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NADIA CHIARLI

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CHARACTERIZATIONS OF CERTAIN SINGULARITIES OF A BRANCHED COVERING*

Nadia Chiarli

Introduction

Let X, Y be locally noetherian schemes, where Y is normal irreducible and X is reduced, and let $\phi: X \rightarrow Y$ be a finite covering of degree n (see definition 1.4). The problem is: how much ramification is allowed in order for X to have nice singularities, in particular in order for X to be seminormal or normal?

We studied essentially the seminormality of X in [4] when $n = 2$, and in [5] when ϕ is locally monogenic of arbitrary degree and X is integral. The purpose of this paper is to give a more general answer to the problem, studying the normal case and generalizing the seminormal case in a way leading also to the unification of the results of [4] and [5].

All the results are obtained by assuming Y to be the spectrum of a discrete valuation ring (see sections 1, 2, 3): they can be globalized (see section 4) in the same way shown in [4] and [5].

In section 1 and 2 we study respectively the normality and the seminormality of X , giving characterizations for both of them in terms of the value of the discriminant sheaf at the points of Y of codimension 1, and showing the relations with the tame ramification over Y of the normalization of X (see 1.2, 1.8, 2.2, 2.7).

In section 3 we study the particular case when X is Gorenstein, and finally in section 4 we discuss the globalization of the previous results.

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Conventions and notations

Rings are assumed to be noetherian, commutative, with identity. In the remainder of this paper, unless stated to the contrary, we make the following assumptions:

(a) A is a discrete valuation ring, with uniformizing parameter t , residue field k and valuation v ;

(b) K is the fraction field of A and L is a reduced K -algebra such that $[L: K] = n$;

(c) B' is a finite A -algebra, with L as total quotient ring;

(d) B is the integral closure of A in L , finite over A ;

(e) for all $\mathfrak{m}_i \in \text{Max } B$, the extensions $k(\mathfrak{m}_i)/k$ are separable;

(f) if M is a sub- A -module of B , free of rank n , then $D_{M/A}$ denotes the discriminant of M over A ([19] §3, p. 59);

(g) l_A denotes the length of an A -module;

(h) for every $\mathfrak{m}_i \in \text{Max } B$, put $f_i = [k(\mathfrak{m}_i): k]$ and $\sum_i f_i = f = l_A(B/\text{rad } B)$; for every $\mathfrak{p}_j \in \text{Max } B'$, put $g_j = [k(\mathfrak{p}_j): k]$ and $\sum_j g_j = g = l_A(B'/\text{rad } B')$. Obviously $g \leq f$.

Remark that from (a), (b), (c), (d) it follows that B and B' are sub- A -modules of B , free of rank n .

For general facts on ramification theory see [1], [8], [9].

1. Normality

PROPOSITION 1.1: *Let A, K, B, L be as above. Suppose M and N are two sub- A -modules of B , free of rank n . Then, for $M \subseteq N$:*

$$v(D_{M/A}) = 2\ell_A(N/M) + v(D_{N/A}).$$

PROOF. By [8] th. 1, p. 26 there exists a basis $\{b_1, \dots, b_n\}$ of N such that $\{t^{r_1}b_1, \dots, t^{r_n}b_n\}$ ($r_h \leq r_{h+1}$) is a basis of M . Therefore ([8] prop. 1, p. 46):

$$\det(\text{Tr}_{M/A}(t^{r_i}b_i t^{r_j}b_j)) = (\det(a_{ij}))^2 \det(\text{Tr}_{N/A}(b_i b_j))$$

where (a_{ij}) is the matrix associated with the A -linear mapping

$$(b_1, \dots, b_n) \rightarrow (t^{r_1}b_1, \dots, t^{r_n}b_n).$$

Then $(\det(a_{ij}))^2 = t^{2\sum_h r_h}$, and $v(D_{M/A}) = 2\sum_h r_h + v(D_{N/A})$. Moreover it

is easy to prove, by induction over $\ell_A(N/M)$, that $\sum_h r_h = \ell_A(N/M)$ and this concludes the proof.

We recall that B is said to be *tamely ramified* over A if for every $\mathfrak{m}_i \in \text{Max } B$ the characteristic of k does not divide the ramification index e_i of \mathfrak{m}_i .

THEOREM 1.2: *If B' is normal, then $v(D_{B'/A}) \geq n - f$.
Moreover the following are equivalent:*

- (i) B' is normal and tamely ramified over A .
- (ii) $v(D_{B'/A}) = n - f$.
- (iii) $v(D_{B'/A}) \leq n - g$.
- (iv) $v(D_{B'/A}) = n - g$.

PROOF: Suppose $B' = B$. We have: $B = \prod_l B_l$, where the B_l 's are normal domains. Therefore ([19] prop. 6, p. 60 and prop. 13, p. 67):

$$D_{B/A} = N(\delta_{B/A}) = N[\prod_s (\mathfrak{q}_{l_s}^{h_{l_s}})] \text{ with } h_{l_s} \geq e_{l_s} - 1,$$

where $\mathfrak{q}_{l_s} \in \text{Max } B_l$ for every s , e_{l_s} is the ramification index of \mathfrak{q}_{l_s} , N is the norm and $\delta_{B/A}$ is the different of B_l over A .

So: $v(D_{B/A}) = \sum_s f_{l_s} h_{l_s} \geq \sum_s f_{l_s} (e_{l_s} - 1)$, where $f_{l_s} = [k(\mathfrak{q}_{l_s}) : k]$. Now, since $D_{B/A} = \prod_l D_{B_l/A}$ ([8] lemma 1, p. 87) we have: $v(D_{B/A}) = \sum_l v(D_{B_l/A}) \geq \sum_l (\sum_s f_{l_s} (e_{l_s} - 1)) = \sum_i f_i (e_i - 1) = n - f$, where the last equality follows from [3] th. 2, p. 147.

(i) \rightarrow (ii) Follows by the previous arguments, after observing that, due to the tame ramification of B over A , we have $h_{l_s} = e_{l_s} - 1$ ([9] prop. 13, p. 67) for every l and s .

(ii) \rightarrow (iii) Follows from $g \leq f$ (see (h) above).

(iii) \rightarrow (iv) By 1.1 since $v(D_{B/A}) \geq n - f$, we have $2\ell_A(B/B') + n - f \leq n - g$, which implies $2\ell_A(B/B') \leq f - g$. But $\ell_A(B/B') \geq \ell_A(B/\text{rad } B) - \ell_A(B'/\text{rad } B') = f - g$, so $f = g$ and $B' = B$. Moreover from $n - g = n - f \leq v(D_{B'/A}) \leq n - g$, it follows $v(D_{B'/A}) = n - g$.

(iv) \rightarrow (i) By 1.1 and the first part of this theorem, we have $n - g \geq 2\ell_A(B/B') + n - f$, so $2\ell_A(B/B') \leq f - g$, which implies, by the same arguments as in (iii) \rightarrow (iv), $f = g$ and B' normal. Moreover B' is tamely ramified over A : in fact (with the same notations as in the first part of this proof) we get $h_{l_s} = e_{l_s} - 1$ for every l and s .

COROLLARY 1.3: (i) $v(D_{B'/A}) \geq n - g$.

(ii) $v(D_{B'/A}) = n - g$ iff B' is normal and tamely ramified over A . (The lower bound $n - g$ for $v(D_{B'/A})$ shall be improved in 2.3 (i): see also remark 2.4).

PROOF: By 1.1 and 1.2 we have: $v(D_{B'/A}) \cong 2\ell_A(B/B') + n - f \cong 2(f - g) + n - f = n + f - 2g \cong n - g$.

(ii) Follows from 1.2.

We will give now a geometrical interpretation of 1.2.

DEFINITION 1.4: Let X, Y be two locally noetherian schemes, with X reduced and Y integral, and denote by $X_i (i = 1, \dots, s)$ the irreducible components of X : let $\phi: X \rightarrow Y$ be a morphism. We say that ϕ is a *finite covering* if ϕ is finite and $\phi|_{X_i}: X_i \rightarrow Y$ is surjective for every i : in this case $\phi|_{X_i}$ induces a natural embedding $k(Y) \hookrightarrow k(X_i)$ for every i . We call *degree* of ϕ the integer $\sum_i [k(X_i): k(Y)]$.

Let now $\phi: X \rightarrow Y$ be a finite covering of degree n between two schemes locally of finite type over an algebraically closed field k : assume that Y is normal and irreducible, and let \mathfrak{D} be the discriminant sheaf of ϕ (see e.g. [4]). Let $Z \subset Y$ be an irreducible closed subscheme of codimension 1, with generic point \mathfrak{q} : assume that $Z \not\subset \text{Sing } Y$ and denote by v_z the valuation associated with the discrete valuation ring \mathfrak{D}_z . Let Z_1, \dots, Z_r be the irreducible components of $\phi^{-1}(Z)$ and, for each i , denote by z_i the generic point of Z_i .

PROPOSITION 1.5: *Assume that for every i we have: $k(z_i)/k(z)$ is separable and \mathcal{O}_{z_i} is tamely ramified over \mathcal{O}_z (e.g. k has characteristic zero). Then the following are equivalent:*

(i) $Z_i \not\subset \text{Sing } X$ for all i 's.

(ii) *There is a non-empty open $U \subset Z$ such that for every closed point $\zeta \in U$ the cardinality of the set $\phi^{-1}(\zeta)$ is equal to $n - v_z(\mathfrak{D}_z)$.*

PROOF: For every i the morphism $\phi_i = \phi|_{Z_i}: Z_i \rightarrow Z$ is a finite covering of degree $d_i = [k(z_i): k(z)]$: by [10] th. 7, p. 117 there is a non-empty open set $U_i \subset Z$ such that $d_i = \# \text{points of } \phi^{-1}(\alpha)$, for all closed points $\alpha \in U_i$.

Hence $\sum_i d_i = \# \text{points of } \phi^{-1}(\zeta)$, where ζ is closed and belongs to the open set $(\cap_i U_i) - [\cup_{i \neq j} \phi(Z_i \cap Z_j)]$ which is non-empty because Z as well as the Z_i 's are irreducible. Now, if we denote by A the local ring of Y at z and by B' the semilocal ring of X at z_1, \dots, z_r , we have by 1.3, $f = \sum_i d_i \leq n - v_z(\mathfrak{D}_z)$, where the equality holds iff B' is normal, i.e. iff $Z_i \not\subset \text{Sing } X$ for all i 's.

COROLLARY 1.6: *Let X, Y be two algebraic curves over an algebraically closed field k of characteristic zero, and let $\phi: X \rightarrow Y$ be a finite covering of degree n . Let $P \in Y$ be a non-singular (closed) point*

and let $\phi^{-1}(P) = \{P_1, \dots, P_s\}$ (as a set). Then:

- (i) $s \cong n - v_P(\mathfrak{D}_P)$.
- (ii) $s = n - v_P(\mathfrak{D}_P)$ iff P_1, \dots, P_s are non-singular.

LEMMA 1.7: $\hat{B} = \overline{\hat{B}'}$ and $[\hat{L} : \hat{K}] = [L : K]$.

PROOF: Since B is semilocal $\hat{B} = \Pi_i \hat{B}_{m_i}$, and since $\dim B' = 1$ $\dim B = 1$ too. Moreover, since B_{m_i} is a discrete valuation ring, \hat{B}_{m_i} is also a discrete valuation ring and therefore \hat{B} is normal. From $B' \subset B \subset L$ it follows: $\hat{B}' = \hat{B}' \otimes_{B'} B' \subset B \otimes_{B'} \hat{B}' \subset L \otimes_{B'} \hat{B}'$ and then $\hat{B}' \subset B \subset L \otimes_{B'} \hat{B}'$, and \hat{B} is finite over \hat{B}' . We have: $L = B'_f$ where $f \in B'$ is a non zero-divisor belonging to $\text{rad } B'$. Therefore: $L \otimes_{B'} \hat{B}' = B'_f \otimes_{B'} \hat{B}' = B'_f$, where, by flatness f is a non zero-divisor in $\text{rad } \hat{B}'$. So $L \otimes_{B'} \hat{B}'$ is the total quotient ring of \hat{B}' . But $L \otimes_B \hat{B}' = L \otimes_B B \otimes_{B'} \hat{B}' = L \otimes_{B'} \hat{B}'$ and then $L \otimes_{B'} \hat{B}'$ is the total quotient ring of \hat{B} , which implies $\hat{B} = \hat{B}'$. Moreover $[\hat{L} : \hat{K}] = [(L \otimes_B \hat{B}') : (K \otimes_A \hat{A})] = [(L \otimes_A \hat{A}) : (K \otimes_A \hat{A})] = [L : K]$.

THEOREM 1.8: If B' is normal and tamely ramified over A , then $v(D_{B'/A}) \leq n - 1$.

The converse holds if either:

- (i) $n = 2$, or
- (ii) B' is local, or
- (iii) there exists a finite group G of automorphisms of B' such that $B'^G = A$.

PROOF: The first claim follows from 1.2.

(i) We have either $f = 2$ or $f = 1$, so the claim follows from 1.1.

(ii) Claim first that $f = g$. By 1.1 and 1.2 we have $n - f + 2(f - g) \leq v(D_{B'/A}) + 2\ell_A(B/B') \leq n - 1$ and so $f - 2g \leq -1$. Now, if k' is the residue field of B' we have: $f = \dim_{k'}(B/\text{rad } B) = g \dim_{k'}(B/\text{rad } B)$. If $f \neq g$, then $\dim_{k'}(B/\text{rad } B) > 1$, which implies $f \geq 2g$; a contradiction. So $f = g$. On the other hand, by 1.1 and 1.2 we have: $n - 1 \geq v(D_{B'/A}) \geq 2\ell_A(B/B') + n - f$ and $\ell_A(B/B') = g\ell_{B'}(B/B')$.

Therefore: $2g\ell_{B'}(B/B') \leq f - 1 = g - 1$, so $g[2\ell_{B'}(B/B') - 1] \leq -1$, which implies $B = B'$.

Moreover B is tamely ramified over A . Indeed, denoting by \mathfrak{m} the unique maximal ideal of B and by e its ramification index, we have: $\delta_{B/A} = \mathfrak{m}^h$ with $h \geq e - 1$. Now, $v(D_{B/A}) = v(N(\delta_{B/A})) = fh$; then $fh \leq n - 1 = ef - 1$ ([3] th. 2, p. 147), so $f(h - e) \leq -1$. This implies $h \leq e - 1$, which concludes the proof ([9] prop. 13, p. 67).

(iii) We have: $B' = \bigoplus_{j=1}^f B'_j$ where all the B'_j 's are local. Moreover

([3] th. 2, p. 42) $[k(\mathfrak{p}_j): k]$, $[k(m_{jl}): k]$, $[L'_j: \hat{K}]$ do not depend on j , for all $\mathfrak{p}_j \in \text{Max } B'$ and all $m_{jl} \in \text{Max } B$ over \mathfrak{p}_j ; and also $[L'_j: \hat{K}] = n/r$ by 1.7.

Since $v(D_{B'/A}) = v(D_{\hat{B}'/\hat{A}})$ ([9] prop. 10, p. 61), we have $v(D_{B'/A}) \leq n - 1$. Therefore: $v(D_{B'_j/\hat{A}}) = (1/r)v(D_{B'/A}) \leq [(n - 1)/r] \leq [L'_j: \hat{K}] - 1$. Since for every B'_j condition (e) is verified because the residue fields do not change by completion, B'_j is normal and tamely ramified over A for every j by (ii); then B' itself is normal and tamely ramified over A .

COROLLARY 1.9: ([4] prop. 1.6). *Suppose that $n = 2$ and that A contains a field of characteristic $\neq 2$. Then B' is normal iff $v(D_{B'/A}) \leq 1$.*

PROOF: B' is tamely ramified over A and the claim follows from 1.8.

REMARK 1.10: In general $v(D_{B'/A}) \leq n - 1$ does not imply B' normal even if B is tamely ramified over A , as shown by the following:

COUNTEREXAMPLE 1.11: Suppose $\text{char } k = 0$, and let $A = k[X]_{(X)}$, $B' = A[Y]/(Y^4 - Y^2 - X^3)$ (B' is the semilocal ring of the points of the curve $F = Y^4 - Y^2 - X^3 = 0$ which are contained in the line $X = 0$). We have: $v(D_{B'/A}) = v(\text{Res}_Y(F, F')) = 3$ (by direct computation), where $\text{Res}_Y(F, F')$ is the resultant of F and its derivative F' with respect to Y .

Therefore $v(D_{B'/A}) \leq 4 - 1$; but B' is not normal.

Moreover we can show, by the following counterexample, that $n - 1$ is the best upper-bound for $v(D_{B'/A})$ in order to grant, under the assumptions of 1.8, the normality of B' and its tame ramification over A .

COUNTEREXAMPLE 1.12: Put $A = \mathbb{R}[T]_{(T)}$, $B = \mathbb{C}[X]_{(X)}$ with the ring homomorphism given by $T \rightarrow X^2$ (so that $n = 4$), and let $B' = \mathbb{R}[X, iX]_{(X, iX)}$. B' is local and moreover there is a finite group G of automorphisms of B' such that $B'^G = A$, given by: $G = \{\sigma_1, \dots, \sigma_4\}$ where $\sigma_1 = \text{id}_{B'}$, $\sigma_2(X, iX) = (X, -iX)$, $\sigma_3(X, iX) = (-X, -iX)$, $\sigma_4 = \sigma_2 \circ \sigma_3$. We have $v(D_{B'/A}) = 4 = n$ (see 3.5), but B' is not normal.

The following example shows that for every $n \in \mathbb{N}$ there exists B' normal and tamely ramified over A such that $v(D_{B'/A}) = 0, 1, \dots, n - 1$; therefore in particular the maximum $n - 1$ is attained.

EXAMPLE 1.13: Let $q_1, \dots, q_u \in k[X]$ be irreducible, $q_i \neq q_j$ and non-associate whenever $i \neq j$; assume $\text{char } k = 0$ and let $S =$

$\{q \in k[X] \mid q_i \text{ does not divide } q \text{ for every } i\}$. Put $A = k[T]_{(T)}$ and $B' = k[X]_S$ with the ring homomorphism given by $T \rightarrow \prod_i q_i^{a_i}$ where the a_i 's are positive integers and $\sum_i a_i = n$. The maximal ideals of B' are the $m_i = (q_i)k[X]_{(q_i)}$ and $[k(\mathfrak{m}):k] = \deg q_i$ ($i = 1, \dots, u$). We have: $K = k(T)$, $L = k(X)$; so $[L:K] = n$. B' is normal and tamely ramified over A , therefore by 1.2 $v(D_{B'/A}) = n - \sum_i \deg q_i$. Now, for a suitable choice of the q_i 's it is possible to obtain every value of $v(D_{B'/A})$ between 0 and $n - 1$ (compare with 1.6).

2. Seminormality

For general facts on seminormality see [6] or [11].

LEMMA 2.1: *The following are equivalent:*

- (i) B' is seminormal.
- (ii) $\text{rad } B' = \text{rad } B$.
- (iii) $\ell_A(B/B') = f - g$.

If moreover $f = g$, then (i), (ii), (iii) are also equivalent to:

- (iv) B' is normal.

PROOF: Let \mathfrak{b} be the conductor of B .

(i) \rightarrow (ii) If B' is seminormal, then B/\mathfrak{b} is reduced ([11] lemma 1.3, p. 588); so, after renumbering the m_i 's, we have: $\mathfrak{b} = m_1 \cap \dots \cap m_s \supset m_1 \cap \dots \cap m_r = \text{rad } B$. But $\mathfrak{b} \subset B'$, so $\text{rad } B \subset \text{rad } B'$ and we are done.

(ii) \rightarrow (i) $\mathfrak{b} \supset \text{rad } B' = \text{rad } B$, therefore $B/\mathfrak{b} = (B/\text{rad } B)/(\mathfrak{b}/\text{rad } B) = (k_1 \times \dots \times k_r)/I$ (I a suitable ideal). Thus $B/\mathfrak{b} = k_i \times \dots \times k_i$, which implies B/\mathfrak{b} reduced and B' seminormal ([6] cor. 2.7, p. 10).

(ii) \leftrightarrow (iii) $\ell_A(B/B') = \ell_A(B/\text{rad } B) - \ell_A(B'/\text{rad } B) = f - g$ iff $\text{rad } B = \text{rad } B'$.

The rest is obvious.

THEOREM 2.2: *Consider the following conditions:*

- (i) B' is seminormal.
- (ii) $v(D_{B'/A}) \geq n + f - 2g$.
- (iii) $v(D_{B'/A}) = n + f - 2g$.
- (iv) $v(D_{B'/A}) \leq n + f - 2g$.
- (v) B' is seminormal and B is tamely ramified over A .

Then: (i) \rightarrow (ii) and (iii), (iv), (v) are equivalent.

PROOF: (i) \rightarrow (ii) By 2.1 we have $\ell_A(B/B') = f - g$, and therefore by 1.1 and 1.2 $v(D_{B'/A}) \geq n + f - 2(f - g) = n + f - 2g$.

(iii) \rightarrow (iv) Trivial.

(iv) \rightarrow (v) Let C be the seminormalization of B' ([11] pp. 585–586): by 1.1 we have $v(D_{C/A}) + 2\ell_A(C/B') = v(D_{B'/A}) \leq n + f - 2g$. But since C is seminormal and the sum of the degrees over k of its residue fields equals g , we have: $n + f - 2g + 2\ell_A(C/B') \leq n + f - 2g$, which implies $\ell_A(C/B') = 0$ and $C = B'$.

Moreover B is tamely ramified over A . In fact: $v(D_{B/A}) + 2\ell_A(B/B') \leq n + f - 2g$ implies, by 2.1, $v(D_{B/A}) + 2(f - g) \leq n + f - 2g$; so $v(D_{B/A}) \leq n - f$ and the claim follows by 1.2.

(v) \rightarrow (iii) Follows from 1.2 and from (i) \rightarrow (ii).

COROLLARY 2.3: (i) $v(D_{B'/A}) \geq n + f - 2g$.

(ii) $v(D_{B'/A}) = n + f - 2g$ iff B' is seminormal and B is tamely ramified over A .

PROOF: (i) Let C be the seminormalization of B' ; by 1.1 we have $v(D_{B'/A}) \geq v(D_{C/A}) \geq n + f - 2g$.

(ii) Follows from 2.2.

REMARK 2.4: Since $f \geq g$, then $n + f - 2g \geq n - g$, with strict inequality whenever $f \neq g$. Therefore 2.3 (i) is an improvement of 1.3 (i).

REMARK 2.5: From 2.2 it follows that if B' is seminormal and B is tamely ramified over A , then $v(D_{B'/A}) \leq 2n - 2$ (the upper bound is obtained when $f = n$ and $g = 1$).

Counterexample 1.11 shows that the converse is false, in general: in fact B' is not seminormal and still $v(D_{B'/A}) \leq 2 \cdot 4 - 2$.

The following example shows that for every $n \in \mathbb{N}$ there exists B' seminormal, with B tamely ramified over A , such that $v(D_{B'/A}) = n, n + 1, \dots, 2n - 2$; therefore, in particular, the maximum $2n - 2$ is attained.

EXAMPLE 2.6: With the same notations as in 1.13 put: $A = k[T]_{(T)}$, $B = k[X]_S$, $B' = k + \text{rad } B$.

B' is seminormal by 2.1 and since B is tamely ramified over A , from 2.2 it follows $v(D_{B'/A}) = n + \sum_i \deg q_i - 2$. Therefore, for a suitable choice of the q_i 's, it is possible to obtain every value of $v(D_{B'/A})$ between n and $2n - 2$.

PROPOSITION 2.7: (i) B' is seminormal iff \hat{B}' is seminormal.

(ii) If $B' = C \oplus D$ (direct sum of rings), then B' is seminormal iff C and D are seminormal.

PROOF: (i) B' is seminormal iff $\text{rad } B' = \text{rad } B$ (see 2.1) iff $\text{rad } \hat{B}' = \text{rad } \hat{B}$ iff $\text{rad } \hat{B}' = \text{rad } \bar{B}'$ (see 1.7) iff \hat{B}' is seminormal (see 2.1).

(ii) If B' is seminormal, then B' is the largest subring of B such that $\text{spec } B \rightarrow \text{spec } B'$ is a homeomorphism with trivial residue field extension. Therefore for C and D the same property holds; and conversely.

In remark 2.5 we pointed out that $v(D_{B'/A}) \leq 2n - 2$ is not a sufficient condition in order for B' to be seminormal. Now we want to find a function $F(n, f, g)$ such that $v(D_{B'/A}) \leq F(n, f, g)$ gives such a sufficient condition (under suitable hypotheses). In the next theorem we show that $F(n, f, g) = n + f - 1$ is the required function.

THEOREM 2.8: *Assume that B is tamely ramified over A . If B' is seminormal, then $v(D_{B'/A}) \leq n + f - 1$.*

The converse holds if either:

- (i) $n = 2$, or
- (ii) B' is local, or
- (iii) *there exists a finite group G of automorphisms of B' such that $B'^G = A$.*

PROOF. By 2.2 $v(D_{B'/A}) = n + f - 2g$: moreover $2g \geq 1$, therefore $v(D_{B'/A}) \leq n + f - 1$, which proves the first part of the theorem.

(i) If B' is local, we can apply (ii). If B' is not local we have $f = g = 2$, which implies B' normal and then seminormal.

(ii) Let C be the seminormalization of B' ; by 2.3 and 1.1 we have: $n + f - 2g \leq v(D_{C/A}) \leq v(D_{B'/A}) \leq n + f - 1$, which implies $v(D_{B'/A}) - v(D_{C/A}) \leq 2g - 1$ and so $2\ell_A(C/B') \leq 2g - 1$. But, since B' is local, $\ell_A(C/B') = g\ell_{B'}(C/B')$: then we have $2g\ell_{B'}(C/B') \leq 2g - 1$, which implies $\ell_{B'}(C/B') = 0$ and B' is seminormal.

(iii) With the same notations as in the proof of 1.8 (ii) we have: $v(D_{B'_j/A}) = (1/r)v(D_{B'/A}) \leq [(n + f - 1)/r] \leq n/r + f/r - 1 = [L'_j: \hat{K}] + [k(\mathfrak{p}_j): k] - 1$, which implies B'_j seminormal for every j , by (ii). Therefore B' itself is seminormal by 2.7.

REMARK 2.9: In general $v(D_{B'/A}) \leq n + f - 1$ does not imply B' seminormal, as shown by counterexample 1.2 of [5].

Moreover we can show, by the following counterexample, that $n + f - 1$ is the best upper bound for $v(D_{B'/A})$ in order to grant, under the assumptions of 2.8, the seminormality of B' .

COUNTEREXAMPLE 2.10: Let $A = \mathbb{R}[T^2]_{(T^2)}$, $B' = \mathbb{C}[T^2, T^3]_{(T^2, T^3)}$ and $B = \mathbb{C}[T]_{(T)}$. We have $n = 4$ and $f = g = 2$. Moreover $\ell_B(B/B') = 1$, so $\ell_A(B/B') = 2$. Since B is tamely ramified over A , from 1.3 it follows $v(D_{B/A}) = 4 - 2 = 2$ and by 1.1 $v(D_{B'/A}) = 4 + 2 = n + f$; but B' is not seminormal, though it is local.

REMARK 2.11: We do not know if, in theorem 2.8, when (i) or (ii) or (iii) are verified and $v(D_{B'/A}) \leq n + f - 1$, B happens to be tamely ramified over A .

3. The Gorenstein case

THEOREM 3.1: *Suppose B' is Gorenstein and B is tamely ramified over A .*

If B' is seminormal, then $v(D_{B'/A}) \leq n$.

The converse holds if either:

- (i) $n = 2$, or
- (ii) B' is local, or
- (iii) *there exists a finite group G of automorphisms of B' such that $B'^G = A$.*

PROOF: For every $\mathfrak{p}_j \in \text{Max } B'$, let $p_j = \ell_A(\overline{B}'_{\mathfrak{p}_j}/\text{rad } \overline{B}'_{\mathfrak{p}_j})$: from [6] th. 8.1, p. 46 it follows $2g_j \geq p_j$ and since $f = \sum_j p_j$ we get $2g = 2 \sum_j g_j \geq f$, and the claim follows from 2.2.

The converse follows from 2.8.

COROLLARY 3.2 ([5] 1.1 and 1.3): *Let $B' = A[x]$ be a domain, and suppose either $\text{char } k = 0$ or $\text{char } k > n$.*

If B' is seminormal, then $v(D_{B'/A}) \leq n$.

The converse holds if B' is local.

PROOF: If G is the characteristic polynomial of x , we have $B' = A[X]/(G)$ and, since $A[X]$ is Gorenstein, B' is also Gorenstein. Moreover from the formula $\sum_{\mathfrak{p}} e_{\mathfrak{p}} f_{\mathfrak{p}} = n$, it follows $e_{\mathfrak{p}} \leq n$ for every $\mathfrak{p} \in \text{spec } B$, which implies that B is tamely ramified over A (obviously $e_{\mathfrak{p}}$ denotes the ramification index of \mathfrak{p} , and $f_{\mathfrak{p}} = [k(\mathfrak{p}) : k]$). Then the claim follows from 3.1.

COROLLARY 3.3 ([4] 1.7): *Assume that $n = 2$, that A contains a field of characteristic $\neq 2$, and that B' is a domain.*

Then B' is seminormal iff $v(D_{B'/A}) \leq 2$.

PROOF: B' is monogenic over A ([4] 1.1), then it is Gorenstein (see proof of 3.2) and B is tamely ramified over A . Then the claim follows from 3.1.

REMARK 3.4: In general $v(D_{B'/A}) \leq n$ does not imply that B' is seminormal, even when B' is Gorenstein and B tamely ramified over A (see counterexample 1.2 of [5]).

In [5] we proved that if B' and k are as in 3.2, and if the characteristic polynomial of x is $X^n - a$ ($a \in A$), then the following are equivalent:

(i) B' is seminormal.

(ii) $v(D_{B'/A}) \leq n$.

(iii) $a = ut^q$, where u is a unit in A and $q \leq n/(n - 1)$. Recently S.S. Abhyankar made us notice that when $n \geq 3$ (i), (ii), (iii) are also equivalent to:

(iv) B' is normal.

In fact, when $n \geq 3$, (iii) implies $v(a) \leq 1$: now, if $v(a) = 0$, then $v(D_{B'/A}) = v(n^n a^{n-1}) = 0$ and B' is normal by 1.1; if $v(a) = 1$, then B' is local and B' is normal by 1.8.

Moreover (iv) \rightarrow (i).

Now, supposing that A contains the n^{th} roots of 1, then there is the group G of automorphisms of B' , $G = \{\sigma_1, \dots, \sigma_n\}$, where $\sigma_i|_A = id_A$ and $\sigma_i(x) = x \cdot \xi^i$ (ξ a fixed primitive n^{th} root of 1) for every i ; for this group obviously $B'^G = A$. Therefore it is natural to conjecture that when $n \geq 3$, B' is Gorenstein and either B' is local or there is a finite group G of automorphisms of B' such that $B'^G = A$, then $v(D_{B'/A}) \leq n$ implies B' normal (notice that by 3.1 B' is seminormal). The following counterexample gives a negative answer to the conjecture.

COUNTEREXAMPLE 3.5: Let A, B, B' be as in 1.12. B' is Gorenstein and seminormal; moreover by 2.2 we have $v(D_{B'/A}) = n + f - 2g = 4 + 2 - 2 \leq 4$. But B' is not normal.

4. Globalization

Suppose X, Y are locally noetherian schemes, with Y integral normal, and let $\phi: X \rightarrow Y$ be a finite covering of degree n . From the going-up and going-down theorems ([3] cor. 2, p. 38 and th. 3, p. 56) it follows that if $x \in X$ is a point of codimension 1, then $y = \phi(x) \in Y$ is

a point of codimension 1, which implies that \mathcal{O}_y is a discrete valuation ring.

Now, when X is S_2 the seminormality and normality of X can be checked in codimension 1 (see [6] th. 2.6, p. 9 and the Krull-Serre criterion, [7] th. 39, p. 125) i.e. it is enough to look at $v_y(\mathcal{D}_y)$ for all $y \in Y$ of codimension 1.

LEMMA 4.1: (i) *If B' is Gorenstein, then B' is S_2 .*

(ii) *If E is any normal domain, and $B' = E[x]$ is a domain integral over A , then B' is S_2 and Gorenstein in codimension 1.*

PROOF: (i) By definition B' is Cohen-Macaulay, hence S_r for all r .

(ii) Since E is normal $\{1, x, \dots, x^{n-1}\}$ is a free basis of B' as an E -module. Now E is S_2 and the fibers of the canonical embedding $E \hookrightarrow B'$ are also S_2 , being 0-dimensional: therefore since B' is faithfully flat over E , B' is S_2 ([7] cor. 2, p. 154).

Moreover for every $\mathfrak{q} \in \text{spec } B'$ of height 1, we have $B'_\mathfrak{q} = (E_\Omega[x])_\mathfrak{q}$ where $\Omega = \mathfrak{q} \cap E$: now $E_\Omega[x]$ is Gorenstein because E_Ω is a discrete valuation ring and $E_\Omega[x]$ is a domain (see proof of 3.2), then $B'_\mathfrak{q}$ is Gorenstein and we are done.

By assuming X to be S_2 we can globalize 1.2, 1.8, 2.2, 2.8: by assuming X to be S_2 and Gorenstein in codimension 1, we can globalize in particular 3.1.

We wish to remark explicitly that when we assume X to be integral and locally monogenic over Y , then by 4.1 (ii) X is both S_2 and Gorenstein in codimension 1: which shows that the result obtained by globalizing 3.1 generalizes the analogous results of [4] and [5].

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Istituto Matematico
Politecnico di Torino
Torino (Italy)