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*p-local unconditional structure of Banach spaces*

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Abstract

There are Banach spaces which fail to have $p$-local unconditional structure (p-l.u.st.) for any $p$, $\infty > p > 0$. In particular, there exist $n$-dimensional Banach spaces $E_n$, $n = 1, 2, \ldots$, whose p-l.u.st. constants are “almost” the largest possible theoretical value $\min\{n^{1/2}, n^p\}$. The p-l.u.st. constant is smaller and not equivalent to the usual l.u.st. constant.

1. Introduction

Given any $\infty > p \geq 0$, let $\eta_p$ be the ideal norm defined in the following manner: If $T \in L(E, F)$ is a bounded operator from a Banach space $E$ to a Banach space $F$ which can be written of the form $Tx = \sum_{i=1}^{\infty} A_i x$ ($x \in E$), where $A_i$ ($i = 1, 2, \ldots$) are in the class $\mathcal{F}(E, F)$ of the finite-rank operators from $E$ to $F$, then

$$\eta_p(T) = \inf \sup \left\| \sum_{i \leq N} \pm (r(A_i))^p A_i \right\|$$

where $r(A)$ denotes the rank of an operator $A$, the supremum ranges over all choices of $\pm$ signs and integers $N$, and the infimum is taken over all the possible representations of the operator $T$.

$\eta_p(T)$ is a non-decreasing function of $p$, and $\eta_p$ is a Banach ideal norm, that is has the following properties:

(1) $\eta_p$ is a norm and $\eta_p(E, F) = \{T \in L(E, F); \eta_p(T) < \infty\}$ is a Banach space under the norm $\eta_p$.

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(2) \( \eta_p(T) = \|T\| \) whenever \( r(T) = 1 \).

(3) If \( u \in L(G, E), T \in \eta_p(E, F), v \in L(F, H) \), then \( vTu \in \eta_p(G, H) \) and \( \eta_p(vTu) \leq \|v\|\|u\|\eta_p(T) \).

We recall some well known facts about a general Banach ideal norm \( \alpha \) which may be found in [9]. If \( T \in L(E, F) \), the adjoint ideal norm \( \alpha^*(T) \) is defined as the least \( C \) such that the inequality

\[
\text{trace}(SuTu) \leq C\|v\|\|u\|\alpha(S)
\]

holds for any finite-dimensional normed spaces \( X \) and \( Y \), \( u \in L(X, E) \), \( v \in L(F, Y) \), and \( S \in L(Y, X) \). If \( X \) and \( Y \) are finite-dimensional normed spaces, the dual space \( (\alpha(X, Y))^\prime \) can be naturally identified with \( \alpha^*(X, Y) = (L(Y, X), \alpha^*) \) via the identity \( \langle T, S \rangle = \text{trace}(ST) \) for \( T \in \alpha(X, Y), S \in \alpha^*(X, Y) \). Hence, if \( T \in L(E, F) \), \( \alpha^{**}(T) = \sup \alpha(vTu) \), where the supremum ranges over all finite-dimensional normed spaces \( X \) and \( Y \), \( u \in L(X, E) \) and \( v \in L(F, Y) \) with \( \|u\| = \|v\| = 1 \). From this we get immediately that \( \eta_p^{**}(T) = \eta_p^{**}(T') \) for every operator \( T \in L(E, F) \).

If in the definition of \( \eta_p(T) \), \( T \) is further restricted only to representations for which \( r(A_i) = 1 \) for all \( i \), then the corresponding resulting norm which is independent of \( p \) was called in [8] the weakly nuclear norm of \( T \) and denoted by \( \eta(T) \). It follows that \( \|\cdot\| \leq \eta_p \leq \eta_q \leq \eta \) for \( 0 \leq p < q < \infty \), and since \( \eta_0 = \|\cdot\| \) on finite dimensional spaces, taking double adjoints we obtain \( \eta_0^{**} = \|\cdot\|^{**} = \|\cdot\| \leq \eta_p^{**} \leq \eta_0^{**} \leq \eta^{**} \).

Using ultraproducts it can be shown (see for example [16]) that \( T \in \eta^{**}(E, F) \) if and only if \( j_FT \) factors through some Banach lattice, more precisely, \( \eta^{**}(T) = \inf\|v\|\|u\| \), where the infimum ranges over all Banach lattices \( L \) and \( u \in L(E, L), v \in L(L, F'') \), satisfying \( j_FT = vu \), where \( j_F : F \to F'' \) is the canonical inclusion. Thus, if \( T \) is a map on, or, into, a norm one complemented subspace of a Banach lattice, then \( \|T\| = \eta_p^{**}(T) = \eta^{**}(T) \). If \( T = I_E \) the identity operator on a Banach space \( E \), \( \eta^{**}(I_E) \) is generally better known as the local unconditional structure (l.u.s.t.) constant of \( E \) which is usually denoted by \( x_u(E) \) [7]. If \( \alpha \) is an ideal norm, \( \alpha(E) \) denotes \( \alpha(I_E) \). For \( p > 0 \), \( \eta_p^{**}(E) \) will be called the \( p \)-l.u.s.t. constant of \( E \). \( x(E) \) will denote the unconditional basis constant of \( E \).

If \( \dim(E) = n \) and \( 0 \leq p < q < \infty \), then trivially we get from the definitions

\[
1 \leq \eta_p(E) \leq \eta_q(E) \leq \eta(E) = x_u(E) \leq x(E) \leq d(E, \ell_2^p) \leq \sqrt{n},
\]

and also, since \( r(A_i) \leq n \), \( \eta_q(E) \leq n^{q-p}\eta_p(E) \). Moreover, the represen-
tation $I_E = I_E$ shows that $\eta_p(E) \leq n^p$, thus $\eta_p(E) \leq \min\{n^{1/2}, n^p\}$ for all $p \geq 0$.

The main result here shows that the last inequality is asymptotically “almost” the best possible. There exists a sequence $E_n$, $n = 1, 2, \ldots$, of $n$-dimensional spaces for which $\eta_p(E_n) \geq \min\{a n^{1/2}, a n^p\} \exp(-\sqrt{\log n})$ where $a$ is an absolute positive constant. Since the exponential factor tends to zero more slowly than any negative power of $n$, this implies that if $p \neq q$ and $0 < p < \frac{1}{2}$, then $\eta_p$ and $\eta_q$ are not equivalent ideal norms, and in particular $\eta_p$ and $\eta$ are not equivalent ideal norms. Since $\eta_p(E_n) \to \infty$ as $n \to \infty$, this also implies there exists a reflexive separable Banach space which fails to have $p$-l.u.st. for all $p > 0$.

Regarding $x_n$ it was proved in [3] that there is an absolute constant $c > 0$ and a sequence of spaces $F_n, \dim(F_n) = n$, such that $x_n(F_n) \geq c \sqrt{n}$. Our result therefore is of interest for the smaller $p$-l.u.st. constants $\eta_p$. We do not know if $\eta_p(E)$ can be asymptotically equivalent to $\min\{n^{1/2}, n^p\}$ for a sequence of spaces $E_n, \dim(E_n) = n$. It is also an open question whether for $q > p \geq \frac{1}{2}$ $\eta_q(E)$ and $\eta_q(E)$ are always equivalent when $\dim(E) < \infty$; the same question is also open for the constants $x_n(E)$ and $x(E)$. It was proved recently by Johnson, Lindenstrauss and Schechtman, that there exists a Banach spaces $E$ with $x_n(E) = \infty$, that is $E$ does not have local unconditional structure, yet $E$ has an unconditional Schauder decomposition into 2-dimensional spaces. This fact implies that $x_n(E)$ and $\eta_p(E)$ are not equivalent since $\eta_p(E)$ is finite for such spaces. Also unknown is whether many of the spaces which fail l.u.st. also fail $p$-l.u.st. for some $p > 0$. Does $L_q(\infty > q \geq 1)$ have a subspace without $p$-l.u.st.? G. Pisier proved that if $p > 2$, $L_q$ has a subspace without l.u.st. (See [15] for $q > 4$; for $2 < q$ we know of an unpublished proof).

To obtain the lower estimates for $\eta_p(E_n)$ we use the characterization of the adjoint norm $\eta_p^*$ proved in the next section and an inequality due to S. Chevet which was communicated to us by G. Pisier who has used the inequality to prove that l.u.st. constant $x_n$ of $\ell_\infty \otimes \ell_1 \otimes \ell_2 \otimes \cdots \otimes \ell_n$ is bigger than $C_n N^\alpha$, where $N = n^{2k}$ and $n, k$ are chosen in some appropriate relation to $N$, and where $\alpha$ is any scalar $<\frac{1}{2}$, and $c_\alpha > 0$ is a constant depending only on $\alpha$. 


2. $p$-local unconditional structure

A characterization of $\eta^*_p$ is given by the following proposition.

**Proposition 1:** If $p, C$ are non-negative constants and $T \in L(E, F)$, then the following statements are equivalent:

1. $\eta^*_p(T) \leq C$.
2. $\sum_{i=1}^n \operatorname{trace}(T A_i) \leq C \max_{i} \|\sum_{i=1}^n \pm (r(A_i))^p A_i\|$ for any choice of $\{A_i\}_{i=1}^n \subset \mathcal{F}(F, E)$.
3. If $K_E$ denotes the $w^*$-closure of the extreme points of the unit ball of $E'$ equipped with the $w^*$ topology, there exists a probability measure on the compact topological product space $K = K_E \times K_{F'}$ such that for every $A \in \mathcal{F}(F, E)$ holds the inequality

$$ \operatorname{trace}(TA) \leq C (r(A))^p \int_K |\langle A'(x'), y'' \rangle| \, d\mu(x', y''). $$

**Proof:** Let $C_i (i = 1, 2, 3)$ denote a constant $C$ which appears in the inequality of statement (i). Let $X, Y$ be finite-dimensional spaces and $\epsilon > 0$, and let $S = \sum_{i=1}^n A_i$ where $A_i \in L(Y, X)$ are chosen to satisfy $(1 + \epsilon) \eta_p(S) \geq \max_{i} \|\sum_{i=1}^n \pm (r(A_i))^p A_i\|$. Then, for any $u \in L(X, E), v \in L(F, Y)$, we get

$$ \operatorname{trace}(SuTv) = \sum_{i=1}^n \operatorname{trace}(A_i v Tu) = \sum_{i=1}^n \operatorname{trace}(u A_i v T) $$

$$ \leq C_2 \max_{i} \left\|\sum_{i=1}^n \pm (r(A_i))^p u A_i v T\right\| $$

$$ \leq C_2 \|u\| \|v\| \max_{i} \left\|\sum_{i=1}^n \pm (r(A_i))^p A_i\right\| $$

$$ \leq C_2 \|u\| \|v\| (1 + \epsilon) \eta_p(S), $$

this implies $\inf C_1 \leq C_2(1 + \epsilon)$, therefore $\inf C_1 \leq \inf C_2$.

Given arbitrary $B_i \in \mathcal{F}(F, E)$, $i = 1, 2, \ldots, n$,

$$ \sum_{i=1}^n \operatorname{trace}(TB_i) \leq C_3 \int_K \sum_{i=1}^n (r(B_i))^p |\langle B'(x'), y'' \rangle| \, d\mu $$

$$ \leq C_3 \sup \left\{\sum_{i=1}^n (r(B_i))^p |\langle B'(x'), y'' \rangle|; \|x''\| = \|y''\| = 1 \right\} $$

$$ = C_3 \max_{i} \left\|\sum_{i=1}^n \pm (r(B_i))^p B_i\right\|, $$

hence $\inf C_2 \leq C_3$. 

If $A \in \mathcal{F}(F, E)$, let $\bar{A} \in C(K)$ be the function defined by:
$$\bar{A}(x', y") = (A'(x'), y") \in (r(A)^p)$$
and denote by $M$ the convex hull of the set \{C2\}; $A \in \mathcal{F}(F, E)$, trace(TA) = 1}. Statement (2) implies that $M$ is disjoint from the set $N = \{f \in C(K), f < 1\}$ which is also convex and contains the open unit ball of $C(K)$, therefore there exists a probability measure $\mu \in M(K) = (C(K))'$ such that $\mu(g) \geq 1$ for all $g \in M$, this shows that $\inf C_2 \leq C_2$.

Let now $\{A_i\}_{i=1}^n \subset \mathcal{F}(F, E)$, and consider the space $X = \text{span}\{A_i(y); y \in F, i = 1, 2, \ldots, n\}$. Let $u : X \to E$ be the inclusion map, and $S = \sum_{i=1}^n A_i$ be the map of $F$ into $X$, $S'$ maps $X'$ to $F'$ and $(S')_a$ will denote the map $S'$ of $X'$ onto $S'(X')$. Let $j$ be the inclusion of $S'(X')$ in $F'$, then $v = j'j_F$ maps $F$ to $Y = (S'(X'))'$. Both $X$ and $Y$ are now finite-dimensional spaces, and

\[ \sum_{i=1}^n \text{trace}(TA_i) = \text{trace}(vTu(S'_a)) \]
\[ \leq C_1 \|u\| \|v\| \eta_p((S')_a) \]
\[ \leq C_1 \max \left\{ \sum_{i=1}^n \pm A_i(r(A_i))^p \right\} \]

the last inequality is because $\|u\| = \|v\| = 1$ and the fact that if we denote by $\bar{A}_i$ the operator $A''_i$ considered as a map of $Y$ to $X$, then

$$(S')_a = \sum_{i=1}^n \bar{A}_i$$

and so

$$\eta_p((S')_a) \leq \max \left\{ \sum_{i=1}^n \pm (r(\bar{A}_i))^p \bar{A}_i \right\} = \max \left\{ \sum_{i=1}^n \pm (r(A_i))^p A_i \right\}.$$ 

Therefore, $\inf C_2 \leq C_1$, and the proof is complete. $\Box$

We need a preliminary lemma which was used in [6].

**Lemma 2:** If $x_i, y_i, i = 1, 2, \ldots, m$, are arbitrary vectors in $\ell_2^n$, then $n \sum_{i=1}^m \Sigma_{j=1}^n \langle x_i, y_i \rangle^2 \geq (\Sigma_{i=1}^m \langle x_i, y_i \rangle)^2$.

**Proof:** Without loss of generality we can assume $\{y_i\}_{i=1}^m$ are fixed such that the operator $T = \sum_{i=1}^n y_i \otimes y_i$ has rank $n$. We shall maximize the function $f(x_1, x_2, \ldots, x_m) = \sum_{i=1}^m \langle x_i, y_i \rangle$ subject to the constraint $\Sigma_{i=1}^m \Sigma_{j=1}^n \langle x_i, y_i \rangle^2 = 1$. At the maximum point, the function

$$\varphi = \sum_{i=1}^m \langle x_i, y_i \rangle - \lambda \left( \sum_{i=1}^m \sum_{j=1}^n \langle x_i, y_j \rangle^2 - 1 \right)$$

is constant, and we have

$$\sum_{i=1}^m \langle x_i, y_i \rangle = \lambda \sum_{i=1}^m \sum_{j=1}^n \langle x_i, y_j \rangle^2.$$
satisfies $\frac{\partial \varphi}{\partial x_{ik}} = 0$, where $x_i = (x_{ik})_{k=1}^n$. This yields $y_i = 2\lambda \sum_{i=1}^n (x_i, y_j) y_j = 2\lambda T(x_i)$, hence $f = \sum_{i=1}^n \langle x_i, y_i \rangle = 2\lambda \sum_{i=1}^n \sum_{j=1}^n \langle x_i, y_j \rangle^2 = 2\lambda. T = \sum_{i=1}^n y_i \otimes y_i = 2\lambda \sum_{i=1}^n y_i \otimes T x_i$, therefore $I_{\lambda} = 2\lambda \sum_{i=1}^n x_i \otimes y_i$ and taking trace, $n = 2\lambda \sum_{i=1}^n \langle x_i, y_i \rangle = 4\lambda^2$, so that $2\lambda = \sqrt{n}$.

Given Banach spaces $E$ and $F$ let $E \otimes_\pi F$ denote the completion of the tensor product space $E \otimes F$ under the $\pi$-norm, that is the ordinary norm induced on it as a subspace of $L(E', F)$. $E \otimes_\epsilon F$ denotes the completion of $E \otimes F$ under the $\epsilon$-norm, that is, on $E \otimes F$ the norm $|\cdot|_\epsilon$ is defined as

$$\left| \sum_{i=1}^n x_i \otimes y_i \right|_\epsilon = \sup \left\{ \sum \langle y_n, B x_i \rangle; \| B \| \leq 1, B \in L(E, F') \right\}.$$ 

If $k$ is a positive integer, $E^k_\epsilon$ will denote the space $E \otimes_\epsilon E \otimes_\epsilon \cdots \otimes_\epsilon E$, and for $\{x_i\}_{i=1}^k \subset E$, $\tilde{x} = \bigotimes_{i=1}^k x_i = x_1 \otimes x_2 \otimes \cdots \otimes x_k$ will be a $k$ tensor in $E^k_\epsilon$. If $u_i \in L(E, E)$ are isometries on $E$ $(i = 1, 2, \ldots, k)$, let $\tilde{u} = \bigotimes_{i=1}^k u_i = u_1 \otimes u_2 \otimes \cdots \otimes u_k$ be the isometry of $E^k_\epsilon$ defined by: $\tilde{u}(\tilde{x}) = \bigotimes_{i=1}^k u_i(x_i)$. This definition makes $\tilde{u}$ also an isometry on $E^k_\pi = E \otimes_\pi \otimes_\pi \cdots \otimes_\pi E$. We shall denote by $\pi_p$ $(1 \leq p < \infty)$ the $p$-absolutely summing ideal norm [14].

**Lemma 3:** Let $E$ have a normalized symmetric basis $\{e_i\}_{i=1}^n$, and let $T : E \rightarrow \ell_2^2$ be the basis to basis map, $T(e_i) = (0, \ldots, 0, 1, 0, \ldots, 0)$, $i = 1, 2, \ldots, n$. Let $A = \sum_{i=1}^n \cdots \sum_{i=k}^n a_{i_1, i_2, \ldots, i_k} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}$ be any norm-one element in $E^k_\epsilon$, then

$$\eta_{1/2}(E^k_\epsilon) \sum |a_{i_1, i_2, \ldots, i_k}| \leq (n \sqrt{2} \pi_1(T))^k.$$ 

**Proof:** By Pietsch [14] there exists a probability measure $\mu$ on $K_E$ such that for every $x = \sum_{i=1}^n x e_i \in E$

$$\|T x\|_2 = \left( \sum_{i=1}^n x_{i}^2 \right)^{1/2} \leq (\pi_1(T) \int_{K_E} |\langle x, x' \rangle| d\mu(x')).$$

Let $d\mu = d\mu \times d\mu \times \cdots \times d\mu$ be the product measure on the set of extreme points of the unit ball of $(E^k_\epsilon)' = (E')^k_\epsilon$,
Let \( u = \sum_{i=1}^{m} A_i \otimes B_i \) be any rank-\( m \) operator in \( L(E^k, E^k) \), where \( A_i \in (E')' \) and \( B_i \in E^k \). Suppose \( A_i \) and \( B_i \) have the representations

\[
A_i = \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} a_{i_1}^{(i)} e_{i_1}^{i} \otimes \cdots \otimes e_{i_k}^{i}, \quad \text{and}
\]

\[
B_i = \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} b_{i_1}^{(i)} \cdots e_{i_k}^{i} \otimes e_{i_k}^{i}.
\]

Let \( \epsilon^{(i)} = (\epsilon^{(i)}_{j})_{j=1}^{n}, \epsilon^{(i)}_{j} = \pm 1, i = 1, 2, \ldots, k, j = 1, 2, \ldots, n \), and \( g_{\epsilon^{(i)}} \) be the isometry of \( E \) defined by \( g_{\epsilon^{(i)}}(e_j) = \epsilon^{(i)}_{j} e_j \). Let \( \pi^{(i)} \) be any permutation of the integers \( \{1, 2, \ldots, n\} \), and \( g_{\pi^{(i)}} \) be the isometry of \( E \) defined by \( g_{\pi^{(i)}}(e_j) = e_{\pi^{(i)}(j)} \). Set \( g_i = g_{\epsilon^{(i)}} g_{\pi^{(i)}} \), and let \( g_{\epsilon, \pi} = \bigotimes_{i=1}^{k} g_i \) be the isometry of \( E^k \).

Denote by \( Av_{\epsilon} \) and \( Av_{\pi} \) the averages with respect to signs and permutations, that is, if \( f(\epsilon^{(1)}, \ldots, \epsilon^{(k)}) \) is a real function then

\[
Av_{\epsilon}(f) = 2^{-nk} \sum_{\epsilon} f(\epsilon^{(1)}, \ldots, \epsilon^{(k)})
\]

where the sum is taken over all possible distinct elements \((\epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(k)})\); and similarly for a function \( h(\pi^{(1)}, \ldots, \pi^{(k)}) \)

\[
Av_{\pi}(h) = (n!)^{-k} \sum_{\pi} h(\pi^{(1)}, \ldots, \pi^{(k)})
\]

where the sum ranges over all \((n!)^k\) possible choices of \((\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(k)})\).

We shall find a lower bound for the integral

\[
I = Av_{\epsilon} Av_{\pi} \int_{K(E^k)} |\langle u'(\bar{x}'), \tilde{g}_{\epsilon, \pi}(A) \rangle|d\tilde{\mu}(\bar{x}')
\]

which will give the claim of the lemma. First observe that

\[
|\langle u'(\bar{x}'), \tilde{g}_{\epsilon, \pi}(A) \rangle| =
\]

\[
= \left| \sum_{\ell_1, \ldots, \ell_k=1}^{n} \langle e_{\ell_1}, x_1^{\ell_1} \rangle \cdots \langle e_{\ell_k}, x_k^{\ell_k} \rangle \sum_{i_1, \ldots, i_k=1}^{n} a_{i_1}^{(i_1)} \cdots \sum_{j_1, \ldots, j_k=1}^{n} b_{i_k}^{(j_k)} \langle g_{\ell_1}(e_{\ell_1}), e_{j_1}^{i_1} \rangle \cdots \langle g_{\ell_k}(e_{\ell_k}), e_{j_k}^{i_k} \rangle \right|
\]

\[
\cdots \langle g_{\ell_k}(e_{\ell_k}), e_{j_k}^{i_k} \rangle a_{i_k}^{(i_k)} \sum_{l=1}^{n} a_{l}^{(i_l)} \cdots \sum_{k=1}^{n} b_{k}^{(j_k)} \cdots \sum_{i_k=1}^{n} \langle g_{\ell_k}(e_{\ell_k}), e_{j_k}^{i_k} \rangle \right|.
\]
Integrating with respect to $d\mu(x'_1)$ first we get

$$\int_{K_E} \left|\langle u'(\bar{x}'), \tilde{g}_{e,\pi}(A) \rangle \right| d\mu(x'_1) \geq (\pi_1(T))^{-1} \left[ \sum_{\ell_1=1}^{\delta} \left( \sum_{i_1} \right)^2 \right]^{1/2}$$

where

$$\sum \equiv \sum_{\ell_2, \ldots, \ell_k=1}^{\delta} \langle e_{\ell_2}, x'_2 \rangle \cdots \langle e_{\ell_k}, x'_k \rangle \sum_{i_1, \ldots, i_k=1}^n \sum_{j_1, \ldots, j_k=1}^n \langle g_1(e_{i_1}), e'_{j_1} \rangle \cdots \langle g_k(e_{i_k}), e'_{j_k} \rangle \sum_{i=1}^m a_{i_1, \ldots, i_k} b_{j_1, \ldots, j_k}.$$ 

Next, integrating with respect to $d\mu(x'_2)$ and using the fact, which we shall also use throughout, that

$$\int \left[ \sum_{\ell_1=1}^{\delta} \left( \sum_{i_1} \right)^2 \right]^{1/2} d\mu(x'_2) \geq \left[ \sum_{\ell_1=1}^{\delta} \left( \int \left| \sum_{i_1} \right| d\mu(x'_2) \right)^2 \right]^{1/2},$$

we obtain

$$\int_{K_E} \int_{K_E} |\langle u'(\bar{x}'), \tilde{g}_{e,\pi}(A) \rangle| d\mu(x'_1) d\mu(x'_2) \geq (\pi_1(T))^{-2} \left[ \sum_{\ell_1, \ell_2=1}^{\delta} \left( \sum_{i_1} \right)^2 \right]^{1/2}$$

where $\Sigma_2 = \Sigma_{\ell_2, \ldots, \ell_k=1}^{\delta} \langle e_{\ell_2}, x'_2 \rangle \cdots$ (the $\ldots$ represent the same terms which appear in $\Sigma_1$).

If we continue to integrate with respect to $d\mu(x'_3)$ and so on, finishing with $d\mu(x'_k)$ we obtain

$$\int_{K_{(E^n)}} |\langle u'(\bar{x}'), \tilde{g}_{e,\pi}(A) \rangle| d\tilde{\mu}(\bar{x}') \equiv (\pi_1(T))^{-k} \left[ \sum_{\ell_1, \ldots, \ell_k=1}^{\delta} \left( \sum_{i_1} \right)^2 \right]^{1/2}$$

where

$$\sum_k = \sum_{i_1, \ldots, i_k=1}^{\delta} \sum_{j_1, \ldots, j_k=1}^{\delta} \langle g_1(e_{i_1}), e'_{j_1} \rangle \cdots \langle g_k(e_{i_k}), e'_{j_k} \rangle a_{i_1, \ldots, i_k} \cdot \sum_{i=1}^m a_{i_1}^{(i)}, \ldots, a_{i_k}^{(i)} b_{j_1, \ldots, j_k}.$$ 

Let $\sigma^{(i)} = \pi^{(i)-1}$, i.e. $\sigma^{(i)}(r) = j$ iff $\pi^{(i)}(j) = r$, then

$$\langle g_i(e_{i_1}), e'_{j_1} \rangle = \langle e'_{j_1}, e_{\pi^{(i)}(j_1)}, e'_{j_1} \rangle = \begin{cases} 0; & i \neq \sigma^{(i)}(j_1) \\ e'_{j_1}; & i = \sigma^{(i)}(j_1) \end{cases}.$$ 

Using Khintchine's inequality [17]
and averaging over all \( \epsilon^{(1)}, \epsilon^{(2)}, \ldots, \epsilon^{(k)} \), we get

\[
2^{k/2} \text{Av}_E \left| \sum_{k} \right| \geq \left[ \sum_{i_1 \ldots i_k = 1}^{k} \left( \sum_{j_1, \ldots, j_k = 1}^{m} (a_{\sigma^{(i)}(j_1), \ldots, \sigma^{(i)}(j_k)})^2 \left( \sum_{j_1, \ldots, j_k = 1}^{m} a_{j_1, \ldots, j_k} b_{\ell}^{(i)} \right)^2 \right)^{1/2} \right].
\]

Hence,

\[
\text{Av}_E \int_{K(E)^+} |(u'(\bar{x}'), \tilde{g}_{e,\pi}(A))| d\tilde{\mu}(\bar{x}') \geq
\]

\[
(\sqrt{2}\pi_1(T))^{-k} \left[ \sum_{\ell_1, \ldots, \ell_k = 1}^{m} \sum_{i_1, \ldots, i_k = 1}^{k} (a_{\sigma^{(i)}(j_1), \ldots, \sigma^{(i)}(j_k)})^2 \right]^{1/2}.
\]

Now we shall average over all permutations, and use the fact that

\[
\text{Av}_E |a_{\sigma^{(i)}(j_1), \ldots, \sigma^{(i)}(j_k)}| = n^{-k} \sum_{i_1, \ldots, i_k} |a_{i_1, \ldots, i_k}|,
\]

this gives the following estimate for I

\[
(n\sqrt{2}\pi_1(T))^k I \geq
\]

\[
\sum_{i_1, \ldots, i_k = 1}^{k} |a_{i_1, \ldots, i_k}| \left[ \sum_{\ell_1, \ldots, \ell_k = 1}^{m} \sum_{j_1, \ldots, j_k = 1}^{m} (a_{j_1, \ldots, j_k})^2 \right]^{1/2} \geq \sum |a_{i_1, \ldots, i_k}| m^{-1/2}|\text{trace}(u)|
\]

the last inequality follows from Lemma 2. The proof is completed by applying Proposition 1 for \( p = \frac{1}{2} \) while noting that \( \tilde{g}_{e,\pi}(A) \) are norm-one elements of \( E_k^* \) and the \( \bar{x}' \) which appear in I are norm-one elements of the dual space \( (E')^*_\pi \).

**Lemma 4:** With the notation of Lemma 3,

\[
(\sqrt{2}\pi_1(T))^k \eta_{1/2}(E_k^*) \geq \sum |a_{i_1, i_2, \ldots, i_k}|.
\]
PROOF: This follows immediately from Lemma 3 and the obvious inequality \(\alpha(F)\alpha^*(F) \geq \dim(F)\) for any ideal norm \(\alpha\) and finite-dimensional space \(F\).

We shall next use the following inequality due to S. Chevet [1], for the sake of completeness we include the proof. If \(\{x_i\}_{i=1}^n \subset E\), we shall denote by \(\varepsilon_2(\{x_i\}) = \sup\{(\Sigma_{i=1}^n |\langle x_i, x'\rangle|^2)^{1/2}; \; x' \in E', \|x'\| = 1\}\).

**LEMMA 5:** If \(\{x_i\}_{i=1}^n\) and \(\{y_j\}_{j=1}^m\) are elements in Banach spaces \(E\) and \(F\) respectively, and \(g_{i,j} (i, j = 1, 2, \ldots, n)\) is a sequence of equidistributed, independent, orthonormal random Gaussian variables, then

\[
\frac{\Lambda}{2} \leq \mathbb{E}\left(\left\|\sum_{i,j} g_{i,j} x_i \otimes y_j \right\|_{E \otimes F}^2\right) \leq \sqrt{2}\Lambda
\]

where \(\Lambda = \varepsilon_2(\{x_i\})\mathbb{E}(\|\Sigma_i g_{i,1} x_i\|) + \varepsilon_2(\{y_j\})\mathbb{E}(\|\Sigma_j g_{j,1} y_j\|).

PROOF: Let \(T = \{(\xi, \eta); \; \xi \in E', \eta \in F', \|\xi\| = \|\eta\| = 1\}\). For each \(t = (\xi, \eta) \in T\), define the random variables

\[
X_t = \sum_{i,j} g_{i,j} (x_i, \xi)(y_j, \eta) \quad \text{and} \quad Y_t = \alpha \sum_j g_{i,1}(y_j, \eta) + \beta \sum_i g_{i,2}(x_i, \xi)
\]

where \(\alpha = \varepsilon_2(\{x_i\}), \beta = \varepsilon_2(\{y_j\})\). It is easy to see that if \(s = (\xi_1, \eta_1) \in T\), then

\[
\mathbb{E}(\|X_t - X_s\|^2) = \sum_{i,j} ((x_i, \xi_1)(y_j, \eta) - (x_i, \xi_1)(y_j, \eta_1))^2
\]

\[
= \sum_{i,j} ((x_i, \xi - \xi_1)(y_j, \eta) + (x_i, \xi_1)(y_j, \eta - \eta_1))^2
\]

\[
\leq 2 \sum_{i,j} ((x_i, \xi - \xi_1)(y_j, \eta))^2 + ((x_i, \xi_1)(y_j, \eta - \eta_1))^2
\]

\[
= 2\mathbb{E}(\|Y_t - Y_s\|^2)
\]

hence \(\mathbb{E}(\|X_t - X_s\|^2) \leq 2\mathbb{E}(\|Y_t - Y_s\|^2)\). By a result due to Sudakov ([2], Corollaire 2.1.3) this implies \(\mathbb{E}(\sqrt{T}X_t) \leq \sqrt{2}\mathbb{E}(\sqrt{T}Y_t)\), from which the right hand side of (*) follows.

For the other side, pick \(\xi_0 \in E', \|\xi_0\| = 1\), such that \(\alpha = \varepsilon_2(\{x_i\})\), and define the random variables \(Z_\eta = \Sigma_{i,j} g_{i,j} (x_i, \xi_0)(y_j, \eta)\) and \(W_\eta = \alpha \Sigma_{i,j} g_{i,j}(y_j, \eta)\). Then

\[
\mathbb{E}(\|Z_\eta - Z_\eta\|^2) = \alpha^2 \sum_j (y_j, \eta - \eta_1)^2 = \mathbb{E}(\|W_\eta - W_\eta\|^2),
\]
so again $E(\sup_{|n|=1} Z_n) = E(\sup_{|n|=1} W_n)$, but

$$E \left( \left\| \sum_{ij} g_{ij} x_i \otimes y_j \right\|_{E_\otimes F} \right) = E \left( \left\| \sum_{ij} g_{ij}(x_i, \xi_0) y_j \right\| \right)$$

$$= E \left( \sup_{|n|=1} Z_n \right) = E \left( \sup_{|n|=1} W_n \right)$$

$$= \alpha E \left( \left\| \sum_j g_{j1} y_j \right\| \right), \text{ and similarly, } \geq \beta E \left( \left\| \sum_i g_{i1} x_i \right\| \right),$$

hence the left side of (*). \(\Box\)

**Theorem 6:** Let \(\{e_i\}_{i=1}^n\) be a symmetric basis for a Banach space \(E\), and let \(T : E \to \ell_2^n\) be the natural basis to basis map. Then, if \(E^k_\epsilon = E \otimes E \otimes \cdots \otimes E\)

$$\eta_{1/2}(E^2_\epsilon)(\|T^{-1}\|\pi_1(T))^2 \geq \sqrt{\frac{2}{\pi n}} n^{2k-2} n^{2-2k-2k-1}. $$

**Proof:** For each integer \(k = 1, 2, \ldots\), let \(I_k\) denote a set consisting of \(n^{2k}\) elements, and let \(\{b\}_{\alpha \in I_k}\) denote the natural basis of \(E^k_\epsilon\) (each \(b\) has the form \(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}\), where \(1 \leq i_j \leq n\)). Consider the random vectors of \(E^k_\epsilon = E^{k-1}_\epsilon \otimes E^{k-1}_\epsilon\) of the form \(A^k = \sum_{\alpha, \beta \in I_k} g_{\alpha, \beta} b_\alpha \otimes b_\beta\). By Lemma 5

$$E(\|A^k\|^2_\epsilon) \leq 2 \sqrt{2} \epsilon_2(\{b\}_{\alpha \in I_k}) E(\|A^{k-1}\|^2_\epsilon). $$

Since \(\{\otimes^{2k-1} x'_i; x'_i \in K_F\}\) is the set of extreme points of the unit ball of \((E^{2k-1}_\epsilon)' = (E'_\epsilon)^{2k-1}\), we have

$$\epsilon_2(\{b\}_{\alpha \in I_k}) = \sup \left\{ \left( \sum_{i_1, \ldots, i_{2k-1}} \langle e_{i_1}, x'_1 \rangle \ldots \langle e_{i_{2k-1}}, x'_{2k-1} \rangle \right)^2 \right\};$$

$$x'_i \in K_F = \left[ \sup \left\{ \left( \sum_{i=1}^{2k-1} \langle e_i, x'^i \rangle \right)^{1/2} ; x' \in K_F \right\} \right]^{2k-1} = \|T^{-1}\|_F^{2k-1},$$

hence we get the reduction formula

$$E(\|A^k\|_\epsilon^2) \leq 2 \sqrt{2} \|T^{-1}\|_F^{2k-1} E(\|A^{k-1}\|_\epsilon^{2k-1}),$$

and so

$$E(\|A^k\|_\epsilon^2) \leq (2 \sqrt{2})^k \|T^{-1}\|_F^{2k-1+2k-2+\cdots+2k} E \left( \sum_{i=1}^k g_i e_i \right)$$
On the other hand, $E(\alpha, \beta | \gamma) = 2n^{2k}$, hence

and the inequality of Lemma 4 establishes the Theorem.

**Remark:** More generally, the same proof can show that if $E_i$ ($i = 1, 2, \ldots, 2^k$) are $n_i$-dimensional spaces with symmetric bases, and if $T_i : E_i \to \ell^2$ are the natural basis to basis maps, then for $F = E_1 \otimes \cdots \otimes E_2$, it is not essential that $2^k$ spaces appear in $F$, however, if the number is not $2^k$ then the bottom line on the right hand side of the inequality will be different.

**Example:** If $E = \ell^2$, then $\|T^{-1}\| = 1$, $\pi_1(T) \leq \sqrt{\pi n/2}$ [5], hence if $E_N = \underbrace{\ell^2 \otimes \cdots \otimes \ell^2}_2$ where $N = n^{2^k}$,

$$
\eta_{1/2}(E_N) \geq \sqrt{2/\pi} \sqrt{n^{2k}/(\sqrt{\pi})^{2k} 2^{k/2 - k} \sqrt{n}} \geq a \sqrt{N} e^{-\sqrt{\log N}}
$$

where $a > 0$ is a constant, and $k$, $n$ are chosen in an appropriate relation to $N$. If $0 < p \leq \frac{1}{2}$, then $\eta_p(E_N) \geq N^{p-1/2} \eta_{1/2}(E_N) \geq a N^p e^{-\sqrt{\log N}} \to \infty$ as $N \to \infty$.

**Corollary 7:** If $E$ is an $n$-dimensional normed space such that $d(E, \ell^2) < (n/\pi)^{1/2}$, then $\eta_{1/2}(E_k) \to \infty$ as $k \to \infty$. 
PROOF: First observe that if \( k > \ell \) then \( E_\ell \) is isometric to a norm one-complemented subspace of \( E_k \), therefore the ideal property of the norm \( \eta_{1/2} \) implies that \( \eta_{1/2}(E_k^*) \), \( k = 1, 2, \ldots \), is a nondecreasing sequence. Clearly \( d(E_k^*, (\ell_2^n)^k) \leq (d(E, \ell_2^n))^k \), hence using the estimates for \( \eta_{1/2}((\ell_2^n)^k) \) we obtain

\[
\eta_{1/2}(E_\ell^2) \geq \eta_{1/2}((\ell_2^n)^k)(d(E, \ell_2^n))^k \geq \sqrt{2/\pi} \left( \frac{\sqrt{n/\pi}}{d(E, \ell_2^n)} \right)^{2k} / 2^{3k/2} \sqrt{n}
\]

which tends to \( \infty \) with \( k \).

REMARKS: (1) It may be true that \( \eta_{1/2}(E_k^*) \sim \eta_k \) as \( k \to \infty \) whenever \( d(E, \ell_2^n) < \sqrt{n} \). It is obviously false if \( d(E, \ell_2^n) = \sqrt{n} \), as in the case \( E = \ell_2^n \).

(2) If \( 1 < q < \infty \), and \( c > |\frac{1}{2} - 1/q| \), there exists \( N \) such that if \( n \geq N \) and \( E \) is any \( m \)-dimensional subspace of \( L_q(\mu) \), then \( \eta_c(E_k^*) \to \infty \) as \( k \to \infty \). The reason for this is that \( d(E, \ell_2^n) \leq n^{1/2 - 1/d} \) by [13], so if \( c \leq \frac{1}{2} \)

\[
\eta_c(E_\ell^2) \geq (c^{-1/2})^{2k} \eta_{1/2}(E_\ell^2)
\]

which tends to \( \infty \) by applying the inequality in the proof of Corollary 7.

THEOREM 8: There exists a reflexive separable Banach space \( E \) with both \( \eta_p(E) \) and \( \eta_p^*(E) \) infinite for all values of \( p > 0 \).

PROOF: Let \( E_N \) be the space in the Example, and let \( E = (\Sigma_i \oplus E_N_i)_{\ell_2} \) where \( N_i \to \infty \), \( N_i = (n_i)^{2k_i} \), which the proper relation maintained between \( n_i \) and \( k_i \) with \( N_i \). Since \( E_N_i \) is norm one-complemented in \( E \), it follows that

\[
\eta_p(E) \geq \eta_p^*(E) \geq \eta_p^*(E_N_i) = \eta_p(E_N_i) \geq a N_i \exp(-\log N_i) \to \infty \quad \text{as} \quad i \to \infty.
\]

In order to estimate \( \pi(T) \) for general spaces it may be useful to apply the following proposition.

PROPOSITION 9: For any \( 0 < p, q, r < \infty \) there exist constants \( a_{r,q} \), \( b_{p,q} > 0 \) such that for any Banach space \( E \) and any operator \( T : E \to \ell_2^q \)

(1) \( b_{p,q}^{-1} \pi_p(T) \leq \sqrt{n} (\int_{S_n} \| T'x \| q dm(x) )^{1/q} \leq a_{r,q}^{-1} \pi_r(T') \), where \( S_n = \{ x \in \ell_2^q ; \| x \| = 1 \} \) and \( dm(x) \) is the rotation invariant normalized measure on \( S_n \).
(2) If \( E' \) is a subspace of an \( L_\infty \)-space, \( 1 \leq s < \infty \), and if \( 0 < p \leq s \leq r < \infty \) and \( 0 < q < \infty \), the inequalities of (1) becomes equivalence relations and the constants of equivalence are independent of \( n, T \) and \( E \).

(3) If \( \dim(E) = n \) and \( E \) has a symmetric basis and \( T : E \to \ell^2_n \) is the basis to basis map, then all the values \( I_q = \left( \int_{S_n} \| T'x \|^q dm(x) \right)^{1/q} \) \( (0 < q < \infty) \) are equivalent and the constants of equivalence are independent of \( n \) and \( E \).

**PROOF:** (1) By [14] there exists a probability measure \( \mu \) on \( S_n \) such that for all \( x \in \ell^2_n \)

\[
\| T'x \| \leq \pi_q(T') \left( \int_{S_n} \langle x, x' \rangle^q d\mu(x') \right)^{1/q}
\]

hence by integrating with respect to \( dm \)

\[
I_q \leq \pi_q(T') \left( \int_{S_n} \int_{S_n} \langle x, x' \rangle^q d\mu(x') dm(x) \right)^{1/q} = \pi_q(T')/\pi_q(\ell^2_n)
\]

since \( \int_{S_n} \langle x, x' \rangle^q dm(x) \right)^{1/q} = \| x \|_2(\pi_q(\ell^2_n))^{-1} \) [5]. Since \( \pi_q(\ell^2_n) \sim \sqrt{n} \), and \( \pi_q(T') \) is a non-increasing function of \( q \), and \( I_q \) is a non-decreasing function of \( q \), the right hand side of (1) readily follows.

Without loss of generality we may assume that \( T' \) is a \( 1 \to 1 \) map, and define the probability measure \( \nu \) on the unit ball \( B_E \) of \( E' \) by

\[
\int_{B_E} f d\nu = \int_{S_n} f(T'x) \| T'x \| \| T'x \|^q dm(x) / \int_{S_n} \| T'x \|^q dm(x)
\]

for \( f \in C(B_E) \). Taking \( f = \langle \xi, \cdot \rangle^q \) where \( \xi \in E \), we obtain

\[
\int_{B_E} \langle \xi, x' \rangle^q d\nu(x') = \int_{S_n} \langle T\xi, x \rangle^q dm(x) / \int_{S_n} \| T'x \|^q dm(x)
\]

\[
= \| T\xi \|^q / (\pi_q(\ell^2_n))^q \int_{S_n} \| T'x \|^q dm(x),
\]

therefore, \( \pi_q(T) \leq \pi_q(\ell^2_n) \left( \int_{S_n} \| T'x \|^q dm(x) \right)^{1/q} \), and as above this proves the left hand side of (1).

(2) Let \( j : E' \to L_s \) be an isometric embedding, then by [12] \( \pi_s(jT') \leq \pi_s(jT'') \), that is

\[
\pi_s(T') = \pi_s(jT') \leq \pi_s(T''j) \leq \pi_s(T'') = \pi_s(T),
\]
and (2) follows from the inequalities

\[
\pi_q(T) \leq \pi_p(T) \leq \pi_{p,q}(T) \leq b_{p,q} I_{q} \tilde{a}^{-1}_{q} \pi_{q}(T') \leq b_{p,q} a^{-1}_{q} \pi_{q}(T').
\]

(3) Set \( |x|_{E} = \|T'x\| \) for vectors \( x \in \ell^q_{n} = (\mathbb{R}^n, \| \cdot \|_2). \) Since \( E \) is symmetric, by \([10]\) \( \tilde{T} \tilde{T}^{-1} = d(E, \ell^2_{n}) \), and \( d(E, \ell^2_{n}) \leq \sqrt{n} \), so \( a \|x\|_2 \leq |x|_E \leq b \|x\|_2 \) for all \( x \in \ell^2_{n} \), where \( b/a \leq \sqrt{n} \). From the remark following Lemma 2.7 in \([4]\), the values \( I_q = \left( \int_{S_n} \| x \|^q_E \, dm(x) \right)^{1/q} \) \((0 < q < \infty)\) are all equivalent to the Levy mean \( M^* \), which is by definition the unique number such that \( m(\{ x \in S_n ; \ |x|_E \leq M^* \}) \leq \frac{1}{2} \) and \( m(\{ x \in S_n ; \ |x|_E \geq M^* \}) \leq \frac{1}{2} \), that is, there exist absolute positive constants \( a_q, b_q \) such that \( a_q M^* \leq I_q \leq b_q M^* \).

\[\text{COROLLARY 10: If } \dim(E) = n, \text{ there are absolute constants } a, b > 0 \text{ such that for any } T : E \to \ell^2_{n} \]

\[a \pi_1(T) \leq \sqrt{n} \int_{S_n} \| T'x \| dm(x) \leq b \pi_1(T) \sqrt{n} x_a(E).\]

**Proof:** Let \( \gamma_p \) denote the best factorization through an \( L_p \)-space norm \([9]\). Interpolation technique as in Theorem 7 \([11]\) shows that \( \pi_p(T') \leq n^{1/p} \gamma_{-n}(T') \). Since \( \gamma_{-n}(T') = \gamma_{1}(T) \leq x_{a}(E) \pi_{1}(T) \) \([7]\), and \( \pi_{p}(\ell^2_{n}) \geq c \sqrt{n/p} \) \([5]\), we obtain

\[
\int_{S_n} \| T'x \| dm(x) \leq \left( \int_{S_n} \| T'x \|^p dm(x) \right)^{1/p} \leq \pi_p(T') / \pi_p(\ell^2_{n}) \leq \sqrt{p/n} c^{-1} n^{1/p} \pi_1(T) x_a(E)
\]

and the estimate follows by taking \( p = \ln n \), and from Proposition 9.

\[\text{COROLLARY 11: Let } E_N = \ell^n_p \bigotimes \ell^2_{e} \cdots \bigotimes \ell^n_p \ell^2_{e}, \ N = n^{2k}, \text{ where } 1 \leq p \leq 2. \text{ There exists } b > 0, \text{ such that } \eta_{1/2}(E_N) \geq b \sqrt{N} e^{-\sqrt{\log N}} \text{ for the proper relation between } n, k \text{ with } N.\]

**Proof:** Factor \( T : \ell^n_p \xrightarrow{A} \ell^1_{e} \xrightarrow{B} \ell^2_{e}, \) where \( A, B \) are the inclusions. Then \( \| T^{-1} \| = n^{1/p-1/2} \), and \( \pi_{1}(T) \leq \| A \| \pi_{1}(B) = \sqrt{2} n^{1-1/p} \), since \( \pi_{1}(B) = \sqrt{2} \) is the Khintchine constant. Therefore, \( \| T^{-1} \| \pi_{1}(T) \leq \sqrt{2n} \). The proof is concluded as in the Example following Theorem 6.
REM A R K: Since the distance between $E_k^c$ and $(\ell_p^n)_k$ is $\leq d(E, \ell_p^n)^k$, it follows from the estimates of Corollary 10, Theorem 6 and the inequality

$$\eta_{1/2}(E_k^c) \geq \eta_{1/2}((\ell_p^n)_k)(d(E, \ell_p^n)^{-k}),$$

that if $E$ is any $n$-dimensional Banach space such that $\inf_{1 \leq p \leq 2} d(E, \ell_p^n) < \sqrt{n}/2$, then $\eta_{1/2}(E_k^c) \to \infty$ as $k \to \infty$. \qed

Denote by $r_i(t)$, the $i$-th Rademacher function on $[0,1]$.

COROLLARY 12: Let $1 < p \leq 2 \leq q < \infty$, $1/p < 1/q + 1/2$. Assume $F$ is a Banach space of type $p$ and cotype $q$. Then, for any $c$ satisfying $c > 1/p - 1/q$, there exists an integer $N$ such that if $n > N$ and if $E$ is any $n$-dimensional symmetric subspace of $F$, then $\eta_c(E_k^c) \to \infty$ as $k \to \infty$.

PROOF: Let $\alpha$ be the type-$p$ constant and $\beta$ be the cotype-$q$ constant of $F$ respectively. Let $\{e_i\}_{i=1}^n$ denote the symmetric basis of a subspace $E \subset F$, and $\{e_i^*\}_{i=1}^n$ be the biorthogonal functionals. The inequality

$$\left(\sum_{i=1}^n |\xi_i|^q \right)^{1/q} \leq \beta \int_0^1 \left\| \sum_{i=1}^n \xi r_i(t) e_i \right\| dt = \beta \left\| \sum_{i=1}^n \xi_i e_i \right\|$$

implies that $\|\Sigma \xi e_i^*\| \leq \beta \|\xi\|_{q'}$. By Proposition 9

$$b_{1, q'} n^{-1/2} \pi_1(T) \leq \left( \int_{S_n} \| T^*x \|_{q'} dm(x) \right)^{1/q'} \leq \beta \left( \int_{S_n} \| \xi \|_{q'} dm(\xi) \right)^{1/q'} = \beta n^{1/q'} (\pi_q(\ell_2^n))^{-1},$$

hence $\pi_1(T) \leq \beta c_{q'} n^{1/q'}$, where $c_{q'}$ is constant. Select scalars $\{x_i\}_{i=1}^n$ so that $\Sigma x_i^2 = 1$ and $\|T^{-1}\| = \|\Sigma x_i e_i\|$. Then,

$$\|T^{-1}\| = \int_0^1 \left\| \sum_{i=1}^n x_i r_i(t) e_i \right\| dt \leq \alpha \|x\|_p \leq \alpha n^{1/p - 1/2},$$

therefore $\|T^{-1}\| \pi_1(T) \leq \alpha \beta c_{n^{1/p - 1/q + 1/2}}$. Combining this with the inequality $\eta_c(E_k^c) \geq \eta_{1/2}(E_k^c)(n^{2k})^{-c/2}$, and Theorem 6, establishes that for $c > 1/p - 1/q$ and $n$ sufficiently large, $\eta_{1/2}(E_k^c)(n^{2k})^{-c/2} \to \infty$ as $k \to \infty$. \qed
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(Obatum 27-VII-1978 & 19-III-1979)