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VANISHING CRITERIA AND THE PICARD GROUP FOR PROJECTIVE VARIETIES OF LOW CODIMENSION

Robert Speiser

Introduction

Let X and Y be closed subschemes of \mathbb{P}^n , where X is irreducible and smooth, and Y is a local complete intersection (characteristic 0) or Cohen-Macaulay (characteristic $p > 0$). We shall be interested in two invariants: first, the cohomological dimension

$$\text{cd}(X - Y) = \sup \left\{ i \left| \begin{array}{l} H^i(X - Y, F) \neq 0 \\ \text{for some coherent} \\ \text{sheaf } F \text{ on } X \end{array} \right. \right\}$$

and second, the Picard group $\text{Pic}(X)$.

Let $s = \dim(X)$ and $t = \dim(Y)$. Our main result for $\text{cd}(X - Y)$ is (2.1), which generalizes some earlier vanishing criteria: we have

$$s + t > n \Rightarrow \text{cd}(X - Y) < s - 1.$$

In particular, we find

$$s + t > n \Rightarrow X \cap Y \text{ is connected,}$$

a result recently proved [6] by Fulton and Hansen, under weaker assumptions. Their method is based on specialization of cycles.

For $\text{Pic}(X)$, assume the characteristic is $p > 0$ and that $s = \dim(X) \geq \frac{1}{2}(n + 2)$. Then (3.1) the Picard group $\text{Pic}(X)$ is a finitely generated abelian group of rank 1, with no torsion prime to p . (The assertion

about the torsion is [5, Cor. 4.6], but the assertion about the rank, although expected, is new.) The analogue in characteristic 0, (compare, for example, Ogus [8]), has been known for some time.

Our proofs are mainly based on ideas used, for example, in [2], [9], [8], [5] and [11]. The approach to (2.1) is through a study of the cohomology of coherent sheaves on a formal completion, with the needed preliminary results in §1. For (3.1) we use Hartshorne's version of the Barth-Lefschetz theorem for l -adic cohomology.

Notation will be standard, except that all schemes will tacitly be assumed separated and of finite type over the spectrum of an algebraically closed field k , of arbitrary characteristic.

§1. Formal Neighborhoods in \mathbb{P}^n

Fix a closed subscheme $X \subset \mathbb{P}^n$, and denote by $\hat{\mathbb{P}}^n$ the formal completion of \mathbb{P}^n along X . For a coherent sheaf F on \mathbb{P}^n , write \hat{F} for the formal completion of F . In particular, the standard invertible sheaf $\mathcal{O}(\nu)$ on \mathbb{P}^n has completion $\hat{\mathcal{O}}(\nu)$.

For a coherent sheaf F on \mathbb{P}^n , the homological dimension $\text{hd}(F)$ is the maximum projective dimension $\dim \text{proj}(F_x)$ of a stalk of F , where x ranges over all scheme points $x \in \mathbb{P}^n$, and the projective dimension of F_x is taken over the local ring of \mathbb{P}^n at x .

For each integer i , define a k -vector space V^i as follows. In characteristic zero, let X_0 be $\mathbb{V}(\mathcal{O}_X(-1))$ minus the zero-section, where of course $\mathcal{O}_X(-1) = \mathcal{O}(-1)|_X$; then set

$$V^i = H_{DR}^i(X_0),$$

the algebraic de Rham cohomology group [13]. In characteristic $p > 0$, set

$$V^i = H^i(X, \mathcal{O}_X),$$

the stable part of $H^i(X, \mathcal{O}_X)$ under the action of the p^{th} -power endomorphism of \mathcal{O}_X .

THEOREM (1.1): *Let $X \subset \mathbb{P}^n$ be a closed subscheme. If the characteristic is zero, suppose X is a local complete intersection, or, if the characteristic is > 0 , that X is Cohen-Macaulay. Given any coherent sheaf F on \mathbb{P}^n there are, for all integers k , natural maps*

$$\beta^k : \bigoplus_{i+j=k} H^i(\mathbb{P}^n, F) \otimes_k V^j \rightarrow H^k(\hat{\mathbb{P}}^n, \hat{F})$$

which are bijective for $k < \dim(X) - \text{hd}(F)$, and injective for $k = \dim(X) - \text{hd}(F)$.

Let $S = k[X_0, \dots, X_n]$ be the homogeneous co-ordinate ring of \mathbb{P}^n . Then

$$M^i = \sum_{\nu \in \mathbb{Z}} H^i(\hat{\mathbb{P}}^n, \hat{\mathcal{O}}(\nu))$$

is a graded S -module under the cup product. With $F = \mathcal{O}(\nu)$ in (1.1), then, summing over ν and taking into account the standard results on the cohomology of invertible sheaves on \mathbb{P}^n , we obtain the following:

COROLLARY (1.2): *Let $X \subset \mathbb{P}^n$ be as in (1.1). We have a natural map of graded S -modules*

$$\beta^i : S \otimes_k V^i \rightarrow M^i$$

which is bijective for $i < \dim(X)$ and injective for $i = \dim(X)$.

PROOF OF (1.1): In characteristic zero, (1.1) is an immediate special case of Ogus' result [8, Th. 2.1, p. 1091]. In characteristic $p > 0$, (1.2) is [2, Cor. 6.6, p. 140], plus the Lemma of Enriques and Severi [2, Ex. 6.13, p. 143]. To deduce (1.1) from (1.2), one can repeat Ogus' argument, once one has the maps β^k , to which we now proceed.

Construction of the β^k

Here we work in the category of graded (S, F) -modules; [2, p. 127–143] contains the definitions and the foundational results we shall need. Since S is regular, the p^{th} -power endomorphism $F : S \rightarrow S$ is flat [loc. cit. Cor. 6.4, p. 138], hence the functor G [loc. cit., p. 132] is left exact.

LEMMA (1.3): *Let M be a graded (S, F) -module such that M_s is finite dimensional over k . Then:*

(a) *there is a natural injection of (S, F) -modules*

$$S \otimes_k (M_0)_s \rightarrow G(M);$$

(b) if $M_\nu = 0$ for $\nu \ll 0$, this map is a bijection.

PROOF: If $M_\nu = 0$ for $\nu \ll 0$, this is precisely [2, Theorem 6.1, p. 133]. If not, consider the functor acting on (S, F) -modules via

$$M \mapsto M^+ = \sum_{\nu \geq 0} M_\nu,$$

where the image M^+ is an (S, F) -module by restriction. We have

$$M_s = (M_0)_s = (M^+)_s,$$

and, since G is left exact, we have a natural inclusion $G(M^+) \subset G(M)$ induced by the inclusion of M^+ in M . Since (b) holds for M^+ , we obtain a composite injective morphism of functors of M :

$$S \otimes_k (M_0)_s = S \otimes_k (M^+)_s \cong G(M^+) \rightarrow G(M).$$

This proves (1.3).

We can now construct the β^k . By [2, Theorem 6.3, p. 135], we have a natural isomorphism

$$M^i \cong G\left(\sum_{\nu \in \mathbb{Z}} H^i(X, \mathcal{O}_X(\nu))\right),$$

hence an injection

$$S \otimes_k V^i = S \otimes_k H^i(X, \mathcal{O}_X)_s \rightarrow M^i$$

by (1.3)(a), since, plainly, $(\sum H^i(X, \mathcal{O}_X(\nu)))_s = H^i(X, \mathcal{O}_X)_s$. This injection restricts to the subspace $1 \otimes V^i$; hence in degree 0 we have an injection

$$V^i \xrightarrow{\alpha} H^i(\hat{P}^n, \hat{\mathcal{O}}_{P^n}) \subset M^i.$$

Finally, composing α with the cup product

$$H^i(P^n, F) \otimes_k H^j(\hat{P}^n, \hat{\mathcal{O}}_{P^n}) \rightarrow H^{i+j}(\hat{P}^n, \hat{F}),$$

we obtain β^k , and this establishes (1.1).

REMARK: Unfortunately for us, [2] only treats the case $F = \mathcal{O}_{\mathbb{P}^n}(\nu)$; we shall need general coherent F , however, in order to prove (2.1) below.

§2. Vanishing Criteria

We shall be concerned for the rest of this section with the following situation: X and Y will be *connected* closed subschemes of \mathbb{P}^n , with $s = \dim(X)$ and $t = \dim(Y)$. We shall assume X is smooth and that Y is a local complete intersection if the characteristic is zero, or, if the characteristic is > 0 , that Y is Cohen-Macaulay. We want bounds on the cohomological dimension $\text{cd}(X - Y)$.

By Lichtenbaum's Theorem (Cf. [7] or [2, Cor. (3.3), p. 98]), $\text{cd}(X - Y) < \dim(X) = s$ if and only if $Y \cap X = \emptyset$; for lower cohomological dimensions the situation is more complicated.

Here is our main result:

THEOREM (2.1): *With closed subschemes X and Y of \mathbb{P}^n as above, suppose $s + t > n$. Then we have*

$$\text{cd}(X - Y) < s - 1;$$

in particular, $X \cap Y$ is connected.

COROLLARY (2.2) (A weak form of Hartshorne's Second Vanishing Theorem [3, Theorem 7.5, p. 444]): *Let $Y \subset \mathbb{P}^n$ be a positive dimensional closed subscheme satisfying the hypotheses of (2.3). Then*

$$\text{cd}(\mathbb{P}^n - Y) < n - 1.$$

COROLLARY (2.3) (Compare [10, Theorem D, p. 179]): *Let $X \subset \mathbb{P}^n$ be a smooth hypersurface, $Y \subset X$ a closed subscheme as in (2.3), of dimension $t > 1$. Then*

$$\text{cd}(X - Y) < n - 2.$$

The corollaries are immediate consequences of (2.1). To prove (2.1), we shall need the following result.

LEMMA (2.4) (Hartshorne [2, Theorem 3.4, p. 96]): *Let X be a smooth projective variety, $Y \subset X$ a closed subset. The following are equivalent:*

- (a) $\text{cd}(X - Y) \leq a$
 (b) *the natural map*

$$\alpha^i: H^i(X, F) \rightarrow H^i(\hat{H}, \hat{F})$$

is bijective for $i < \dim(X) - a - 1$ and injective for $i = \dim(X) - a - 1$, for all locally free sheaves F on X .

PROOF OF (2.1): Let F be a locally free sheaf on X . Hence $\text{hd}(F) = \text{hd}(\mathcal{O}_X) = n - s$. Denote by “ $\hat{}$ ” the operation of formal completion along Y . Applying (1.1) to the t -dimensional subscheme $Y \subset \mathbb{P}^n$, we find that the natural maps

$$\beta^k: \bigoplus_{i+j=k} H^i(\mathbb{P}^n, F) \otimes_k V^j \rightarrow H^k(\hat{\mathbb{P}}^n, \hat{F})$$

are bijective for $k = 0$ and injective for $k = 1$. Hence, since Y is connected, $V^0 = \underline{k}$. Thus the bijection β^0 reduces to the natural map

$$\alpha^0: H^0(X, F) \rightarrow H^0(\hat{X}, \hat{F});$$

indeed, X is the support of F , so we can replace $\hat{\mathbb{P}}^n$ with \hat{X} . Similarly the injection β^1 induces

$$\alpha^1: H^1(X, F) \rightarrow H^1(\hat{X}, \hat{F})$$

on the summand corresponding to $i = 0, j = 1$. Since α^0 is bijective and α^1 is injective, our assertion about $\text{cd}(X - Y)$ follows from (2.4). Finally, to see that $X \cap Y$ is connected, one applies [2, Cor. 3.9, p. 101].

EXAMPLE: Let $X = \mathbb{P}^m \times \mathbb{P}^1$, $Y = \mathbb{P}^m \times \{P\}$ for a closed point $P \in \mathbb{P}^1$. By [12, (1.3)], $\text{cd}(\mathbb{P}^m \times \mathbb{P}^1 - Y) = m$. (Since $Y \neq \emptyset$, we can't have $\text{cd} = m + 1$, by Lichtenbaum's Theorem.) We therefore obtain the bound

$$n \geq 2m + 1$$

for any \mathbb{P}^n containing $\mathbb{P}^m \times \mathbb{P}^1$. Now any smooth projective variety of dimension $m + 1$ can be projected isomorphically into $\mathbb{P}^{2(m+1)+1} = \mathbb{P}^{2m+3}$; Hartshorne, however, shows [4, pp. 1025–1026] that $\mathbb{P}^m \times \mathbb{P}^1$ can be projected isomorphically into \mathbb{P}^{2m+1} , two dimensions less. In other

words, the inequality of (2.1) actually gives the embedding dimension of $\mathbb{P}^m \times \mathbb{P}^1$, hence is best possible.

REMARK: Related techniques (compare [5], [8], [9]) give other, perhaps better known, results. For example, with $F = \mathcal{O}_X(1)$ in (1.1), we find

$$V^j = \begin{cases} k & \text{if } j = 0 \\ 0 & \text{if } 0 < j \leq 2 \dim(X) - n. \end{cases}$$

Then a straightforward inspection of the β^k , with F locally free on X , gives the inequality

$$(2.5) \quad \text{cd}(X - Y) < n + s - t - \inf(s, t).$$

With $s \leq t$, we obtain $\text{cd}(X - Y) < n - t$. On the other hand, with $t \leq s$ (e.g., if $Y \subset X$), we find $\text{cd}(X - Y) < n + s - 2t$.

Taking $X = \mathbb{P}^n$, the last inequality gives

$$\text{cd}(\mathbb{P}^n - Y) < 2n - 2t,$$

a result due originally to Barth [1, §7, Cor. of Th. III] in the complex case.

§3. The Main Result for $\text{Pic}(X)$

THEOREM (3.1): *Let $X \subset \mathbb{P}^n$ be a smooth closed subscheme of dimension s , over the spectrum of an algebraically closed field of characteristic $p > 0$. If $s \geq \frac{1}{2}(n + 2)$, then $\text{Pic}(X)$ is a finitely generated abelian group of rank 1, with no torsion prime to p .*

PROOF: $\text{Pic}(\mathbb{P}^n) \rightarrow \text{Pic}(X)$ is injective; if not, $\mathcal{O}_{\mathbb{P}^n}(d)|_X$ would be trivial for some $d > 0$, but $\mathcal{O}_{\mathbb{P}^n}(d)|_X$ is very ample. By [5, Cor. 4.6, p. 74], $\text{Pic}(X)$ has no torsion prime to p , so $\underline{\text{Pic}}^0(X)_{\text{red}}$, an abelian variety, is trivial. Hence

$$\text{Pic}(X) = \underline{\text{Pic}}(X)/\underline{\text{Pic}}^0(X) = \text{NS}(X),$$

a finitely generated abelian group, by the Theorem of the Base.

For the rest of the proof, let l be a prime different from p , and consider the l -adic étale cohomology of X . Then, by [4, Remark 2, p.

1021], the natural maps

$$H^i(\mathbb{P}_{\underline{ét}}^n, \mathbb{Q}_l) \xrightarrow{\gamma^i} H^i(X_{\underline{ét}}, \mathbb{Q}_l)$$

are bijective for $i \leq 2s - n$. In particular, γ^2 is bijective. Recall that, as functors, we have

$$H^i(*_{\underline{ét}}, \mathbb{Q}_l) = H^i(*_{\underline{ét}}, Z_l) \otimes_{Z_l} \mathbb{Q}_l,$$

where

$$H^i(*_{\underline{ét}}, Z_l) = \lim_{\rightarrow r} H^i(*_{\underline{ét}}, Z/l^r Z).$$

Hence, bijectivity of γ^2 implies that $H^2(X_{\underline{ét}}, Z_l)$ has rank 1.

Since the base field is algebraically closed, we can make a non-canonical identification $\mu_{l^r} = Z/l^r Z$. Then the Kummer sequence reads

$$0 \rightarrow Z/l^r Z \rightarrow G_m \xrightarrow{l^r} G_m \rightarrow 1.$$

Passing to cohomology, we obtain the exact sequence

$$H^1(X_{\underline{ét}}, G_m) \xrightarrow{l^r} H^1(X_{\underline{ét}}, G_m) \rightarrow H^2(X_{\underline{ét}}, Z/l^r Z).$$

Since $H^1(X_{\underline{ét}}, G_m) = \text{Pic}(X)$ we have a natural inclusion

$$\text{Pic}(X)/l^r \text{Pic}(X) \hookrightarrow H^2(X_{\underline{ét}}, Z/l^r Z)$$

compatible with the reduction maps on both sides as r varies. Letting $r \rightarrow \infty$, we find

$$\text{rank}(\text{Pic}(X)) \leq \text{rank}(H^2(X_{\underline{ét}}, Z_l)) = 1,$$

and this completes the proof.

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