J. H. M. Steenbrink

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COHOMOLOGICALLY INSIGNIFICANT DEGENERATIONS

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Introduction

The following two problems in singularity theory appear to be closely related. On the one hand, given a complete singular variety $X$ over $\mathbb{C}$, to construct a filtered complex of sheaves $(\Omega_X, F)$ on $X$, which computes the Hodge filtration on the cohomology of $X$ (see the next section for a more precise statement). This problem has been treated by Philippe du Bois [1]. On the other hand one can ask, for which flat map germs $f: (\mathcal{X}, X) \rightarrow (x, 0)$ with $f^{-1}(0) = X$, the Hodge numbers $h_{pq}^n$ of $H^*(X)$ and the limit Hodge structure on $H^*(\mathcal{X}_x)$ (cf. [5, 7]) are equal for all $p, q, n, \geq 0$ with $pq = 0$. If this is the case, such a degeneration is called cohomologically insignificant. The preceding paper [4] of Igor Dolgachev contains many results on these.

We prove the following local criterion:

**Theorem 2:** Suppose $X$ is a complete algebraic variety over $\mathbb{C}$ such that $\mathcal{O}_X \cong \Omega_X^0$. Then every proper and flat degeneration $f$ over the unit disk $S$ with $f^{-1}(0) = X$ is cohomologically insignificant.

**Example:** If in a degeneration of curves, $X$ is a multiple elliptic fibre, then $X$ is cohomologically insignificant, but $\mathcal{O}_X \not\cong \Omega_X^0$. See [4], Theorem (3.10).

In [4], Igor Dolgachev conjectures, that every family over the disk, whose singular fibre is reduced and has only insignificant limit singularities in the sense of Mumford and Shah (cf. [6]), is cohomologically insignificant.
QUESTION: Suppose $X$ is an algebraic variety over $\mathbb{C}$ which has only insignificant limit singularities. Is it true that $\mathcal{O}_X \cong \Omega^0_X$?

Using Theorem 3 one checks easily that this the case for those from the list of J. Shah [6].

**The filtered De Rham complex of a singular variety**

According to Du Bois [1], for every algebraic variety $X$ over $\mathbb{C}$ there exists a complex $\Omega^*_X$ of analytic sheaves on $S$, whose differentials are first order differential operators, together with a decreasing filtration $F$ on it, such that the following properties are satisfied:

(i) the complex $\Omega^*_X$ is a resolution of the constant sheaf $\mathbb{C}$ on $X$;
(ii) the differential in the graded complex $\text{Gr}_p\Omega^*_X$ is $\mathcal{O}_X$ linear;
(iii) the pair $(\Omega^*_X, F)$ is functorial in $X$ (in a suitable derived category);
(iv) there exists a natural morphism of filtered complexes

$$
\lambda : (\Omega^*_X, \sigma) \to (\Omega^*_X, F)
$$

where $\Omega^*_X$ is the holomorphic De Rham complex and $\sigma$ its "filtration bête" (cf. [2], Definition (1.4.7)); if $X$ is smooth then $\lambda$ is a filtered quasi-isomorphism.

(v) if $X$ is complete, then the spectral sequence

$$
E_1^{pq} = H^{p+q}(X, \text{Gr}_p^q\Omega^*_X) \Rightarrow H^{p+q}(X, \mathbb{C})
$$

degenerates at $E_1$ and abuts to the Hodge filtration of $H^*(X, \mathbb{C})$, which carries Deligne’s mixed Hodge structure (cf. [3]).

Let $\Omega^*_X$ denote the complex $\text{Gr}_p^0\Omega^*_X$.

**Theorem 1:** Let $f : X \to S$ be a proper and flat morphism of complex algebraic varieties. For $s \in S$, let $X_s$ denote the fibre $f^{-1}(s)$ over $s$. If for all $s \in S$ the map

$$
\text{Gr}_p^0(\lambda) : \mathcal{O}_{X_s} \to \Omega^*_X
$$

is a quasi-isomorphism, then for all $i \geq 0$ the sheaf $R^i(f)_*\mathcal{O}_X$ is locally free on $S$ and for all $s \in S$ the natural map
is an isomorphism.
Cf. [1], Théorème 4.6.

If $X$ is a complete algebraic variety, let us denote the numbers $h_{p,q}^n$ are the Hodge numbers of $H^n(X, \mathbb{C})$.

Then one clearly has

$$
\sum_{q \geq 0} h_{p,q}^n(X) = \dim_c \text{Gr}_p^n H^n(X, \mathbb{C})
$$

for all $p, n \geq 0$. Hence if $X$ is complete and $\mathcal{O}_X \cong \mathcal{O}_X^0$, then in view of property (v) one obtains

$$
\dim_c H^n(X, \mathcal{O}_X) = \sum_{q \geq 0} h_{p,q}^n(X) = \sum_{q \geq 0} h_{p,q}^0(X).
$$

In the next theorem we consider degenerations with singular fibre $X$, that is flat projective mappings $f : \mathcal{X} \to S$ where $\mathcal{X}$ is a complex space, $S$ is the unit disk in the complex plane and $f$ is smooth over the punctured disk $S^* = S \setminus \{0\}$, and $X = f^{-1}(0)$.

Let $H$ denote the universal covering of $S^*$, i.e. the upper half plane, and let $X_\infty$ denote the family $\mathcal{X}_\infty H$ over $H$. We endow $H^*(X_\infty)$ with the limit Hodge structure (cf. [5], [7]). One has a natural map

$$
s_p : H^*(X) \to H^*(X_\infty)
$$

which is a morphism of mixed Hodge structures.

**Theorem 2:** Let $f : \mathcal{X} \to S$ be a degeneration with singular fibre $X$, satisfying $\mathcal{O}_X \cong \mathcal{O}_X^0$. Then for all $n \geq 0$:

$$
\text{Gr}_p^0 (s_p) : \text{Gr}_p^0 H^n(X) \to \text{Gr}_p^0 H^n(X_\infty).
$$

In other words: $f$ is a cohomologically insignificant degeneration.

**Proof:** As $X$ is a deformation retract of $\mathcal{X}$, the map

$$(R^n f_* C_\mathcal{X})_0 \to H^*(X, \mathbb{C})$$
is an isomorphism for all \( n \geq 0 \). Because \( \mathcal{O}_X \cong \Omega^0_X \) and \( X \) is complete, the map

\[ H^n(X, \mathbb{C}) \to H^n(X, \mathcal{O}_X) \]

is surjective. Hence there exist sections \( \sigma_1, \ldots, \sigma_h \) of \( R^nf_*\mathcal{O}_X \) over \( S \) such that their images in \( H^n(X, \mathcal{O}_X) \) form a basis. Let \( \tilde{\sigma}_i \) denote the image of \( \sigma_i \) under the natural map

\[ R^nf_*\mathcal{O}_X \to R^nf_*\mathcal{O}_S. \]

Because \( R^nf_*\mathcal{O}_X \) is locally free, the sections \( \tilde{\sigma}_1, \ldots, \tilde{\sigma}_h \) give a basis on some small neighborhood of 0 in \( S \). This means, that the map

\[ Gr^0_\mathcal{T}H^n(X, \mathbb{C}) \to Gr^0_\mathcal{T}H^n(X, \mathbb{C}) \]

is an isomorphism for \(|t|\) sufficiently small. In particular the images of \( \sigma_1, \ldots, \sigma_h \) in \( H^n(X_\infty, \mathbb{C}) \) are linearly independent; because morphisms of mixed Hodge structures are strictly compatible with the Hodge filtrations, the images of \( \sigma_1, \ldots, \sigma_h \) in \( Gr^0_\mathcal{T}H^n(X_\infty, \mathbb{C}) \) are also linearly independent. Moreover the fact that \( R^nf_*\mathcal{O}_X \) is locally free implies that for \( t \neq 0 \):

\[
\dim_{\mathbb{C}} Gr^0_\mathcal{T}H^n(X, \mathbb{C}) = \dim_{\mathbb{C}} H^n(X, \mathcal{O}_X) \\
= \dim_{\mathbb{C}} H^n(X, \mathcal{O}_X) = \dim_{\mathbb{C}} Gr^0_\mathcal{T}H^n(X_\infty, \mathbb{C}) \\
= \dim_{\mathbb{C}} Gr^0_\mathcal{T}H^n(X_\infty, \mathbb{C}).
\]

Hence \( Gr^0_\mathcal{T}(sp) \) is an isomorphism.

**Examples where \( \mathcal{O}_X \equiv \Omega^0_X \).**

(a) If \( X \) is a reduced curve, then \( \mathcal{O}_X \equiv \Omega^0_X \) if and only if at every singular point of \( X \) the branches are smooth and their tangent directions are independent. If \( X \) lies on a smooth surface, it can only have ordinary double points; more generally, if \( X \) has embedding dimension \( n \) at \( x \in X \), then

\[ \mathcal{O}_{X,x} \equiv \mathbb{C}[[z_1, \ldots, z_n]]/(z_i z_j : i \neq j). \]

See [1], Proposition 4.9.

(b) Suppose \( X \) is a normal surface, \( \pi: \tilde{X} \to X \) a resolution of its
singularities, \( E_x = \pi^{-1}(x)_{\text{red.}} \) for \( x \in X \). Then \( \mathcal{O}_X \cong \Omega^0_X \) if and only if 
\( (\text{R}^1\pi_*\mathcal{O}_X)_x \cong H^1(E_x, \mathcal{O}_{E_x}) \) for all \( x \in \text{Sing}(X) \). See [1], Proposition 4.13 and its proof.

Hence if \( X \) has embedding dimension three, its singularities can only be rational double points, simple-elliptic or cusp singularities. See [4], Corollary 4.11.

(c) If \( X \) has only quotient singularities, then \( \mathcal{O}_X \cong \Omega^0_X \). See [1], Théorème (5.3).

(d) Suppose \( X \) is a complex variety, \( p : \tilde{X} \to X \) its normalisation, 
\( \mathcal{E} = \text{Ann}_{\mathcal{O}_X}(p_*\mathcal{O}_{\tilde{X}}/\mathcal{O}_X) \) the conductor ideal sheaf.
Let \( \Delta = V(\mathcal{E}) \) be the subscheme of \( X \) defined by \( \mathcal{E} \) and let \( \tilde{\Delta} = p^{-1}(\Delta) \).
Let \( q = P|_{\tilde{\Delta}} \).

**Theorem 3**: With the above notations, suppose that \( \mathcal{O}_X \cong \Omega^0_X \), 
\( \mathcal{O}_\Delta \cong \Omega^0_\Delta \) and \( \mathcal{O}_{\tilde{\Delta}} \cong \Omega^0_{\tilde{\Delta}} \). Then

\[
\mathcal{O}_X \cong \Omega^0_X.
\]

**Proof**: One has a commutative diagram

\[
\begin{array}{ccc}
0 \to \mathcal{O}_X & \xrightarrow{u} & \mathcal{O}_\Delta \bigoplus p_*\mathcal{O}_{\tilde{X}} & \xrightarrow{v} & q_*\mathcal{O}_{\tilde{\Delta}} & \to 0 \\
 & \downarrow{\lambda_X} & (\lambda_\Delta, \lambda_{\tilde{\Delta}}) & & \downarrow{\lambda_{\tilde{\Delta}}} \\
0 \to \Omega^0_X & \xrightarrow{u} & \Omega^0_\Delta \bigoplus p_*\Omega^0_{\tilde{X}} & \xrightarrow{v} & q_*\Omega^0_{\tilde{\Delta}} & \to 0
\end{array}
\]

where \( u(f) = (f|_{\tilde{\Delta}}, p^*f) \) and \( v(g, h) = q^*(g) - h|_{\tilde{\Delta}} \).

Exactness of the top row is a general fact, while exactness of the bottom row follows from [1], Proposition (4.11) and the remark that \( p \) and \( q \) are finite morphisms. The assumptions of the theorem mean that \( (\lambda_\Delta, \lambda_{\tilde{\Delta}}) \) and \( \lambda_{\tilde{\Delta}} \) are quasi-isomorphisms. Hence \( \lambda_X \) is a quasi-isomorphism.

**Corollary**: If \( X \) is a general projection surface (see [4], Definition (4.16)) then \( \mathcal{O}_X \cong \Omega^0_X \). For in that case, \( \tilde{X} \) is smooth and \( \Delta \) and \( \tilde{\Delta} \) are curves with only singularities of the type mentioned in (a).

**Remark**: Application of Theorem 2 in the cases (a), (b) and (d) generalizes some of the theorems from [4] to the case of degenerations whose total space is not necessarily smooth.
REFERENCES


Wassenaarseweg 80
2333 AL LEIDEN, The Netherlands