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## EACH SCHWARTZ FRÉCHET SPACE IS A SUBSPACE OF A SCHWARTZ FRÉCHET SPACE WITH AN UNCONDITIONAL BASIS

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### Abstract

It is shown that for  $1 \leq p \leq \infty$ , {Schwartz Fréchet spaces embeddable in a product of  $\ell_p$ -spaces} = {subspaces of Schwartz Fréchet Köthe  $\ell_p$ -sequence spaces}. As a corollary, we obtain the fact that every Schwartz Fréchet space is a subspace of Schwartz Fréchet space with an unconditional basis. Furthermore, the Schwartz variety has a universal generator which is separable, barreled and bornological. These results are shown to be best possible.

It is well known that there are Banach spaces which are not isomorphic to any subspace of a Banach space with an unconditional basis. The author, in [4], has produced examples of Fréchet Montel spaces which are not isomorphic to any subspace of a Fréchet space (Montel or otherwise) with an unconditional basis. On the other hand, each nuclear Fréchet space is isomorphic to a subspace of  $s^N$  (a countable product of spaces isomorphic to the space of rapidly decreasing sequences) which is a nuclear Fréchet and it has an unconditional basis [10]. As the title indicates, the results of this paper show how to embed any Schwartz Fréchet space into a Schwartz Fréchet space with an unconditional basis.

The Theorem (below) actually yields more. For  $1 \leq p \leq \infty$ , let  $S_p = \{\text{Schwartz spaces that are isomorphic to a subspace of a product of } \ell_p\text{-spaces}\}$  and let  $\mathcal{B}_p$  be the set of Fréchet spaces in  $S_p$ . Then each space in  $\mathcal{B}_p$  is isomorphic to a subspace of some Schwartz Fréchet Köthe  $\ell_p$ -sequence space. There are two weaker results in the literature of this kind. Randtke [13] has shown that every space in  $\mathcal{B}_\infty$  is isomorphic to a subspace of a compact projective limit of a sequence

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of spaces, each isomorphic to  $c_0$ , and in [2] there is a similar result for  $\mathcal{B}_2$ . Both of these results depend on special properties of  $c_0$  or  $\ell_2$ , and their proofs do not generalize to  $\mathcal{B}_p$ . The Theorem is a much stronger result whose proof is surprisingly easy.

As a corollary, we obtain a separable barreled and bornological universal generator for  $S_p$ . The Proposition (below) shows that this is best possible, in the sense that there are no Fréchet universal generators for  $S_p$ . This generalizes a result of Moscatelli [9], who proved it for  $p = \infty$ . Moreover, the proof below avoids the use of bornologies in [9] by the use of approximate dimension rather than diametral dimension (see [10]). The previously known universal generators for  $S_p$  (see [6], [12], [13], [9], [2] and [3]) are not even  $\sigma$ -barreled (= bounded sequences in the dual are equicontinuous).

## 1. Terminology

A LCS is a locally convex topological vector space which is Hausdorff. If  $\rho$  is a continuous semi-norm on the LCS  $E$ , then  $E_\rho$  is the normed space  $(E/\ker \rho, \rho)$ . The LCS  $E$  is *Schwartz* (respectively, *nuclear*) if for each continuous semi-norm  $\rho$ , there is a continuous semi-norm  $\mu$ , with  $\rho \leq \mu$  and so that the identity on  $E$  induces a precompact (respectively nuclear) map  $E_\mu \rightarrow E_\rho$ . A *Fréchet* space is a complete metrizable LCS. A Fréchet space is *Montel* if each closed bounded set is compact. For Fréchet spaces, nuclear implies Schwartz implies Montel, but neither implication can be reversed.

Let  $N$  denote the set of natural numbers. For  $1 \leq p \leq \infty$ ,  $\ell_p$  denotes the Banach space of all  $p$ -sumable scalar sequences  $(\alpha_n)$  with norm  $\|(\alpha_n)\| = [\sum |\alpha_n|^p]^{1/p}$ . We will use  $\ell_\infty$  to denote the Banach space of bounded scalar sequences  $(\alpha_n)$  with norm  $\|(\alpha_n)\| = \sup |\alpha_n|$  and  $c_0$  will be its subspace of null-scalar sequences.

A *prevariety* [1] is a collection of LCS's (locally convex spaces) closed with respect to the formation of subspaces and arbitrary products. A *variety* [5] is a prevariety which is also closed under the formation of quotients by closed subspaces. The collections  $S_p$ ,  $1 \leq p \leq \infty$  are prevarieties [3], while the collections (nuclear spaces), {Schwartz spaces} =  $S_\infty$ , and  $S_2$  are varieties [5], [2]. A LCS  $X$  is a *universal generator* for the prevariety  $\mathcal{P}$  if  $x \in \mathcal{P}$  and each  $Y \in \mathcal{P}$  is isomorphic to a subspace of  $X^I$ , for some index set  $I$ .

Suppose  $1 \leq p \leq \infty$  and let  $[a_n^k]_{k,n=1}^\infty$  be a matrix of non-negative reals. Define for each  $k$  and scalar sequence  $(\alpha_n)$ ,  $\|(\alpha_n)\|_k = \|\sum_n a_n^k \alpha_n\|_p$ , where  $\|\cdot\|_p$  is the  $\ell_p$ -norm. The set of sequences  $(\alpha_n)$ , with  $\|(\alpha_n)\|_k < \infty$ ,

for each  $k$  along with the semi-norms  $\{\|\cdot\|_k\}$  defines a Fréchet space which is called a Köthe  $\ell_p$ -sequence space [8, p. 419]. Let  $1 \leq p \leq \infty$  and  $[a_n^k]$  be as above with  $a_n^{k+1} \geq a_n^k$  and  $Y$  be the resulting Köthe  $\ell_p$ -sequence space. It is well known that  $Y$  is Schwartz if and only if, for each  $k$ , there is a  $j$ , with  $\lim_n(a_n^k/a_n^j) = 0$ . (Here  $0/0 = 0$ .)

Finally we note that we use the notion of approximate dimension of LCS's without definition. The definition and its basic properties may be found in Pietsch [10].

### 2. Results

**THEOREM:** *Each Fréchet space in  $S_p$  is isomorphic to a subspace of a Schwartz Fréchet Köthe  $\ell_p$ -sequence space.*

**PROOF:** Let  $X \in \mathcal{B}_p$ . Since  $\ell_p$  is isomorphic to its square, there is a neighborhood basis  $\{U(n)\}$  of the origin in  $X$  with the following properties:

- (1) For each  $n$ ,  $X_n$  (the quotient space  $X/\ker \rho_{U(n)}$  with norm  $\rho_{U(n)}$ ) is isomorphic to a subspace of  $\ell_p$ , say by the map  $i_n: X_n \rightarrow \ell_p$ .
- (2) For each  $n$ , the canonical map  $j_n: X_{n+1} \rightarrow X_n$  is precompact.
- (3)  $X$  is isomorphic to a subspace of the projective limit of

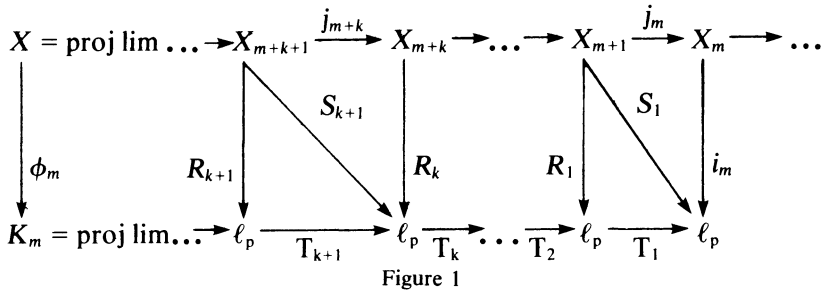
$$\cdots \rightarrow X_{n+1} \xrightarrow{j_n} X_n \rightarrow \cdots \rightarrow X_1.$$

For each  $m$ , we construct a Köthe  $\ell_p$ -sequence space  $K_m$  as follows. We define  $R_k, S_k, T_k$  by induction. Let  $S_1: X_{m+1} \rightarrow \ell_p$  be the composite  $i_m j_m$ . Since  $S_1$  is compact, by Lemma 2.1 of [3] there are compact maps  $R_1: X_{m+1} \rightarrow \ell_p$  and  $T_1: \ell_p \rightarrow \ell_p$  with  $S_1 = T_1 R_1$  and  $T_1$  is a diagonal map. Suppose  $R_k: X_{m+k} \rightarrow \ell_p$ ,  $S_k: X_{m+k} \rightarrow \ell_p$  and  $T_k: \ell_p \rightarrow \ell_p$  have been chosen. Let  $S_{k+1} = R_k j_{m+k+1}$ , and again factor  $S_{k+1} = R_{k+1} T_{k+1}$  so that  $T_{k+1}$  is a diagonal compact map:  $\ell_p \rightarrow \ell_p$  and  $R_{k+1}: X_{m+k+1} \rightarrow \ell_p$  is compact. Let  $K_m$  be the projective limit of

$$\cdots \rightarrow \ell_p \xrightarrow{T_k} \ell_p \rightarrow \cdots \xrightarrow{T_1} \ell_p$$

and let  $\phi_m$  be the induced map:  $X \rightarrow K_m$  (see Figure 1).

By construction,  $K_m$  is a Schwartz Fréchet Köthe  $\ell_p$ -sequence space, the map  $\phi_m: X \rightarrow K_m$  is continuous, and  $\phi_m(U(m))$  is relatively open in  $K_m$ . Thus the map  $\phi: X \rightarrow \prod_{m=1}^\infty K_m$  is an isomorphism into if



$\phi(x) = (\phi_1(x), \phi_2(x), \dots)$ . It follows from [8, p. 409] that this product space is a Köthe  $\ell_p$ -sequence space and clearly it is Schwartz and Fréchet.

REMARKS 1: If  $X$  is a Banach space with a f.d.d. and  $X \oplus X$  is isomorphic to  $X$ , then there is an analogous theorem between  $S_X$  and the Schwartz Fréchet sequence spaces modeled on  $X$  (see [3]).

2. Let  $D$  and  $D'$  be compact diagonal maps:  $\ell_p \rightarrow \ell_p$ , if  $e_n$  is the usual basis for  $\ell_p$ , we will say  $D$  is weaker than  $D'$  if  $D'e_n \neq 0 \Rightarrow De_n \neq 0$  and  $\|De_n\|/\|D'e_n\| \rightarrow \infty$  as  $n \rightarrow \infty$ . The proof of Theorem remains valid if we replace  $T_k$  (any or all) by weaker compact diagonal maps:  $\ell_p \rightarrow \ell_p$ .

It is well known that there is no countable cofinal subset in this ordering. Hence there is a  $D: \ell_p \rightarrow \ell_p$  which is weaker than any of the  $T_k$ 's. Let  $K_p(D)$  be the resulting Köthe sequence space that is obtained in the Theorem with  $D$  replacing all the  $T_k$ 's. We have:

COROLLARY 1: For each  $E \in \mathcal{B}_p$ , there is a diagonal compact map  $D: \ell_p \rightarrow \ell_p$ , so that  $E$  is isomorphic to a subspace of  $K_p(D)$ .

COROLLARY 2: Each Schwartz Fréchet space is a subspace of a Schwartz Fréchet space with an unconditional basis.

PROOF: Schwartz-Köthe  $\ell_p$ -sequence spaces have an unconditional basis.

COROLLARY 3: Let  $1 \leq p \leq \infty$ , then  $S_p$  has a universal generator which is separable barreled and bornological.

PROOF: Let  $X \in S_p$ . Since  $S_p$  is an operator defined prevariety [1] ( $X$  is  $\mathcal{F}$ -space in the sense of [11] with  $\mathcal{F}$  = compact maps which factor through a subspace of  $\ell_p$ ) and by a Proposition 3.1 of [1],  $X$  is isomorphic to a subspace of a product of Fréchet spaces in  $S_p$ . Thus

by Corollary 1,  $X$  is a subspace of a product of  $U_p = \prod\{K_p(D) : D \text{ diagonal compact: } \ell_p \rightarrow \ell_p\}$ , and  $U_p$  is a universal generator for  $S_p$ .

Each  $K_p(D)$  is separable, since it is a Fréchet Montel space, which is separable by [8, p. 370]. The cardinality of  $(D : \ell_p \rightarrow \ell_p \text{ diagonal compact}) = c = \text{cardinality of the Reals}$ , since  $\text{card}(c_0) = c$ . Hence by [7, p. 32],  $U_p$  is separable and is easily seen to be barreled and bornological.

The Proposition below shows that Corollary 3 cannot be improved to produce a Fréchet universal generator. By Theorem 5.4 of [3],  $S_\infty \supset S_p \supset S_2$  for  $1 \leq p \leq \infty$ .

**PROPOSITION:** *Let  $\mathcal{P}$  be any prevariety satisfying  $S \supset \mathcal{P} \supset S_2$ , then  $\mathcal{P}$  has no Fréchet universal generator.*

**PROOF:** The proof requires the notion of approximate dimension. We refer the reader to [10, pp. 160–165] for notation and definitions.

Suppose  $E$  is a Fréchet universal generator for  $\mathcal{P}$ . Then  $F = E^N$  is a Fréchet space in  $\mathcal{P}$ , which contains every Fréchet space in  $\mathcal{P}$  as a subspace. Let  $\{U(m)\}$  be a basis of neighborhoods of the origin of  $F$  with  $U(m+1)$  precompact with respect to the seminorm  $\rho_{U(m)}$ . Let  $\phi_n(\epsilon) = M_\epsilon(U(n+1), U(n))$ , we note that  $\phi_n(\epsilon)$  is defined for  $\epsilon > 0$  and  $\phi_n(\epsilon) \rightarrow \infty$  as  $\epsilon \rightarrow 0$ . There is a  $\phi(\epsilon)$ , defined on  $(0, \infty)$ , so that  $\phi^{-1}(\epsilon)\phi_n(\epsilon) \rightarrow 0$  as  $\epsilon \rightarrow 0$ , for  $n = 1, 2, \dots$  and  $\phi(\epsilon) \rightarrow \infty$  as  $\epsilon \rightarrow 0$ . ( $\phi(\epsilon) = \epsilon^{-1} \sup\{\phi_m(\epsilon) : m \leq n\}$  for  $\epsilon \in (1/n + 1, 1/n)$ .) Hence  $\phi(\epsilon) \in \Phi_U(F)$ , and by Proposition 1 of [10, p. 165],  $\phi(\epsilon) \in \Phi_U(X)$  for each Fréchet space  $X \in \mathcal{P}$ . We complete the proof by constructing a Fréchet space  $X$  in  $S_2$  with  $\phi(\epsilon) \notin \Phi_U(X)$  obtaining a contradiction.

Let  $N_0 = 0$  and inductively choose  $N_n > N_{n-1}$  so that  $N_n > \phi(1/n^2)$ . Let  $\lambda_k = 1/n$  for  $N_{n-1} < k \leq N_n$  and let  $T : \ell_p \rightarrow \ell_p$  be the diagonal map which sends  $e_n$  to  $\lambda_n e_n$ . Let  $\delta = 1/m$ ,  $n \geq m$  and  $U = \text{unit ball of } \ell_2$ . It is easy to see that  $M_{1/n^2}(T(\delta U), U) \geq N_n = K$  since  $\delta \lambda_K \geq (1/m)(1/n) \geq 1/n^2$ . Thus  $\phi^{-1}(\epsilon)M_\epsilon(T(\delta U), U) \not\rightarrow 0$  as  $\epsilon \rightarrow 0$ . Let  $\mu^k = (\mu_n^k)$  be the sequences with  $\mu_n^k > 0$  and  $(\mu_n^k)^{2k} = \lambda_n$  and let  $T_k : \ell_2 \rightarrow \ell_2$  be the diagonal map which sends  $e_n$  to  $\mu_n^k e_n$ . Let  $X \in S_2$  be the projective limit:

$$\begin{array}{ccccc}
 \ell_2 & \xrightarrow{\theta} & X = \text{proj lim} \dots & \xrightarrow{T_2} & \ell_2 & \xrightarrow{T_1} & \ell_2 \\
 \downarrow \text{id} & & & & & & \downarrow \text{id} \\
 \ell_2 & & & \xrightarrow{T} & & & \ell_2
 \end{array}$$

and let  $\theta: \ell_2 \xrightarrow{\theta} X$  that makes Figure 2 a commutative diagram. To see that  $\phi(\epsilon) \notin \Phi_U(X)$ , let  $U =$  unit ball in the  $\ell_2$  in Figure 2 which is the range of  $T$ , and suppose  $V$  is a neighborhood of  $X$ . There is  $\delta = 1/m$  so that  $\theta(\delta U) \subset V$  and thus  $\phi^{-1}(\epsilon)M_\epsilon(V, U) \geq \phi^{-1}(\epsilon)M_\epsilon \phi^{-1}(\epsilon)M_\epsilon(T(\delta U), U) \not\rightarrow 0$  as  $\epsilon \rightarrow 0$ . This completes the proof.

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