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ON THE LIMITING DISTRIBUTION OF NON NEGATIVE ADDITIVE FUNCTIONS

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Let A be a set of positive integers and let for $N \geq 2$

$$A(N) = \sum_{n \in A, n \leq N} 1.$$

Let N_0 be the smallest element of A.
Let f be an arithmetic function and define for $N \geq N_0$,

$$F_{N,A}(t) = \frac{1}{A(N)} \sum_{\substack{n \leq N, n \in A \\ f(n) \leq t}} 1.$$

When A is fixed, or $A = N^*$, the set of positive integers, we write simply $F_N(t)$.

We say that f has a limiting distribution of the set A if there exists a non-decreasing function F satisfying $\lim_{t \to -\infty} F(t) = 0$, $\lim_{t \to +\infty} F(t) = 1$ and, at every continuity point, t, of F, $F_N(t)$ tends, when $N \to +\infty$, to $F(t)$.

Erdös and Wintner [2] showed that an additive function f (i.e. $f(m \cdot n) = f(m) + f(n)$ for every coprime m and n) has a limiting distribution on the set of all integers if and only if the following series converge

$$\sum_{|f(p)| \leq 1} \frac{f(p)}{p}, \quad \sum_{|f(p)| \leq 1} \frac{|f(p)|^2}{p}, \quad \sum_{|f(p)| > 1} \frac{1}{p}.$$

Kátai [3] proved that if these series converge, then the additive function f has a limiting distribution on the set $\{p + 1 \mid p \text{ is prime}\}$.

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Elliott [1] proved that if $f(p) \geq 0$ for every prime p and $f(p^r) = f(p)$ if $r \geq 1$ and if f has a limiting distribution on the set $\{p+1\}$ these series converge. We shall be concerned in the following with non negative additive functions (i.e. $f(p^r) \geq 0$ for every prime power p^r).

THEOREM: *Let A be a set of positive integers satisfying*
(i) *for every* $d \in \mathbb{N}^*$,

$$\lim_{N\to\infty} \frac{1}{A(N)} \sum_{\substack{d|n, n \leq N \\ n \in A}} 1$$

exists and is equal to $\omega(d)/d$.
(ii) ω *is a multiplicative function satisfying*

$$\sum_p \sum_{r \geq 2} \frac{\omega(p^r)}{p^r} < \infty.$$

Let f be a non negative additive function satisfying

$$\sum_{0 \leq f(p) \leq 1} \frac{f(p)\omega(p)}{p} + \sum_{1 < f(p)} \frac{\omega(p)}{p} = +\infty$$

then f has not a limiting distribution on the set A and more precisely

$$\lim_{N\to+\infty} F_N(t) = 0 \quad \textit{for every } t.$$

COROLLARY 1: *If* $A = \{p+1\}$ *and if f is a non negative additive function satisfying*

$$\sum_{0 \leq f(p) \leq 1} \frac{f(p)}{p} = +\infty \quad \textit{or} \quad \sum_{1 < f(p)} \frac{1}{p} = \infty$$

then f has not a limiting distribution on A and $\lim_{N\to\infty} F_N(t) = 0$ *for every t.*

This corollary contains Elliott's result.

COROLLARY 2: *Let A be a set of positive integers such that*

$$\liminf \frac{A(N)}{N} > 0.$$

If f is a non negative additive function and if

$$\sum_{0\le f(p)\le 1} \frac{f(p)}{p} = +\infty \quad \textit{or} \quad \sum_{1<f(p)} \frac{1}{p} = +\infty$$

then f has not a limiting distribution on A and $\lim_{N\to\infty} F_{N,A}(t) = 0$.

PROOF OF THE COROLLARIES: Corollary 1 is immediate by remarking that for every

$$d \in \mathbb{N}^*, \ \frac{1}{A(N)} \sum_{\substack{p+1\le N \\ d\mid p+1}} 1 \text{ tends to } \frac{1}{\varphi(d)} \text{ that is } \omega(d) = \frac{d}{\varphi(d)}$$

where φ is Euler's function.

For the proof of Corollary 2, let

$$F_{N,A}(t) = \frac{1}{A(N)} \sum_{\substack{n\le N, n\in A \\ f(n)\le t}} 1$$

and

$$F_N(t) = \frac{1}{N} \sum_{\substack{n\le N \\ f(n)\le t}} 1.$$

From the theorem with $A = \mathbb{N}^*$ we see that $\lim_{N\to\infty} F_N(t) = 0$. As

$$F_{N,A}(t) \le \frac{N}{A(N)} F_N(t)$$

we get $\lim F_{N,A}(t) = 0$ for every t.

PROOF OF THE THEOREM: We first remark that

$$\sum_p \frac{\omega(p)(1-e^{-f(p)})}{p} = +\infty.$$

This is easily deduced from the following inequalities:

$$(1-1/e)t \le 1-e^{-t} \le t \quad \text{if } 0 \le t \le 1$$

$$(1-1/e) \le 1-e^{-t} \le 1 \quad \text{if } t > 1$$

and the hypothese on f.

For $y \ge 2$, define the non negative additive function f_y by

$$f_y(p^r) = \begin{cases} f(p^r) & \text{if } p^r \le y \\ 0 & \text{if } p^r > y \end{cases} \quad \text{for every prime power } p^r.$$

Let $g_y = e^{-f_y} * \mu$ where μ is the Möbius function. Clearly g_y is multiplicative, $g_y(p^r) = e^{-f_y(p^r)} - e^{-f_y(p^{r-1})}$ which shows that $|g_y(p^r)| \le 2$ and that $g_y(n) = 0$ except on a finite set of integers S_y, say.

Let $\Pi_y = \sum \frac{g_y(n)}{n} \omega(n)$. Then one sees easily that

$$\Pi_y = \prod_{p \le y} \left(1 - \frac{(1 - e^{-f(p)})\omega(p)}{p} + \sum_{r \ge 2} \frac{g_y(p^r)\omega(p^r)}{p^r}\right).$$

As $\sum_p \frac{(1 - e^{-f(p)})\omega(p)}{p} = +\infty$ and $\sum_p \sum_{r \ge 2} \frac{\omega(p^r)}{p^r} < \infty$ we get $\lim_{y \to \infty} \Pi_y = 0$.

Now, as $e^{-f_y(n)} = \sum_{d \mid n} g_y(d)$ we have

$$\frac{1}{A(N)} \sum_{\substack{n \le N \\ n \in A}} e^{-f_y(n)} = \sum_{d \in S_y} g_y(d) \frac{1}{A(N)} \sum_{\substack{n \le N \\ n \in A, d \mid n}} 1$$

and so

$$\lim_{N \to \infty} \frac{1}{A(N)} \sum_{\substack{n \le N \\ n \in A}} e^{-f_y(n)} = \sum \frac{g_y(d)\omega(d)}{d} = \Pi_y.$$

Remarking that

$$e^{-f(n)} = \prod_{p^r \| n} e^{-f(p^r)} \le \prod_{\substack{p^r \| n \\ p^r \le y}} e^{-f(p^r)} = e^{-f_y(n)}$$

we obtain

$$\varlimsup_{N \to \infty} \frac{1}{A(N)} \sum_{\substack{n \le N \\ n \in A}} e^{-f(n)} \le \Pi_y$$

and by taking the limit when $y \to \infty$ we get

$$\varlimsup_{N \to \infty} \frac{1}{A(N)} \sum_{\substack{n \le N \\ n \in A}} e^{-f(n)} = 0.$$

As

$$\frac{1}{A(N)} \sum_{\substack{n \le N \\ n \in A}} e^{-f(n)} = \int_0^\infty e^{-x} \, \mathrm{d}F_N(x)$$

and for every t

$$\int_0^\infty e^{-x}\, \mathrm{d}F_N(x) \geq e^{-t}F_N(t)$$

we obtain

$$\overline{\lim_{N\to\infty}}\, F_N(t) = 0$$

and so the theorem.

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