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COHOMOLOGICALLY INSIGNIFICANT DEGENERATIONS

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Introduction

The following two problems in singularity theory appear to be closely related. On the one hand, given a complete singular variety X over $\mathbb{C}$, to construct a filtered complex of sheaves $(\underline{\Omega}^{\cdot}_X, F)$ on X, which computes the Hodge filtration on the cohomology of X (see the next section for a more precise statement). This problem has been treated by Philippe du Bois [1]. On the other hand one can ask, for which flat map germs $f: (\mathscr{X}, X) \to (x, 0)$ with $f^{-1}(0) = X$, the Hodge numbers h^{pq}_n of $H^*(X)$ and the limit Hodge structure on $H^*(\mathscr{X}_\infty)$ (cf. [5, 7]) are equal for all $p, q, n, \geq 0$ with $pq = 0$. If this is the case, such a degeneration is called cohomologically insignificant. The preceding paper [4] of Igor Dolgachev contains many results on these.

We prove the following local criterion:

THEOREM 2: *Suppose X is a complete algebraic variety over $\mathbb{C}$ such that $\mathcal{O}_X \cong \underline{\Omega}^0_X$. Then every proper and flat degeneration f over the unit disk S with $f^{-1}(0) = X$ is cohomologically insignificant.*

EXAMPLE: If in a degeneration of curves, X is a multiple elliptic fibre, then X is cohomologically insignificant, but $\mathcal{O}_X \not\cong \underline{\Omega}^0_X$. See [4], Theorem (3.10).

In [4], Igor Dolgachev conjectures, that every family over the disk, whose singular fibre is reduced and has only insignificant limit singularities in the sense of Mumford and Shah (cf. [6]), is cohomologically insignificant.

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QUESTION: Suppose X is an algebraic variety over $\mathbb{C}$ which has only insignificant limit singularities. Is it true that $\mathcal{O}_X \cong \underline{\Omega}^0_X$?

Using Theorem 3 one checks easily that this the case for those from the list of J. Shah [6].

The filtered De Rham complex of a singular variety

According to Du Bois [1], for every algebraic variety X over $\mathbb{C}$ there exists a complex $\underline{\Omega}^{\cdot}_X$ of analytic sheaves on S, whose differentials are first order differential operators, together with a decreasing filtration F on it, such that the following properties are satisfied:

(i) the complex $\underline{\Omega}^{\cdot}_X$ is a resolution of the constant sheaf $\mathbb{C}$ on X;
(ii) the differential in the graded complex $Gr_F\underline{\Omega}^{\cdot}_X$ is $\mathcal{O}_X$ linear;
(iii) the pair $(\underline{\Omega}^{\cdot}_X, F)$ is functorial in X (in a suitable derived category);
(iv) there exists a natural morphism of filtered complexes

$$\lambda : (\Omega^{\cdot}_X, \sigma) \to (\underline{\Omega}^{\cdot}_X, F)$$

where $\Omega^{\cdot}_X$ is the holomorphic De Rham complex and σ its "filtration bête" (cf. [2], Definition (1.4.7)); if X is smooth then λ is a filtered quasi-isomorphism.
(v) if X is complete, then the spectral sequence

$$E_1^{pq} = \mathrm{H}^{p+q}(X, Gr_F^p\underline{\Omega}^{\cdot}_X) \Rightarrow H^{p+q}(X, \mathbb{C})$$

degenerates at E_1 and abuts to the Hodge filtration of $H^*(X, \mathbb{C})$, which carries Deligne's mixed Hodge structure (cf. [3]).
Let $\underline{\Omega}^0_X$ denote the complex $Gr_F^0\underline{\Omega}^{\cdot}_X$.

THEOREM 1: *Let $f: X \to S$ be a proper and flat morphism of complex algebraic varieties. For $s \in S$, let X_s denote the fibre $f^{-1}(s)$ over s. If for all $s \in S$ the map*

$$Gr_F^0(\lambda) : \mathcal{O}_{X_s} \to \underline{\Omega}^0_{X_s}$$

is a quasi-isomorphism, then for all $i \geq 0$ the sheaf $R^if_*\mathcal{O}_X$ is locally *free on S and for all $s \in S$ the natural map*

$$R^i f_* \mathcal{O}_X \otimes_{\mathcal{O}_S} | \mathrm{k}(s) \to H^i(X_s, \mathcal{O}_{X_s})$$

is an isomorphism.
Cf. [1], *Théorème* 4.6.

If X is a complete algebraic variety, let us denote

$$h_n^{pq}(X) = \dim_{\mathbb{C}} Gr_F^p Gr_{p+q}^W H^n(X, \mathbb{C});$$

the numbers h_n^{pq} are the Hodge numbers of $H^n(X, \mathbb{C})$.
Then one clearly has

$$\sum_{q \geq 0} h_n^{pq}(X) = \dim_{\mathbb{C}} Gr_F^p H^n(X, \mathbb{C})$$

for all p, $n \geq 0$. Hence if X is complete and $\mathcal{O}_X \cong \underline{\Omega}_X^0$, then in view of property (v) one obtains

$$\dim_{\mathbb{C}} H^n(X, \mathcal{O}_X) = \sum_{q \geq 0} h_n^{0,q}(X) = \sum_{q \geq 0} h_n^{q,0}(X).$$

In the next theorem we consider degenerations with singular fibre X, that is flat projective mappings $f: \mathscr{X} \to S$ where $\mathscr{X}$ is a complex space, S is the unit disk in the complex plane and f is smooth over the punctured disk $S^* = S \backslash \{0\}$, and $X = f^{-1}(0)$.

Let H denote the universal covering of S^*, i.e. the upper half plane, and let X_∞ denote the family $\mathscr{X} x_S H$ over H. We endow $H^*(X_\infty)$ with the limit Hodge structure (cf. [5], [7]). One has a natural map

$$sp: H^*(X) \to H^*(X_\infty)$$

which is a morphism of mixed Hodge structures.

THEOREM 2: *Let $f: \mathscr{X} \to S$ be a degeneration with singular fibre X, satisfying $\mathcal{O}_X \cong \underline{\Omega}_X^0$. Then for all $n \geq 0$:*

$$Grf_F^0(sp): Gr_F^0 H^n(X) \tilde{\to} Gr_F^0 H^n(X_\infty).$$

In other words: f is a cohomologically insignificant degeneration.

PROOF: As X is a deformation retract of $\mathscr{X}$, the map

$$(R^n f_* \mathbb{C}_{\mathscr{X}})_0 \to H^n(X, \mathbb{C})$$

is an isomorphism for all $n \geq 0$. Because $\mathcal{O}_X \cong \underline{\Omega}^0_X$ and X is complete, the map

$$H^n(X, \mathbb{C}) \to H^n(X, \mathcal{O}_X)$$

is surjective. Hence there exist sections $\sigma_1, \ldots, \sigma_h$ of $R^n f_* \mathbb{C}_{\mathscr{X}}$ over S such that their images in $H^n(X, \mathcal{O}_X)$ form a basis. Let $\bar{\sigma}_i$ denote the image of σ_i under the natural map

$$R^n f_* \mathbb{C}_{\mathscr{X}} \to R^n f_* \mathcal{O}_{\mathscr{X}}.$$

Because $R^n f_* \mathcal{O}_{\mathscr{X}}$ is locally free, the sections $\bar{\sigma}_1, \ldots, \bar{\sigma}_h$ give a basis on some small neighborhood of 0 in S. This means, that the map

$$Gr^0_F H^n(X, \mathbb{C}) \to Gr^0_F H^n(X, \mathbb{C})$$

is an isomorphism for $|t|$ sufficiently small. In particular the images of $\sigma_1, \ldots, \sigma_h$ in $H^n(X_\infty, \mathbb{C})$ are linearly independent; because morphisms of mixed Hodge structures are strictly compatible with the Hodge filtrations, the images of $\sigma_1, \ldots, \sigma_h$ in $Gr^0_F H^n(X_\infty, \mathbb{C})$ are also linearly independent. Moreover the fact that $R^n f_* \mathcal{O}_{\mathscr{X}}$ is locally free implies that for $t \neq 0$:

$$\begin{aligned} \dim_{\mathbb{C}} Gr^0_F H^n(X, \mathbb{C}) &= \dim_{\mathbb{C}} H^n(X, \mathcal{O}_X) \\ &= \dim_{\mathbb{C}} H^n(X_t, \mathcal{O}_{X_t}) = \dim_{\mathbb{C}} Gr^0_F H^n(X_t, \mathbb{C}) \\ &= \dim_{\mathbb{C}} Gr^0_F H^n(X_\infty, \mathbb{C}). \end{aligned}$$

Hence $Gr^0_F(sp)$ is an isomorphism.

Examples where $\mathcal{O}_X \cong \underline{\Omega}^0_X$.

(a) If X is a reduced curve, then $\mathcal{O}_X \cong \underline{\Omega}^0_X$ if and only if at every singular point of X the branches are smooth and their tangent directions are independent. If X lies on a smooth surface, it can only have ordinary double points; more generally, if X has embedding dimension n at $x \in X$, then

$$\hat{\mathcal{O}}_{X,x} \cong \mathbb{C}[[z_1, \ldots, z_n]]/(z_i z_j : i \neq j).$$

See [1], Proposition 4.9.

(b) Suppose X is a normal surface, $\pi: \tilde{X} \to X$ a resolution of its singularities, $E_x = \pi^{-1}(x)_{\text{red.}}$ for $x \in X$. Then $\mathcal{O}_X \cong \underline{\Omega}^0_X$ if and only if $(R^1\pi_*\mathcal{O}_{\tilde{X}})_x \cong H^1(E_x, \mathcal{O}_{E_x})$ for all $x \in \text{Sing}(X)$. See [1], Proposition 4.13 and its proof.

Hence if X has embedding dimension three, its singularities can only be rational double points, simple-elliptic or cusp singularities. See [4], Corollary 4.11.

(c) If X has only quotient singularities, then $\mathcal{O}_X \cong \underline{\Omega}^0_X$. See [1], Théorème (5.3).

(d) Suppose X is a complex variety, $p : \tilde{X} \to X$ its normalisation, $\mathscr{C} = \text{Ann}_{\mathcal{O}_X}(p_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X)$ the conductor ideal sheaf.
Let $\Delta = V(\mathscr{C})$ be the subscheme of X defined by $\mathscr{C}$ and let $\tilde{\Delta} = p^{-1}(\Delta)$.
Let $q = P|\tilde{\Delta}$.

THEOREM 3: *With the above notations, suppose that* $\mathcal{O}_{\tilde{X}} \cong \underline{\Omega}^0_{\tilde{X}}$, $\mathcal{O}_\Delta \cong \underline{\Omega}^0_\Delta$ *and* $\mathcal{O}_{\tilde{\Delta}} \cong \underline{\Omega}^0_{\tilde{\Delta}}$. *Then*

$$\mathcal{O}_X \cong \underline{\Omega}^0_X.$$

PROOF: One has a commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \mathcal{O}_X & \xrightarrow{u} & \mathcal{O}_\Delta \oplus p_*\mathcal{O}_{\tilde{X}} & \xrightarrow{v} & q_*\mathcal{O}_{\tilde{\Delta}} & \to 0 \\
& \downarrow{\scriptstyle \lambda_X} & & \downarrow{\scriptstyle (\lambda_\Delta, \lambda_{\tilde{X}})} & & \downarrow{\scriptstyle \lambda_{\tilde{\Delta}}} & \\
0 \to & \underline{\Omega}^0_X & \xrightarrow{u} & \underline{\Omega}^0_\Delta \oplus p_*\underline{\Omega}^0_{\tilde{X}} & \xrightarrow{v} & q_*\underline{\Omega}^0_{\tilde{\Delta}} & \to 0
\end{array}$$

where $u(f) = (f_{|\Delta}, p^*f)$ and $v(g, h) = q^*(g) - h_{|\tilde{\Delta}}$.

Exactness of the top row is a general fact, while exactness of the bottom row follows from [1], Proposition (4.11) and the remark that p and q are finite morphisms. The assumptions of the theorem mean that $(\lambda_\Delta, \lambda_{\tilde{X}})$ and $\lambda_{\tilde{\Delta}}$ are quasi-isomorphisms. Hence λ_X is a quasi-isomorphism.

COROLLARY: If X is a general projection surface (see [4], Definition (4.16)) then $\mathcal{O}_X \cong \underline{\Omega}^0_X$. For in that case, $\tilde{X}$ is smooth and Δ and $\tilde{\Delta}$ are curves with only singularities of the type mentioned in (a).

REMARK: Application of Theorem 2 in the cases (a), (b) and (d) generalizes some of the theorems from [4] to the case of degenerations whose total space is not necessarily smooth.

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