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An application of the building to orbital integrals


<http://www.numdam.org/item?id=CM_1980__42_3_417_0>
Let \( G \) denote the set of \( F \)-points of a connected, semi-simple, algebraic group defined over a \( p \)-adic field \( F \). Let \( T \) be a Cartan subgroup of \( G \) and denote the set of regular elements in \( T \) by \( T' \). Let \( T_s \) be the maximal \( F \)-split torus contained in \( T \) and let \( dg \) be a \( G \)-invariant measure on the quotient \( T_s \backslash G \). For \( f \in C_c(G) \), the smooth functions of compact support on \( G \), and \( x \in T' \), the integral converges and is called an orbital integral. Let \( \Omega \) be the set of unipotent conjugacy classes in \( G \) and for each \( u \in \Omega \), let \( d\mu_u \) be a \( G \)-invariant measure on \( u \). The integral \( \Lambda_u(f) = \int f d\mu_u \) converges for \( f \in C_c(G) \). According to a theorem of [6], there are functions \( \Gamma^T(u) \) on \( T' \), one for each \( u \in \Omega \), called germs with the following property: for all \( f \in C_c(G) \), there is a neighborhood \( N(f) \) of \( 1 \) in \( G \) such that

\[
\Phi(x, f) = \sum_{u \in \Omega} \Lambda_u(f) \Gamma^T_u(x) \quad \text{for all } x \in N(f) \cap T'.
\]

Denote the germ associated to \( u = \{1\} \) by \( \Gamma^T_1 \) and define \( \Lambda_1(f) = f(1) \).

The theorem which we state below and prove in this paper was conjectured by Harish-Chandra [4] and Shalika [6].

**Theorem:** Let \( \pi_0 \) denote the special representation of \( G \) and let \( d(\pi_0) \) be its formal degree. Assume that \( T \) is a compact Cartan subgroup.
Then:

$$I^T_1 = \frac{(-1)^r}{d(\pi_0)}$$

where $r$ is the $F$-rank of $G$.

In [5], Howe proved, in the case $G = GL(n)$, that $I^T_1$ is a constant which is independent of the compact Cartan subgroup and Harish-Chandra extended his result to arbitrary $G$ in [4]. Our method is entirely different from the methods of [4] and [5]. The main tool used here is the Bruhat–Tits building associated to $G$. We assume that the reader is familiar with the theory and terminology of buildings as presented in [3]. The assumption that $F$ is of characteristic zero is essential because the exponential map is needed to prove the main lemmas.

Let $X$ be the Bruhat–Tits building associated to the simply-connected covering group of $G$ and let $X'$ be the set of vertices in $X$. If $p \in X$, we denote the stabilizer of $p$ in $G$ by $G_p$ and if $W$ is a subset of $G$, the set of points in $X$ which are fixed by all of the elements in $W$ is denoted by $S(W)$. If $M$ is any set, $\#(M)$ will denote the cardinality of $M$.

**Lemma 1:** Let $g \in G$ be an elliptic regular element. Then $S(g)$ is a compact subset of $G$.

**Proof:** Let $Y$ be the building of parabolic subgroups associated to $G$. Theorem 5.4 of [2] asserts that there is a topology on the set $Z = X \sqcup Y$ which extends the topology defined by the metric on $X$ and with respect to which $Z$ is compact and the action of $G$ is continuous. Suppose that $g \in G$ is elliptic and regular. Certainly $S(g)$ is a closed subset of $X$. If it is not bounded, there is a sequence $p_j$, $j = 1, 2, \ldots$, of points in $S(g)$ which is contained in no bounded subset of $X$. But since $Z$ is compact, there is a subsequence of the $p_j$ which converges to a point $z \in Y$. The action of $g$ on $Z$ being continuous, $g$ fixes $z$ and hence lies in a parabolic subgroup of $G$. This contradicts the assumption that $g$ is elliptic and regular. Therefore $S(g)$ is bounded and hence compact.

Assume from now on that $T$ is a compact Cartan subgroup of $G$. Let $\mathfrak{G}$ be the Lie algebra of $T$, let $O_F$ be the ring of integers of $F$, and choose a prime element $\tau$ in $O_F$. There is an open neighborhood $\mathfrak{G}^*$ of $O$ in $\mathfrak{G}$ such that $O_F \mathfrak{G}^* \subseteq \mathfrak{G}^*$ and such that $\exp: \mathfrak{G}^* \to T$ is defined. Choose $x \in T'$ in the image $\exp(\mathfrak{G}^*)$, say $x = \exp(H)$ for $H \in \mathfrak{G}^*$. For each non-negative integer $m$, put $U_m = \exp(\tau^m O_F H)$. If $m_1 \geq m_2$, then $U_{m_1} \subseteq U_{m_2}$ and $[U_{m_2}: U_{m_1}] = q^{m_2 - m_1}$ where $q$ is the cardinality of the residue field of $F$. Furthermore, $\bigcap_{m \geq 0} U_m = 1$. Since $U_0$ is a compact
For a subgroup of $G$, it stabilizes a point $p_0 \in X'$.

For $p$ and $q$ in $X$, let $d(p, q)$ be the geodesic distance from $p$ to $q$. Restricted to any apartment of $X$, $d(\cdot, \cdot)$ is a Euclidean metric [3]. For $d \geq 0$, $B_d$ will denote the set $\{p \in X : d(p, p_0) \leq d\}$.

**Lemma 2:** For each $d \geq 0$, there is a positive integer $m$ such that $U_m$ fixes all points $p \in B_d$.

**Proof:** Let $W$ be the set of vertices of $x$ which lie in some chamber which intersects $B_d$. Since $\#(W)$ is finite, $U_0 \cap \left( \bigcap_{p \in W} G_p \right)$ is an open subgroup of $U_0$, hence contains $U_m$ for some $m$. So $U_m$ fixes pointwise all chambers which intersect $B_d$ and in particular, all points in $B_d$.

**Lemma 3:** Let $x \in U_0$ and assume that $x \neq 1$. Then there is an integer $k \geq 0$ such that $S(xU_k) = S(x)$ and if $xp = p$ for some $p \in X$ and some $y \in U_k$, then $p \in S(x)$.

**Proof:** Since $x \neq 1$, it is elliptic regular and $S(x)$ is compact by lemma 1. By lemma 2, there is a $d \geq 0$ and an integer $k \geq 0$ such that $U_k$ fixes all points in $B_d$ and such that $S(x)$ is contained in the interior of $B_d$. For this $k$, $S(x) \subseteq S(xU_k)$. Now suppose that $p \in X$ is fixed by $xy$ for some $y \in U_k$. We must show that $p \in S(x)$. This is clearly so if $p \in S(U_k)$. If $p \not\in S(U_k)$, let $L$ be the geodesic line joining $p$ and $p_0$. It is fixed by $xy$ since $xy$ fixes $p_0$ and lies in an apartment $A$ of $X$. Furthermore, $L$ passes through a point on the boundary of the Euclidean ball $B_d \cap A$, say $q$. Then $xy$ and $y$ both fix $q$, hence $x$ does also -- a contradiction to the assumption on $B_d$.

**Corollary:** If a sequence $\{x_j\}$ of elements of $U_0$ converges to $x \neq 1$, then there is an $N \geq 0$ such that $S(x_j) = S(x)$ for all $j \geq N$.

**Proof:** If $x_j \to vx$, then the sequence $y_j = x^{-1}x_j$ approaches $1$. By the previous lemma, there is a $k \geq 0$ such that $S(x_j) = S(x)$ if $y_j \in U_k$. Choose $N$ so that $y_j \in U_k$ for all $j \geq N$.

**Lemma 4:** For each positive integer $m$, there is a $d \geq 0$ such that $G_p \cap U_0 \subseteq U_m$ for all $p \in X$ such that $p \not\in B_d$.

**Proof:** It suffices to show that for each infinite sequence $\{p_j\}$ of points in $X$ which is not bounded, there is an $N \geq 0$ such that
For each positive integer s, there is a d ≥ 0 such that

\[ \#(U_0p) \equiv 0 \mod q^t \] for all p ∈ X such that p ∉ B_d.

**Proof:** By lemma 4, there is a d ≥ 0 such that \( G_p \cap U_0 \subseteq U_s \) for all \( p \notin B_d \). Hence \( q^t = [U_0 : U_s] \) divides \( \#(U_0p) \) if \( p \notin B_d \).

When \( T \) is compact, \( T_s = \{1\} \) and the orbital integral is defined by giving a normalization of Haar measure on \( G \). The statement of the theorem is independent of this choice because the germs are proportional and the formal degrees are inversely proportional to a change of normalization of \( dg \). Let \( I \) be a fixed Iwahori subgroup of \( G \) and let \( C_I \) be the chamber in \( X \) which is pointwise fixed by \( I \). We choose the Haar measure \( dg \) on \( G \) which assigns measure one to \( I \). Let \( G_0 \) be the largest subgroup of \( G \) which acts on \( X \) by special automorphisms, i.e., which preserve the type of a face. Then \( G_0 \) is normal and of finite index in \( G \) [1]; let \( \#(G/G_0) = n \) and let \( \{g_0 = 1, g_1, \ldots, g_{n-1}\} \) be a set of representatives for \( G/G_0 \). We may assume that the \( g_i \) normalize \( I \) because the Iwahori subgroups of \( G \) are all conjugate under the action of \( G_0 \) [1]. For the rest of the paper, put \( x = \exp(H) \) for some regular \( H \in \mathbb{S}^* \), and put \( x_t = \exp(t^2H) \) for \( t \in O_F \). Let \( f_0 \) be the characteristic function of \( I \).

**Lemma 6:** Let \( c(t) \) = the number of chambers in \( X \) which are fixed by \( x_t \). Then \( \Phi(x_t, f_0) = nc(t) \).

**Proof:** First of all, \( I \) is contained in \( G_0 \), so

\[
\int_{G_0} f_0(g^{-1}x_t g) \, dg = \sum_{y \in G_0I, y^{-1}x_t} \gamma = c(t)
\]

since all Iwahori subgroups of \( G \) are conjugate in \( G_0 \) and, in particular, have measure one. Thus

\[
\Phi(x_t, f_0) = \sum_{j=0}^{n-1} \int_{G_0} f_0((gg_j)^{-1}x_t(gg_j)) \, dg = n \int_{G_0} f_0(g^{-1}x_t g) \, dg = nc(t)
\]

because of the assumption that the \( g_i \) normalize \( I \).
Let $d(u)$ be the dimension of $u$ for $u \in \Omega$. We recall from [4] that the $\Gamma_u^T$ satisfy the following property:

\[(*) \quad \Gamma_u^T(x_t) = |t|_F^{d(u)} \Gamma_u^T(x)\]

for all $t \in O_F$. For $t \in O_F$, $v(t)$ will denote the valuation of $t$, so that $|t|_F = q^{-v(t)}$. Let $m_j = \sum_{d(u)=j} \Lambda_u(f_0) \Gamma_u^T(x)$. There are only finitely many unipotent conjugacy classes in $G$. Let $M = \sup_{u \in \Omega} d(u)$. Furthermore, there is only one unipotent conjugacy class of dimension zero, hence $m_0 = \Gamma^T_1(x)$ since $f_0(1) = 1$. By lemma 6, $(*)$, and the germ expansion principle, there exists a $\delta > 0$ such that

\[(**) \quad \Phi(x_t, f_0) = \sum_{j=1}^M m_j q^{j(1)} + m_0 = nc(t) \text{ if } |t|_F < \delta.\]

**Lemma 7:** Let $Q$ be the rational numbers and let $Z^+$ be the set of positive integers. Let $a_0, \ldots, a_N$ be complex numbers and suppose that $F(n) = \sum_{j=0}^N a_j q^{jn}$ lies in $Q$ for almost all $n \in Z^+$. Then $a_j \in Q$ for $j = 0, 1, \ldots, n$.

**Proof:** We use induction on the degree, $N$, of $F(n)$. The lemma is certainly true if $N = 0$. If $N > 0$, let

\[F'(n) = q^{-n}(F(n) - F(n-1)) = \sum_{j=1}^{N} a_j(1 - q^{-j})q^{(j-1)n}.\]

$F'(n)$ has degree $N - 1$ and $F'(n) \in Q$ for almost all $n \in Z^+$ since this is true for $F$. By induction, $a_j \in Q$ for $j = 1, \ldots, N$ and this also implies that $a_0 \in Q$.

We apply lemma 7 to $(**)$ to conclude that the $m_j \in Q$: $nc(t)$ is obviously an integer for all $t \in O_F$ and $(**)$ holds if $v(t)$ is sufficiently large. The next lemma follows immediately.

**Lemma 8:** Let $p$ be the rational prime dividing $q$. Then the $p$-adic limit $\lim_{|t|_p \to 0} \Phi(x_t, f_0)$ exists and is equal to $m_0$.

Let $(W, S)$ be the Coxeter system associated to the Tits system for $G_0$ [1]. As in [1], let $T = \{t_s\} \subseteq S$ be a family of indeterminates indexed by elements of $S$ and for each $w \in W$, let $t_w = t_{s_1} \ldots t_s$ where $(s_1, \ldots, s)$ is a reduced decomposition for $w$, $s_i \in S$. The monomial $t_w$ is independent of the reduced decomposition of $w$. The formal power series $W(T) = \sum_{w \in W} t_w$ is called the Poincaré series of $(W, S)$. For $w \in W$, let
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$q_w = \#(I_0wI_0/I_0)$; it is a power of $q$ and the value $t_w(Q)$ is equal to $q_w$, where $Q$ denotes the substitution $t_s = q_s$.

**Lemma 9**: 1) $W(T)$ is a rational function of $T$ which is defined at the points $Q$ and $Q^{-1}$.  
2) $W(Q^{-1}) = (-1)^r W(Q)$.  
3) $d(\pi_0) = 1/nw(Q^{-1}) = (-1)^rnW(Q)$.

**Proof**: 1) and 2) are due to Serre [7], and 3) appears in [1].

The series $G = \sum_{w \in W} q_w$ converges in the $p$-adic topology because $q_w$ is a power of $q$ which tends to infinity as the length $l(w)$ (the number of elements in a reduced decomposition of $w$) approaches infinity. As a formal power series, $W(T)$ is equal to a rational function which is defined at $T = Q$ by the previous lemma. It is easy to see from this that the series $G$ converges $p$-adically to the value $W(Q)$.

To complete the proof of the theorem, we shall show that the $p$-adic limit, as $|t|_p \to 0$, of $c(t)$ is equal to $W(Q)$. This is sufficient, in view of lemma 8 which says that the $p$-adic limit, as $|t|_p \to 0$, of $nc(t)$ is equal to $I^r_1(x)$.

Let $B(d)$ be the union of all closed chambers in $X$ which are of the form $C = gC_t$ for some $g \in IwI$ with $l(w) \leq d$. Then $B(d) \subseteq B(d')$ if $d \leq d'$ and $\bigcup_{d \geq 0} B(d) = X$. It is clear that for each $d \geq 0$, there is a $d' \geq 0$ such that $B(d) \subseteq B(d')$ and for each $d \geq 0$, there is a $d' \geq 0$ such that $B_d \subseteq B(d')$. Therefore all of the lemmas involving $B_d$ also hold for $B(d) - \text{mutatis mutandis}$. Let $N(d)$ be the number of chambers contained in $B(d)$. Then $N(d)$ is a partial sum of the series $G$; it is equal to $\sum_{w \in W} q_w$ and hence $\lim_{d \to \infty} N(d) = W(Q)$ in the $p$-adic topology.

We may assume, without loss of generality, that $U_0 \subseteq I$. Let $Ch(t)$ be the set of chambers in $X$ which are fixed by $x_t$; $\#(Ch(t)) = c(t)$. Then

$$Ch(t) = (Ch(t) \cap B(d)) \cup (Ch(t) - (Ch(t) \cap B(d)))$$

Since $U_0$ commutes with $x_t$, it stabilizes the set $Ch(t)$ and the above assumption on $U_0$ implies that the action of $U_0$ on $Ch(t)$ preserves the two sets in the disjoint union of (**). Lemma 5 implies that, given a positive integer $s$, there is a positive $d_s$ which tends to infinity with $s$, such that $\#(U_0C)$ is divisible by $q^s$ for all $C \subseteq B(d_s)$. By lemma 2, there is a positive $\epsilon_s \to 0$ as $s \to \infty$, such that $x_t$ fixes all of the
chambers in $B(d_i)$ for $|t|_F \leq \epsilon_i$. Let $s$ tend to infinity and apply (***) to $d_i$ and $t_i$ where $|t_i|_F \leq \epsilon_i$. We have shown that the cardinality of the first term on the right hand side of (***\) approaches $W(Q)$ $p$-adically while the cardinality of the second term approaches zero $p$-adically.

QED.

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