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THE MODULI AND THE GLOBAL PERIOD MAPPING OF SURFACES WITH $K^2 = p_g = 1$: A COUNTEREXAMPLE TO THE GLOBAL TORELLI PROBLEM

F. Catanese*

0. Introduction

In the theory of algebraic curves one of the most celebrated results is the theorem of R. Torelli (see [1], [22]), stating, in modern language, that two curves are isomorphic iff their Jacobians are isomorphic as polarized Abelian varieties.

In the general trend to extend this kind of result to higher dimensional varieties some difficulties are encountered (e.g. for surfaces of general type with $q = p_g = 0$ the Hodge structure is trivial though they depend on some moduli), but some positive results have been obtained [18], [11] and some problems have been raised [9], [10] on the restrictions to impose in order that Torelli type theorems should be valid in the theoretical set up of Griffiths [8], [10].

Here we exhibit the pathology of a class of simply connected surfaces of general type for which the moduli variety is a rational variety of dimension 18, and the period mapping, assigning to the isomorphism class of a surface S the class of its polarized Hodge structure, is a generally finite map of degree at least 2.

In this paper we essentially draw the consequences of the study of the geometry of these surfaces and the local behaviour of the period mapping, pursued in our previous paper [3].

We mention again that our interest was aroused by Griffiths who showed us a paper of Kynef [14] where was constructed a particular surface of this kind such that at the point corresponding to it the

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differential of the local period mapping was not invertible. Another particular example of these surfaces was also known [5], [6]: in [3] we proved, using results of [2], that the canonical models X of our surfaces S are all the weighted complete intersections (w.c.i.) of type $(6, 6)$ in the weighted projective space $\mathbb{P}(1, 2, 2, 3, 3)$ (see [4], [15] as a reference about the theory of w.c.i.) with at most rational double points as singularities, and proved, among other things, that the local period map was invertible outside a hypersurface of the local moduli space.

For the reader's convenience we reproduce some proofs of [3], since we need to analyze closely the isomorphisms of two surfaces of our family (to ensure that the ramification of the local period mapping is not, generally, due to automorphisms).

We then show that an open dense set of the coarse moduli variety M is isomorphic to a Zariski open set of the affine space of dimension 18, over it lies a family which is the universal deformation family of Kodaira–Spencer–Kuranishi, and the restriction of the period map to this open set is a (generally finite) map of degree at least 2. We refer to [3] for more details on our surfaces, as an explicit description of the "special" surfaces, those for which the bicanonical map is a Galois covering of the plane, of their geometric construction (giving as a corollary the result that they are simply connected). However, in [3], the discussion of the restriction of the period mapping Ψ to the subvariety N representing special surfaces was misleading. In fact theorem 5 of [3] is false, hence the conclusions of theorem 6 do not hold. Indeed, an easy extension of the arguments employed in the proof of theorem 6 gives the result that $\Psi|_N$ is a fibration with smooth rational fibres of dimension 2.

We want to mention that this last fact has been shown by A. Todorov in a research independent of ours: he views the special surfaces as double ramified covers of K3 surfaces to infer that when the branch curve moves the periods are not changing.

Notations throughout the paper

S is a minimal smooth surface with $p_g = K^2 = 1$,

$s_0 \in H^0(S, \mathcal{O}(K))$ the unique (up to constants) non zero section,

$C = \operatorname{div}(s_0)$ the canonical curve,

R the graded ring $\mathbb{C}[x_0, y_1, y_2, z_3, z_4]$, where $\deg x_0 = 1$, $\deg y_i = 2$, $\deg z_j = 3$ $(i = 1, 2,\ j = 3, 4)$,

$x_0^2 = y_0$, $x_0^3 = z_0$, $x_0 y_1 = z_1$, $x_0 y_2 = z_2$ as elements of R,

R_m the graded part of degree m of R,

$Q = Q(1, 2, 2, 3, 3) = \operatorname{proj}(R)$,

$\mathbb{P} = \mathbb{P}(1, 2, 2, 3, 3) = Q - \{x_0 = z_3 = z_4 = 0\} - \{x_0 = y_1 = y_2 = 0\}$, the non singular part of Q,

$R(S) = \bigoplus_{m=0}^{\infty} H^0(S, \mathcal{O}_S(mK))$ the canonical ring of S,

$h^i(S, L) = \dim H^i(S, L)$ if L is a coherent sheaf on S,

$i : Q \to Q$ the automorphism which permutes z_3 with z_4, ρ the one which permutes y_1 with y_2.

1. Surfaces with $K^2 = p_g = 1$, their canonical models and their isomorphisms

LEMMA 1.1: *$P_m = h^0(S, \mathcal{O}_S(mK)) = \frac{1}{2}m(m-1) + 2$, and S has no torsion.*

PROOF: $P_m = \frac{1}{2}m(m-1)K^2 + \chi(S, \mathcal{O}_S)$ (see [2] p. 185, or [12]) and by Theorems 11, 15 of [2] $q = 0$ and S has no torsion, so in particular $\chi(S, \mathcal{O}_S) = 2$.

One can choose therefore s_1, s_2, s_3, s_4 such that s_0^2, s_1, s_2 are a basis of $H^0(S, \mathcal{O}_S(2K))$, and $s_0^3, s_0s_1, s_0s_2, s_3, s_4$, are a basis of $H^0(S, \mathcal{O}_S(3K))$.

Write now $C = \operatorname{div}(s_0) = \Gamma + Z$, where Γ is irreducible and $K \cdot \Gamma = 1$, $K \cdot Z = 0$.

LEMMA 1.2: *If $D \in |2K|$ and $D \geq \Gamma$, $D = 2C$.*

PROOF: Write $D = D' + \Gamma$ and let D'' be the movable part of $|D'| : D'' \cdot K = 1$, so by the index theorem either D'' is homologous (hence linearly equivalent, as S has no torsion) to K, or $D''^2 \leq -1$, hence in both cases $h^0(S, \mathcal{O}_S(D'')) = 1$.

COROLLARY 1.3: $H^0(S, \mathcal{O}_S(4K)) = s_0 \cdot H^0(S, \mathcal{O}_S(3K)) \oplus \mathbb{C}s_1^2 \oplus \mathbb{C}s_1s_2 \oplus \mathbb{C}s_2^2$.

PROOF: Because $P_4 = 8$ it is enough to prove that the two vector subspaces have no common line. Supposing the contrary, there would exist a section $s \in H^0(S, \mathcal{O}_S(3K))$, and constants $\lambda_1, \mu_1, \lambda_2, \mu_2$, such that $s_0 \cdot s = (\lambda_1 s_1 + \lambda_2 s_2)(\mu_1 s_1 + \mu_2 s_2)$.

Taking the associated divisors, $C + \operatorname{div}(s) = D_1 + D_2$, where $D_i \in |2K|$, and one of the D_i, say D_1, is therefore $\geq \Gamma$.

By lemma 2 $D_1 = 2C$, hence $\lambda_1 s_1 + \lambda_2 s_2 = cs_0^2$ for a suitable $c \in \mathbb{C}$, contradicting the independence of s_0^2, s_1, s_2.

THEOREM 1.4: *$|2K|$ has no base points, so that $\Phi = \Phi_{2K}: S \to \mathbb{P}^2$ is a morphism of degree* 4.

PROOF: If b were a base point of $|2K|$, then s_0, s_1, s_2 would vanish at b; by Corollary 3b would be a base point of $|4K|$, contradicting Theorem 2 of [2].

Define an homomorphism $\alpha^*: R \to R(S)$ by sending x_0 to $s_0, \ldots, z_4$ to s_4: by Theorem 1.4. α^* induces a morphism $\alpha: S \to Q = Q(1,2,2,3,3)$. Remark that Q is smooth outside the two $\mathbb{P}^1$'s $\{x_0 = z_3 = z_4 = 0\}$, $\{x_0 = y_1 = y_2 = 0\}$, and on their complement $\mathbb{P} = \mathbb{P}(1,2,2,3,3)$ $\mathcal{O}_Q(m)$ is an invertible sheaf for every integer m and $\forall a, b \in \mathbb{Z}$ one has an isomorphism $\mathcal{O}_Q(a) \otimes \mathcal{O}_Q(m)^{\otimes b} \to \mathcal{O}_Q(a + bm)$ (compare [15], exp. pages 619–624, and also cf. [4]).

Denote by I the ideal $\ker \alpha^*$: because $\dim R_6 = 19$, $P_6 = \dim R(S)_6 = 17$ there exist two independent elements $f, g \in I_6$.

PROPOSITION 1.5: *f, g are irreducible and $\alpha(S) = Y = \{f = g = 0\}$. In particular $I_5 = 0, |3K|$ has no base points, so $\alpha(S) \subset \mathbb{P}$.*

PROOF: If f is reducible, by Corollary 1.3 $f = x_0 \cdot f'$, $f' \in I_5$, f' irreducible, and $\alpha(S) \subset \{f' = g = 0\}$. Denote now by $p: Q \to \mathbb{P}^2$ the rational map given by (x_0^2, y_1, y_2): clearly $\Phi = p \circ \alpha$. Now one gets a contradiction considering that

(i) $p: \{f' = g = 0\} \to \mathbb{P}^2$ is of degree ≤ 2, because $f' = 0$ is irreducible and the variables z_3, z_4 appear at most quadratically in g, and linearly in f' (observe that $(-x_0, y_1, y_2, -z_3, -z_4) \cong (x_0, y_1, y_2, z_3, z_4)$).

(ii) α is a birational because the tricanonical map is such ([2], p. 202).

(iii) Φ is of degree four.

As $I_5 = 0$, s_0, s_1, s_2, s_3, s_4 generate $H^0(S, \mathcal{O}_S(5K))$ so a base point for $|3K|$ would be a base point of $|5K|$, against Theorem 2 of [2].

Finally Y has dimension 2, contains $\alpha(S)$ ($\alpha(S) \subset \mathbb{P}$) and has its same degree, 1 ([15], Proposition 3.2): therefore $\alpha(S) = Y$.

PROPOSITION 1.6: *The subscheme of* $\mathbb{P}$ *weighted complete intersection of type* (6, 6), *$Y = \{f = g = 0\}$ is isomorphic to the canonical model X of S. Therefore I is generated by f, g, and α^* induces an isomorphism $\alpha^*: R' = R/I \to R(S)$.*

PROOF: $\alpha: S \to Y$ is a desingularization such that the pull-back of the dualizing sheaf on Y is the canonical bundle K of S (as $\omega_Y =$

$\mathcal{O}_Q(1)$ by [15], Proposition 3.3): therefore Y has only rational double points as singularities and is the canonical model of S (cf. [2], [16]).

THEOREM 1.7: *The canonical models of minimal surfaces with* $K^2 = p_g = 1$ *correspond to weighted complete intersections* X *of type* $(6, 6)$, *contained in* $\mathbb{P}(1, 2, 2, 3, 3)$, *with at most rational double points as singularities, and two surfaces are isomorphic iff their canonical models are projectively equivalent in* $\mathbb{P}$.

PROOF: If X is as above, $\mathcal{O}_X(1)$ is the canonical sheaf and by Proposition 3.2 of [15] $\mathcal{O}_X(1)^2 = 1$; again by Proposition 3.3 of [15] R'_m is isomorphic to $H^0(X, \mathcal{O}_X(m))$, so our first assertion follows immediately.

Note that an isomorphism of two surfaces corresponds to an isomorphism of their canonical rings: this means that their canonical models are related by an invertible transformation of the following form

$$x_0 \to dx_0$$

$$y_i \to d_{i1}y_1 + d_{i2}y_2 + d_{i0}x_0^2 = \sum_{j=0}^{2} d_{ij}y_j \quad (i = 1, 2)$$

$$z_j \to c_{j3}z_3 + c_{j4}z_4 + c_{j0}x_0^3 + c_{j1}x_0y_1 + c_{j2}x_0y_2 = \sum_{h=0}^{4} c_{jh}z_h. \quad (j = 3, 4).$$

PROPOSITION 1.8: *There exists a projective change of coordinates such that* X *is defined by* 2 *equations in canonical form*

$$\begin{cases} f = z_3^2 + x_0z_4(a_0x_0^2 + a_1y_1 + a_2y_2) + F_3(x_0^2, y_1, y_2) = 0 \\ g = z_4^2 + x_0z_3(b_0x_0^2 + b_1y_1 + b_2y_2) + G_3(x_0^2, y_1, y_2) = 0 \end{cases}$$

where F_3, G_3 are cubic forms.

PROOF: Write

$$\begin{cases} f = Q_1(z_3, z_4) + \cdots \text{(terms of deg} \le 1 \text{ in the } z_j\text{'s)} \\ g = Q_2(z_3, z_4) + \cdots \end{cases}$$

I claim that the quadratic forms Q_1, Q_2 are not proportional: otherwise, by taking a linear combination of the 2 equations one would have $Q_2 = 0$, but then $p : X \to \mathbb{P}^2$ would have degree 2 and not 4.

By a transformation $z_j \to c_{j3}z_3 + c_{j4}z_4$ one can suppose $Q_1 = z_3^2$, $Q_2 =$

z_4^2: this is immediate if both Q_1, Q_2 have rank 1, while if, say, Q_1 has rank 2, one proceeds as follows.

First take coordinates such that $Q_1 = z_3 \cdot z_4$, then, subtracting to g a multiple of f, one can get $Q_2 = m_3 z_3^2 + m_4 z_4^2$.

If m_3 and m_4 are $\neq 0$, one takes first new variables $\sqrt{m_3}\, z_3$, $\sqrt{m_4}\, z_4$, so that for $f/\sqrt{m_3 m_4}$ and g, Q_1 and Q_2 have now respectively the form $z_3 z_4$, $z_3^2 + z_4^2$: then one takes variables z_3', z_4' with $z_3 = z_3' - z_4'$, $z_4 = z_3' + z_4'$ so that $Q_1 = z_3'^2 - z_4'^2$, $Q_2 = 2(z_3'^2 + z_4'^2)$, and finally $\dfrac{2Q_1 + Q_2}{4}$, $\dfrac{Q_2 - 2Q_1}{4}$ are in the desired form.

If, say, m_4 is zero, one can suppose $Q_2 = z_3^2$: but then we have a contradiction because the point $(0, 0, 0, 0, 1)$ would satisfy $f = g = 0$, against the fact that $X \subset \mathbb{P}$.

Finally, if now $f = z_3^2 + x_0 z_3(\tilde{a}_0 x_0^2 + \tilde{a}_1 y_1 + \tilde{a}_2 y_2) + \cdots$ one kills the $\tilde{a}_j$'s by completing the square, i.e. by taking $z_3 + \frac{1}{2} x_0(\tilde{a}_0 x_0^2 + \tilde{a}_1 y_1 + \tilde{a}_2 y_2)$ as new z_3 coordinate, and analogously one then does for g, acting on the z_4 variable.

Let H be the connected projective subgroup of $\mathrm{Aut}(\mathbb{P})$ consisting of the transformations whose matrix is of the form

$$\begin{pmatrix} d & d_{10} & d_{20} & 0 & 0 \\ 0 & d_{11} & d_{21} & 0 & 0 \\ 0 & d_{12} & d_{22} & 0 & 0 \\ 0 & 0 & 0 & c_{33} & 0 \\ 0 & 0 & 0 & 0 & c_{44} \end{pmatrix}$$

The transformation i which permutes z_3 with z_4 centralizes every element of H, and denote by $\tilde{H}$ the subgroup generated by H and i.

Proposition 1.9: *If X and X' are defined by two canonical forms, they are isomorphic iff their canonical equations are equivalent under the projective subgroup $\tilde{H}$.*

Proof: Set $y_0 = x_0^2$, as usual, and suppose X is given by $f = g = 0$ where

$$f = z_3^2 + x_0 z_4 \left(\sum_{i=0}^{2} a_i y_i \right) + F_3(y_0, y_1, y_2) = z_3^2 + x_0 z_4 \alpha(y) + F(y),$$

$$g = z_4^2 + x_0 z_3 \left(\sum_{i=0}^{2} b_i y_i \right) + G_3(y_0, y_1, y_2) = z_4^2 + x_0 z_3 \beta(y) + G(y)$$

and X' is given by $f' = g' = 0$.

X is isomorphic to X' iff $\exists \delta \in \mathrm{Aut}(\mathbb{P})$, λ, μ, λ, $\mu' \in \mathbb{C}$ such that

$$\text{(A)} \qquad \begin{cases} f \circ \delta = \lambda f' + \mu g' \\ g \circ \delta = \lambda' f' + \mu' g'. \end{cases}$$

Writing $z_0 = x_0^3$, $z_1 = x_0 y_1$, $z_2 = x_0 y_2$ suppose δ is given by $x_0 \to dx_0$, $y_i \to \sum_{j=0}^{2} d_{ij} y_j$ $(i = 1, 2)$, $z_j \to \sum_{h=0}^{4} c_{jh} z_h$ $(j = 3, 4)$,

$$\text{(A) implies in particular (B)} \begin{cases} (c_{33}z_3 + c_{34}z_4)^2 = \lambda z_3^2 + \mu z_4^2 \\ (c_{43}z_3 + c_{44}z_4)^2 = \lambda' z_3^2 + \mu' z_4^3, \end{cases}$$

hence $c_{33}c_{34} = c_{43}c_{44} = 0$ and (B) is equivalent to either

(B1) $\quad c_{34} = 0 \Rightarrow c_{43} = 0,\ \lambda = c_{33}^2,\ \mu = \lambda' = 0,\ \mu' = c_{44}^2$

or

(B2) $\quad c_{33} = 0 \Rightarrow c_{44} = 0,\ \lambda = \mu' = 0,\ \mu = c_{34}^2,\ \lambda' = c_{43}^2.$

By looking further at only those terms where z_3, z_4 do appear and denoting by D the transformation induced by δ on the $\mathbb{P}^2$ of homogeneous coordinates (y_0, y_1, y_2),

$$\begin{aligned} \text{(B1)} \quad \Rightarrow c_{33}^2 z_3^2 + 2c_{33}z_3x_0\left(\sum_{j=0}^{2} c_{3j}y_j\right) &+ \mathrm{d}x_0c_{44}z_4\alpha(D(y)) \\ &= c_{33}^2(z_3^2 + x_0z_4\alpha'(y)), \\ c_{44}z_4^2 + 2c_{44}z_4x_0\left(\sum_{j=0}^{2} c_{4j}y_j\right) &+ \mathrm{d}x_0c_{33}z_3\beta(D(y)) \\ &= c_{44}(z_4^2 + x_0z_3\beta'(y)) \end{aligned}$$

and these last two equalities are equivalent to $c_{3j} = c_{4j} = 0$ for $j = 0, 1, 2$, and to

$$\text{(C1)} \qquad \begin{cases} \alpha'(y) = \dfrac{\mathrm{d}c_{44}}{c_{33}^2}\, \alpha(D(y)) \\[2ex] \beta'(y) = \dfrac{\mathrm{d}c_{33}}{c_{44}^2}\, \beta(D(y)). \end{cases}$$

Notice that if δ satisfies (B2), then $i \circ \delta$ satisfies (B1) and the assertion is thus proven.

COROLLARY 1.10: *If, as in Proposition* 1.9, *X*, *X′ are given by* *canonical equations $f=g=0$, $f'=g'=0$ respectively, and are isomorphic under H, then*

$$\alpha'(y)=\frac{dc_{44}}{c_{33}^2}\alpha(D(y)),\quad \beta'(y)=\frac{dc_{33}}{c_{44}^2}\beta(D(y)),$$

$$F'(y)=F(D(y))/c_{33}^2,\quad G'(y)=G(D(y))/c_{44}^2.$$

LEMMA 1.11: If X is given by $f=g=0$, *and* $a_1b_2-a_2b_1\neq 0$, *then one can assume, after a projective change of coordinates, that* $\alpha(y)=y_1$, $\beta(y)=y_2$.

PROOF: The transformation leaving x_0, z_3, z_4 invariant and sending $y_1\to\alpha(y)$, $y_2\to\beta(y)$ is invertible iff $a_1b_2-a_2b_1\neq 0$.

LEMMA 1.12: *Suppose X, X' are given by pairs of canonical equations (f,g), (f',g') with $\alpha=\alpha'=y_1$, $\beta=\beta'=y_2$, and they are isomorphic. Denote by ρ the transformation which permutes y_1 and y_2. Then either f,g, (f',g') are equivalent under a diagonal matrix δ with $d_{11}=\dfrac{c_{33}^2}{dc_{44}}$, $d_{22}=\dfrac{c_{44}^2}{dc_{33}}$, or $(g\circ\rho\circ i, f\circ\rho\circ i)$, (f',g') are such.*

PROOF: By Proposition 1.9, X, X' are isomorphic iff either (f,g), (f',g') are equivalent under H or $(f\circ i, g\circ i)$, (f',g') are such.

In the first case by Corollary 1.10 $\begin{cases} c_{33}^2y_1=dc_{44}\left(\sum\limits_{j=0}^{2} d_{1j}y_j\right)\\ c_{44}^2y_2=dc_{33}\left(\sum\limits_{j=0}^{2} d_{2j}y_j\right)\end{cases}$

so that $d_{21}=d_{20}=d_{10}=d_{12}=0$, $d_{11}\,dc_{44}=c_{33}^2$, $d_{22}\,dc_{33}=c_{44}^2$. In the second case observe that ρ commutes with i, ρ belongs to H, and

$$\begin{cases} f\circ i\circ\rho=z_4^2+x_0z_3y_2+F(y_0,y_2,y_1)\\ g\circ i\circ\rho=z_3^2+x_0z_4y_1+G(y_0,y_2,y_1).\end{cases}$$

COROLLARY 1.13: *In the hypotheses of Lemma 1.12 (f,g) (f',g') are projectively equivalent iff exist non zero constants d, c_{33}, c_{44} such that $D:\mathbb{P}^2\to\mathbb{P}^2$ being the transformation such that $y_0\to d^2y_0$, $y_1\to(c_{33}^2/dc_{44})y_1$, $y_2\to(c_{44}^2/dc_{33})y_2$, being $D'=D\circ\rho=\rho\circ D$, either*

$$\begin{cases} F'(y)=c_{33}^{-2}F(D(y))\\ G'(y)=c_{44}^{-2}G(D(y))\end{cases}\quad\text{or}\quad\begin{cases} F'(y)=c_{33}^{-2}G(D'(y))\\ G'(y)=c_{44}^{-2}F(D'(y))\end{cases}$$

Supposing further that the coefficient in F, F', of $y_0^2y_1$ is 1, *that the coefficient in G, G' of $y_0^2y_2$ is* 1, *then necessarily $c_{33} = c_{44} = d^3$, therefore either $F = F'$, $G = G'$, or $F' = G \circ \rho$, $G' = F \circ \rho$.*

PROOF: The first part is a restatement of the conclusions of Lemma 1.12. The second follows from the equations stated in the first part, looking at the coefficients of $y_0^2y_1$, $y_0^2y_2$: $c_{33}^2 = d^4(c_{33}^2/dc_{44})$, $c_{44}^2 = d^4(c_{44}^2/dc_{33})$.

We end this section by giving a model in $\mathbb{P}^3$ for our surfaces:

PROPOSITION 1.14: *A surface S with $K^2 = p_g = 1$, whose canonical model is given by a pair of canonical equations f, g, with $\alpha(y_0, y_1, y_2) \neq 0$, is birational to a sextic surface N in $\mathbb{P}^3$ defined by the following equation in the homogeneous coordinates (z_0, z_1, z_2, z_3): $(F(z_0, z_1, z_2) + z_3^2z_0)^2 + \alpha(z_0, z_1, z_2)^2z_0 \cdot (z_3z_0\beta(z_0, z_1, z_2) + G(z_0, z_1, z_2)) = 0$.*

PROOF: Eliminate z_4 from the two equations f, g, and multiply the resultant by x_0^6 to obtain the above equation. Equation f, as $\alpha \neq 0$, guarantees that z_4 is a rational function of (z_0, z_1, z_2, z_3).

REMARK 1.15: N has as double curve, generally, a smooth plane cubic, (given by $\alpha = F + z_3^2z_0 = 0$), which passes through the point $0 = (0, 0, 0, 1)$, image point of the whole canonical curve. This is against the argument of Enriques [5] that the double curve should consist of 3 coplanar lines passing through 0.

2. Moduli of surfaces with $\mathbf{K^2 = p_g = 1}$

LEMMA 2.1: *If X is given as in Proposition* 1.8 *by two canonical equations*

$$\begin{cases} f = z_3^2 + x_0z_4\alpha(y) + F(y) = 0 \\ g = z_4^2 + x_0z_3\beta(y) + G(y) = 0, \end{cases}$$

for general choice of α, β, F, G, X is smooth (and contained in $\mathbb{P}$).

PROOF: $X \subset \mathbb{P}$ means that $\{x_0 = z_3 = z_4 = 0\} \cap X = \emptyset$, i.e. that the resultant of $F(0, y_1, y_2)$ and $G(0, y_1, y_2)$ is $\neq 0$ ($X \cap \{x_0 = y_1 = y_2 = 0\} = \emptyset$ is automatically satisfied). If you set $\alpha = \beta = 0$, X is smooth (and contained in $\mathbb{P}$) iff the two cubics F, G, and the line $\{y_0 = 0\}$, in $\mathbb{P}^2$, are smooth and have transversal intersections (and have no point of common intersection), as it is easy to verify.

Let U be the Zariski open set in the affine space of dimension 26, parametrizing the coefficients of α, β, F and G, such that for $(\alpha, \beta, F, G) \in U$ the corresponding X (given as in Lemma 2.1) is contained in $\mathbf{P}$ and has at most rational double points as singularities; let $p: \mathscr{X} \to U$ be this family of canonical models in $\mathbf{P}$.

Before going on, however, we mention some more results contained in [3] (the following being Theorem 3, ibidem).

PROPOSITION 2.2: *S is such that $\Phi: S \to \mathbf{P}^2$ is a Galois covering iff X admits canonical equations with $\alpha = \beta = 0$. The Galois group is $Z/2Z \oplus Z/2Z$, acting by the two involutions $z_3 \mapsto -z_3$, $z_4 \mapsto -z_4$ (their product being the involution $x_0 \mapsto -x_0$).*

THEOREM 2.2: *All the surfaces with $K^2 = p_g = 1$ are diffeomorphic, and simply connected.*

PROOF: By Theorem 1.7 every canonical model of such surfaces occurs in the family $\mathscr{X}$, whose base U is smooth and connected. By the results of [19] all the non singular models are deformation of each other, hence diffeomorphic. The other assertion (Proposition 13 of [3]) follows from the fact that the surfaces S for which Φ is a Galois cover are simply connected (Proposition 10, ibidem).

In order to come to the main result of this section, some more notations are needed. Let V be the Zariski open set, in the affine space of dimension 18 parametrizing the pairs of cubics (F, G) with $f_{001} = g_{002} = 1$ (where $F = \Sigma_{0 \le i \le j \le k \le 2} f_{ijk} y_i y_j y_k$, and similarly for G), such that for $(F, G) \in V$ the weighted complete intersection (w.c.i.) given by

$$\begin{cases} z_3^2 + x_0 z_4 y_1 + F(y) = 0 \\ z_4^2 + x_0 z_3 y_2 + G(y) = 0 \end{cases}$$

is smooth and $\subset \mathbf{P}$.

Denote by $\pi: \mathscr{S} \to V$ the smooth family that one thus obtains, by $\rho^*: V \to V$ the linear involution such that $\rho^*(F, G) = (G \circ \rho, F \circ \rho)$ (remember Corollary 1.13), whose fixed set has dimension 9. Remark further that if $\tilde{U}$ is the open set in U such that for $u = (\alpha, \beta, F, G) \in \tilde{U}$, X_u is smooth, $a_1 b_2 - a_2 b_1$ is $\neq 0$ and, after the change of variables of

Lemma 1.11, f_{001} and g_{002} are $\neq 0$, then for $u \in \tilde{U}$ exists $v \in V$ such that X_u is isomorphic to S_v.

THEOREM 2.3: *The coarse moduli variety M for the surfaces with $K^2 = p_g = 1$ is a rational variety of dimension* 18.

PROOF: First of all M exists and is a quasi-projective variety by Gieseker's theorem [7]. By the universal property of M there exist unique maps $U \xrightarrow{\varphi} M$, $V \xrightarrow{\psi} M$: we know that φ is onto, therefore M is irreducible, moreover, by the above remark $\psi(V) = \varphi(\tilde{U})$, ψ factors via $V \to V/\rho^* \xrightarrow{\tilde{\psi}} M$ and $\tilde{\psi}$ is injective. The conclusion is that M is birational to V/ρ^*, a rational variety.

3. Periods of surfaces S with $K^2 = p_g = 1$

In this section we adopt notations and terminology from [10], [8], to which we refer.

The Hodge structure of our surfaces is all in dimension 2, where $H^2(S, \mathbb{C}) = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$, once fixed a Hodge metric, and is completely determined by the 1-dimensional subspace $H^{2,0}$ (cf. [8]). If K_S is ample (i.e. the canonical model is smooth) there is a natural polarization on the cohomology, and here the classifying domain D for the polarized Hodge structure on the primitive cohomology is $\simeq \mathrm{SO}(2, 18, \mathbb{R})/U(1) \times \mathrm{SO}(18, \mathbb{R})$, its complex structure being given by its realization as an open set in a smooth quadric hypersurface in $\mathbb{P}^{19}$.

The global period mapping is defined on M', the Zariski open set in M parametrizing surfaces with K ample, and goes to the classifying space D/Γ, where Γ is $\mathrm{SO}(2, 18, \mathbb{Z})$, a group operating properly discontinuously on D [10], [8].

We want to show that this map is not injective.

We denote then by B the base of the local universal deformation of a surface S (see [13], [21]): its tangent space at the point $b \in B$ corresponding to S is naturally identified with $H^1(S, T_S)$ (T_S being the tangent sheaf of S). The local period mapping is defined on B and goes holomorphically to D, and (the tangent space to D at the image point of b being canonically identified to a subspace of $\mathrm{Hom}(H^0(S, \Omega^2_S), H^1(S, \Omega^1_S))$ its differential μ (by [8]) is obtained via the bilinear mapping $H^1(S, T_S) \times H^0(S, \Omega^2_S) \to H^1(S, T_S \otimes \Omega^2_S)$ and the natural isomorphism $\Omega^1_S \cong T_S \otimes \Omega^2_S$.

We quote, without reproducing the proof, Theorem 4 of [3].

THEOREM 3.1: *Suppose S is a smooth weighted complete intersection of type* (6, 6) *given by the vanishing of two canonical equations*

$$\begin{cases} f = z_3^2 + x_0 z_4 \left(\sum_{i=0}^{2} a_i y_i\right) + \sum_{0 \le i \le j \le k \le 2} f_{ijk} y_i y_j y_k, \\ g = z_4^2 + x_0 z_3 \left(\sum_{i=0}^{2} b_i y_i\right) + \sum_{0 \le i \le j \le k \le 2} g_{ijk} y_i y_j y_k. \end{cases}$$

Then the local universal deformation of S has a smooth base of dimension 18, and μ is injective iff for S is $\neq 0$ the determinant $\Delta(\alpha, \beta, F, G)$ of the following generally invertible matrix (actually more holds: dim ker μ = corank of the matrix)

a_1	0	0	$3f_{111}$	0	f_{112}
0	a_2	0	f_{122}	0	$3f_{222}$
a_2	a_1	0	$2f_{112}$	0	$2f_{122}$
b_1	0	$3g_{111}$	0	g_{112}	0
0	b_2	g_{122}	0	$3g_{222}$	0
b_2	b_1	$2g_{112}$	0	$2g_{122}$	0

As a corollary, we have

THEOREM 3.2: *$\pi : \mathcal{S} \to V$ is, for $v \in V$, the universal deformation of S_v.*

PROOF: S_v is smooth, hence by the previous theorem the base B of its universal deformation is smooth: but the dimensions of V, B, are the same and there is locally a unique map h of $V \to B$ inducing the family $\mathcal{S}$. Observe that h is injective if $\rho^*(v) \neq v$, and if $\rho^*(v) = v$ it has at most degree two: in the first case then h is obviously an isomorphism, in the second case if it were not a local isomorphism h would factor through $V \to V/\rho^*$; this is impossible however since V/ρ^* has a singular set of dimension 9, hence cannot go injectively to B (smooth of the same dimension).

THEOREM 3.3: *The global period mapping of surfaces with* $K^2 = p_g = 1$ *has degree at least* 2.

PROOF: If $v \in V$ is a point such that $\rho^*(v) \neq v$, $\Delta(v) = 0$, $\ker \mu$ has dimension 1 and is not tangent to $\{\Delta = 0\}$, then a small neighbourhood W of v goes (by ψ) injectively into the moduli variety M (by Theorem 2.3 and Corollary 1.13), while the local period mapping from W to D is a finite covering of degree at least 2 with $\{\Delta = 0\}$ as ramification locus, being a map between manifolds of the same dimension.

Let's show that such a point v exists, i.e. that Δ vanishes on V (then we are done, since its locus of zeros is a hypersurface, while the fixed locus of ρ^* has codimension 9, the locus of points where $\dim \ker \mu = 2$ has codimension 4, in particular is a proper subvariety of $\{\Delta = 0\}$), and that $\ker \mu$ is generally not tangent to $\{\Delta = 0\}$.

We have seen (Lemma 2.1) that for F, G general the points of U of the form $(0, 0, F, G)$, where Δ does vanish, correspond to smooth canonical models: so, if we restrict to the subspace U' of U where $\alpha = a_1y_1 + a_2y_2$, $\beta = b_1y_1 + b_2y_2$, by the remark preceding Theorem 2.3 it is enough to prove that every component of $\{\Delta = 0\}$ meets $\tilde{U} \cap U'$.

But the complement of $\tilde{U}$ in U' is given by the equations $a_1b_2 - a_2b_1 = b_2f_{001} - b_1f_{002} = a_1g_{002} - a_2g_{001} = 0$, so it suffices to prove that these polynomials do not divide Δ. For the first set $a_1 = a_2 = b_1 = b_2 = 1$, $f_{112} = g_{122} = 1$, and the other variables zero: then the value of Δ is 4. For the other two notice that the variables f_{001}, f_{002}, g_{001}, g_{002} do not appear in Δ.

Finally it is easy to verify that the restriction of Δ to the affine space of V is irreducible and that at the point $v = (F, G) = (y_1^3 + y_2^3 + y_0^2y_1, y_1^3 + y_0^2y_2)$ $\ker \mu$ is spanned by the tangent vector $\xi = (6y_0y_2^2 - y_0y_1^2, 5y_2^3 + 2y_1^3 + 2y_0^2y_1 + 2y_0^3)$, such that $\xi(\Delta) = 135$.

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