2 on $\ker(d \cdot V)$ by $V^2(x) = C \cdot Vx$, where C is the Cartier operator. He defines V^3 on $\ker(d \cdot V^2)$ similarly, and so on. He shows that $\cap \ker(d \cdot V^n)$ is isomorphic to the dual of the Dieudonné module of the finite group scheme ${}_p\text{Pic}^\tau(X)$, where ${}_p\text{Pic}^\tau(X)$ is the kernel of multiplication by p on $\text{Pic}^\tau(X)$. If all 1-forms on X are closed, then it is clear that Oda's subspace is all of $H^1_{DR}(X)$. In the case of the Enriques surface, the explicit computation of $\text{Pic}^\tau(X)$ is Bombieri–Mumford III shows that Oda's subspace is 0 if $p \neq 2$ and is 1-dimensional if $p = 2$. Therefore, we need only prove the following lemma.

LEMMA: *If X is an Enriques surface, all 1-forms on X are closed.*

PROOF: This is obvious if $p_g = 0$, therefore we need only check the singular and supersingular cases in characteristic 2. In these cases, K_X is trivial and therefore $h^0(\Omega^2_X) = 1$. Suppose this lemma is not true. Then $d: H^0(\Omega^1_X) \rightarrow H^0(\Omega^2_X)$ is surjective. This implies that $d: H^2(0_X) \rightarrow H^2(\Omega^1_X)$ is injective. (This implication is a special case of a well-known duality theorem in de Rham cohomology which is proved as follows. If X is a variety of dimension n , the conjugate spectral sequence shows that $H^{2n}_{DR}(X) \simeq k$. Therefore, the differential in the Hodge–de Rham spectral sequence $d_1: H^n(\Omega^{n-1}) \rightarrow H^n(\Omega^n)$ is zero. Now let $a \in H^p(\Omega^q)$ and let $b \in H^{n-p}(\Omega^{n-q-1})$. Then $0 = d_1(a \cup b) = da \cup b \pm a \cup db$. This shows that $d: H^p(\Omega^q) \rightarrow H^p(\Omega^{q+1})$ is (up to sign) the transpose of $d: H^{n-p}(\Omega^{n-q-1}) \rightarrow H^{n-p}(\Omega^q)$.) Let a be a non-zero class in $H^1(0_X)$. Then $a \cup a \neq 0$ in $H^2(0_X)$, since the Bockstein operation is injective ([1], Lemma 3.1). Now (using characteristic 2) we see that $d(a \cup a) = a \cup da + da \cup a = 2(a \cup da) = 0$, contradiction. Q.E.D.

This proves our assertions about $H^1_{DR}(X)$. To finish the proof, we need only remark that a class in $H^1(0_X)$ fixed by Frobenius lives forever in the Hodge–de Rham spectral sequence, so $H^0(\Omega^1)$ must be zero in the singular case; in the supersingular case, it is known that $h^0(\Omega^1) \geq 1$.

We may use Theorem 2 to compute $h^i(\theta_X)$ in the singular and supersingular cases in characteristic 2, since K_X is trivial, and therefore θ_X is isomorphic to Ω^1_X . We get

	$h^0(\theta_X)$	$h^1(\theta_X)$	$h^2(\theta_X)$
Singular	0	10	0
Supersingular	1	12	1

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