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AN APPLICATION OF THE BUILDING TO ORBITAL INTEGRALS

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Let G denote the set of F-points of a connected, semi-simple, algebraic group defined over a p-adic field F. Let T be a Cartan subgroup of G and denote the set of regular elements in T by T'. Let T_s be the maximal F-split torus contained in T and let $d\dot{g}$ be a G-invariant measure on the quotient $T_s\backslash G$. For $f \in C_c^\infty(G)$, the smooth functions of compact support on G, and $x \in T'$, the integral

$$\Phi(x, f) = \int_{T_s\backslash G} f(g^{-1}xg)dg$$

converges and is called an *orbital integral.* Let Ω be the set of unipotent conjugacy classes in G and for each $u \in \Omega$, let $d\mu_u$ be a G-invariant measure on u. The integral $\Lambda_u(f) = \int f d\mu_u$ converges for $f \in C_c^\infty(G)$. According to a theorem of [6], there are functions Γ_u^T on T', one for each $u \in \Omega$, called *germs* with the following property: for all $f \in C_c^\infty(G)$, there is a neighborhood $N(f)$ of 1 in G such that

$$\Phi(x, f) = \sum_{u\in\Omega} \Lambda_u(f)\Gamma_u^T(x) \quad \text{for all } x \in N(f) \cap T'.$$

Denote the germ associated to $u = \{1\}$ by Γ_1^T and define $\Lambda_1(f) = f(1)$.

The theorem which we state below and prove in this paper was conjectured by Harish–Chandra [4] and Shalika [6].

THEOREM: *Let π_0 denote the special representation of G and let $d(\pi_0)$ be its formal degree. Assume that T is a compact Cartan subgroup.*

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Then:

$$\Gamma_1^T = \frac{(-1)^r}{d(\pi_0)} \quad \text{where } r = \text{the } F\text{-rank cf } G.$$

In [5], Howe proved, in the case $G = GL(n)$, that Γ_1^T is a constant which is independent of the compact Cartan subgroup and Harish–Chandra extended his result to arbitrary G in [4]. Our method is entirely different from the methods of [4] and [5]. The main tool used here is the Bruhat–Tits building associated to G. We assume that the reader is familiar with the theory and terminology of buildings as presented in [3]. The assumption that F is of characteristic zero is essential because the exponential map is needed to prove the main lemmas.

Let X be the Bruhat–Tits building associated to the simply-connected covering group of G and let X' be the set of vertices in X. If $p \in X$, we denote the stabilizer of p in G by G_p and if W is a subset of G, the set of points in X which are fixed by all of the elements in W is denoted by $S(W)$. If M is any set, $\#(M)$ will denote the cardinality of M.

LEMMA 1: *Let $g \in G$ be an elliptic regular element. Then $S(g)$ is a compact subset of G.*

PROOF: Let Y be the building of parabolic subgroups associated to G. Theorem 5.4 of [2] asserts that there is a topology on the set $Z = X \amalg Y$ which extends the topology defined by the metric on X and with respect to which Z is compact and the action of G is continuous. Suppose that $g \in G$ is elliptic and regular. Certainly $S(g)$ is a closed subset of X. If it is not bounded, there is a sequence p_j, $j = 1, 2, \ldots$, of points in $S(g)$ which is contained in no bounded subset of X. But since Z is compact, there is a subsequence of the p_j which converges to a point $z \in Y$. The action of g on Z being continuous, g fixes z and hence lies in a parabolic subgroup of G. This contradicts the assumption that g is elliptic and regular. Therefore $S(g)$ is bounded and hence compact.

Assume from now on that T is a compact Cartan subgroup of G. Let $\mathfrak{H}$ be the Lie algebra of T, let O_F be the ring of integers of F, and choose a prime element τ in O_F. There is an open neighborhood $\mathfrak{H}^*$ of O in $\mathfrak{H}$ such that $O_F\mathfrak{H}^* \subseteq \mathfrak{H}^*$ and such that $\exp: \mathfrak{H}^* \to T$ is defined. Choose $x \in T'$ in the image $\exp(\mathfrak{H}^*)$, say $x = \exp(H)$ for $H \in \mathfrak{H}^*$. For each non-negative integer m, put $U_m = \exp(\tau^m O_F H)$. If $m_1 \geq m_2$, then $U_{m_1} \subseteq U_{m_2}$ and $[U_{m_2}: U_{m_1}] = q^{m_1 - m_2}$ where q is the cardinality of the residue field of F. Furthermore, $\bigcap_{m \geq 0} U_m = 1$. Since U_0 is a compact subgroup of G, it stabilizes a point $p_0 \in X'$.

For p and q in X, let $d(p, q)$ be the geodesic distance from p to q. Restricted to any apartment of X, $d(\ ,\ )$ is a Euclidean metric [3]. For $d \geq 0$, B_d will denote the set $\{p \in X : d(p, p_0) \leq d\}$.

LEMMA 2: *For each $d \geq 0$, there is a positive integer m such that U_m fixes all points $p \in B_d$.*

PROOF: Let W be the set of vertices of x which lie in some chamber which intersects B_d. Since $\#(W)$ is finite, $U_0 \cap (\bigcap_{p \in W} G_p)$ is an open subgroup of U_0, hence contains U_m for some m. So U_m fixes pointwise all chambers which intersect B_d and in particular, all points in B_d.

LEMMA 3: *Let $x \in U_0$ and assume that $x \neq 1$. Then there is an integer $k \geq 0$ such that $S(xU_k) = S(x)$ and if $xyp = p$ for some $p \in X$ and some $y \in U_k$, then $p \in S(x)$.*

PROOF: Since $x \neq 1$, it is elliptic regular and $S(x)$ is compact by lemma 1. By lemma 2, there is a $d \geq 0$ and an integer $k \geq 0$ such that U_k fixes all points in B_d and such that $S(x)$ is contained in the interior of B_d. For this k, $S(x) \subseteq S(xU_k)$. Now suppose that $p \in X$ is fixed by xy for some $y \in U_k$. We must show that $p \in S(x)$. This is clearly so if $p \in S(U_k)$. If $p \not\in S(U_k)$, let L be the geodesic line joining p and p_0. It is fixed by xy since xy fixes p_0 and lies in an apartment A of X. Furthermore, L passes through a point on the boundary of the Euclidean ball $B_d \cap A$, say q. Then xy and y both fix q, hence x does also – a contradiction to the assumption on B_d.

COROLLARY: *If a sequence $\{x_j\}$ of elements of U_0 converges to $x \neq 1$, then there is an $N \geq 0$ such that $S(x_j) = S(x)$ for all $j \geq N$.*

PROOF: If $x_j \to \nu x$, then the sequence $y_j = x^{-1}x_j$ approaches 1. By the previous lemma, there is a $k \geq 0$ such that $S(x_j) = S(x)$ if $y_j \in U_k$. Choose N so that $y_j \in U_k$ for all $j \geq N$.

LEMMA 4: *For each positive integer m, there is a $d \geq 0$ such that $G_p \cap U_0 \subseteq U_m$ for all $p \in X$ such that $p \not\in B_d$.*

PROOF: It suffices to show that for each infinite sequence $\{p_j\}$ of points in X which is not bounded, there is an $N \geq 0$ such that

$G_{p_j} \cap U_0 \subseteq U_m$ for all $j \geq N$. If not, there is such a sequence p_j and elements $x_j \in U_0 - U_m$ such that x_j fixes p_j. Since U_0 is compact, we may, passing to a subsequence if necessary, assume that x_j converges to $x \in U_0 - U_m$. By the previous corollary, there is an $N \geq 0$ such that $S(x_j) = S(x)$ for all $j \geq N$. But $S(x)$ is compact – contradiction.

LEMMA 5: *For each positive integer s, there is a $d \geq 0$ such that $\#(U_0 p) \equiv 0 \bmod q^s$ for all $p \in X$ such that $p \notin B_d$.*

PROOF: By lemma 4, there is a $d \geq 0$ such that $G_p \cap U_0 \subseteq U_s$ for all $p \notin B_d$. Hence $q^s = [U_0 : U_s]$ divides $\#(U_0 p)$ if $p \notin B_d$.

When T is compact, $T_s = \{1\}$ and the orbital integral is defined by giving a normalization of Haar measure on G. The statement of the theorem is independent of this choice because the germs are proportional and the formal degrees are inversely proportional to a change of normalization of dg. Let I be a fixed Iwahori subgroup of G and let C_I be the chamber in X which is pointwise fixed by I. We choose the Haar measure dg on G which assigns measure one to I. Let G_0 be the largest subgroup of G which acts on X by special automorphisms, i.e., which preserve the type of a face. Then G_0 is normal and of finite index in G [1]; let $\#(G/G_0) = n$ and let $\{g_0 = 1, g_1, \ldots, g_{n-1}\}$ be a set of representatives for G/G_0. We may assume that the g_j normalize I because the Iwahori subgroups of G are all conjugate under the action of G_0 [1]. For the rest of the paper, put $x = \exp(H)$ for some regular $H \in \mathfrak{H}^*$, and put $x_t = \exp(t^2 H)$ for $t \in O_F$. Let f_0 be the characteristic function of I.

LEMMA 6: *Let $c(t)$ = the number of chambers in X* which are fixed by x_t. *Then $\Phi(x_t, f_0) = nc(t)$.*

PROOF: First of all, I is contained in G_0, so

$$\int_{G_0} f_0(g^{-1}x_t g)\, \mathrm{d}g = \sum_{\substack{y \in G_0/I \\ x_t \in yIy^{-1}}} 1 = c(t)$$

since all Iwahori subgroups of G are conjugate in G_0 and, in particular, have measure one. Thus

$$\Phi(x_t, f_0) = \sum_{j=0}^{n-1} \int_{G_0} f_0((gg_j)^{-1} x_t (gg_j))\, \mathrm{d}g = n \int_{G_0} f_0(g^{-1}x_t g)\, \mathrm{d}g = nc(t)$$

because of the assumption that the g_j normalize I.

Let $d(u)$ be the dimension of u for $u \in \Omega$. We recall from [4] that the Γ_u^T satisfy the following property:

$$(*) \qquad \Gamma_u^T(x_t) = |t|_F^{-d(u)} \Gamma_u^T(x)$$

for all $t \in O_F$. For $t \in O_F$, $v(t)$ will denote the valuation of t, so that $|t|_F = q^{-v(t)}$. Let $m_j = \sum_{d(u)=j} \Lambda_u(f_0)\Gamma_u^T(x)$. There are only finitely many unipotent conjugacy classes in G. Let $M = \sup_{u \in \Omega} d(u)$. Furthermore, there is only one unipotent conjugacy class of dimension zero, hence $m_0 = \Gamma_1^T(x)$ since $f_0(1) = 1$. By lemma 6, (*), and the germ expansion principle, there exists a $\delta > 0$ such that

$$(**) \qquad \Phi(x_t, f_0) = \sum_{j=1}^{M} m_j q^{jv(t)} + m_0 = nc(t) \quad \text{if } |t|_F < \delta.$$

Lemma 7: *Let Q be the rational numbers and let Z^+ be the set of positive integers. Let $a_0, \ldots, a_N$ be complex numbers and suppose that $F(n) = \sum_{j=0}^{N} a_j q^{jn}$ lies in Q for almost all $n \in Z^+$. Then $a_j \in Q$ for $j = 0, 1, \ldots, n$.*

Proof: We use induction on the degree, N, of $F(n)$. The lemma is certainly true if $N = 0$. If $N > 0$, let

$$F'(n) = q^{-n}(F(n) - F(n-1)) = \sum_{j=1}^{N} a_j(1 - q^{-j})q^{(j-1)n}.$$

$F'(n)$ has degree $N - 1$ and $F'(n) \in Q$ for almost all $n \in Z^+$ since this is true for F. By induction, $a_j \in Q$ for $j = 1, \ldots, N$ and this also implies that $a_0 \in Q$.

We apply lemma 7 to (**) to conclude that the $m_j \in Q$: $nc(t)$ is obviously an integer for all $t \in O_F$ and (**) holds if $v(t)$ is sufficiently large. The next lemma follows immediately.

Lemma 8: *Let p be the rational prime dividing q. Then the p-adic limit $\lim_{|t|_F \to 0} \Phi(x_t, f_0)$ exists and is equal to m_0.*

Let (W, S) be the Coxeter system associated to the Tits system for G_0 [1]. As in [1], let $T = \{t_s\}_{s \in S}$ be a family of indeterminates indexed by elements of S and for each $w \in W$, let $t_w = t_{s_1} \ldots t_s$ where $(s_1, \ldots, s)$ is a reduced decomposition for w, $s_i \in S$. The monomial t_w is independent of the reduced decomposition of w. The formal power series $W(T) = \sum_{w \in W} t_w$ is called the Poincaré series of (W, S). For $w \in W$, let

$q_w = \#(I_0wI_0/I_0)$; it is a power of q and the value $t_w(Q)$ is equal to q_w, where Q denotes the substitution $t_s = q_s$.

LEMMA 9: 1) *$W(T)$ is a rational function of T which is defined at the points Q and Q^{-1}.*
2) $W(Q^{-1}) = (-1)^r W(Q)$.
3) $d(\pi_0) = 1/nw(Q^{-1}) = (-1)^r/nW(Q)$.

PROOF: 1) and 2) are due to Serre [7], and 3) appears in [1].

The series $G = \sum_{w\in W} q_w$ converges in the p-adic topology because q_w is a power of q which tends to infinity as the length $1(w)$ (the number of elements in a reduced decomposition of w) approaches infinity. As a formal power series, $W(T)$ is equal to a rational function which is defined at $T = Q$ by the previous lemma. It is easy to see from this that the series G converges p-adically to the value $W(Q)$.

To complete the proof of the theorem, we shall show that the p-adic limit, as $|t|_F \to 0$, of $c(t)$ is equal to $W(Q)$. This is sufficient, in view of lemma 8 which says that the p-adic limit, as $|t|_F \to 0$, of $nc(t)$ is equal to $\Gamma_1^T(x)$.

Let $B(d)$ be the union of all closed chambers in X which are of the form $C = gC_I$ for some $g \in IwI$ with $1(w) \le d$. Then $B(d) \subseteq B(d')$ if $d \le d'$ and $\bigcup_{d\ge 0} B(d) = X$. It is clear that for each $d \ge 0$, there is a $d' \ge 0$ such that $B(d) \subseteq B_{d'}$ and for each $d \ge 0$, there is a $d' \ge 0$ such that $B_d \subseteq B(d')$. Therefore all of the lemmas involving B_d also hold for $B(d)$ – *mutatis mutandis*. Let $N(d)$ be the number of chambers contained in $B(d)$. Then $N(d)$ is a partial sum of the series G; it is equal to $\sum_{\substack{w\in W\\ 1(w)\le d}} q_w$ and hence $\lim_{d\to\infty} N(d) = W(Q)$ in the p-adic topology.

We may assume, without loss of generality, that $U_0 \subseteq I$. Let $\mathrm{Ch}(t)$ be the set of chambers in X which are fixed by x_t; $\#(\mathrm{Ch}(t)) = c(t)$. Then

$$(**) \qquad \mathrm{Ch}(t) = (\mathrm{Ch}(t) \cap B(d)) \cup (\mathrm{Ch}(t) - (\mathrm{Ch}(t) \cap B(d)).$$

Since U_0 commutes with x_t, it stabilizes the set $\mathrm{Ch}(t)$ and the above assumption on U_0 implies that the action of U_0 on $\mathrm{Ch}(t)$ preserves the two sets in the disjoint union of (***). Lemma 5 implies that, given a positive integer s, there is a positive d_s which tends to infinity with s, such that $\#(U_0C)$ is divisible by q^s for all $C \not\subseteq B(d_s)$. By lemma 2, there is a positive $\epsilon_s \to 0$ as $s \to \infty$, such that x_t fixes all of the chambers in $B(d_s)$ for $|t|_F \leq \epsilon_s$. Let s tend to infinity and apply (***) to d_s and t_s where $|t_s|_F \leq \epsilon_s$. We have shown that the cardinality of the first term on the right hand side of (***) approaches $W(Q)$ p-adically while the cardinality of the second term approaches zero p-adically.

QED.

REFERENCES

[1] A. BOREL: Admissible Representations of a Semi-simple Group over a Local Field *Inventiones Mathematicae* 35 (1976) 233–259.

[2] A. BOREL and J.P. SERRE: Cohomologie D'Immeubles et de Groupes S-Arithmetiques. *Topology* 15, No. 3, (1976) 211–232.

[3] F. BRUHAT and J. TITS: Groupes Réductifs sur un Corps Local. Publ. *Math. IHES 25* (1972).

[4] HARISH-CHANDRA: Admissible Invariant Distributions on Reductive p-adic Groups. *Lie Theories and applications, Queen's papers in pure and applied mathematics 48* (1978) 281–347.

[5] R. HOWE: The Fourier Transform and Germs of Characters. *Math. Annalen 208* (1974) 305–322.

[6] J.A. SHALIKA: A Theorem on Semi-simple p-adic Groups. *Annals of Math. 95, No.* 1 (1972) 226–242.

[7] J.P. SERRE: Cohomologie des Groupes Discrets. Prospects in Mathematics *Annals of Math. Studies 70.* Princeton University Press 1970.

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