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ON THE LIMITING DISTRIBUTION OF NON NEGATIVE ADDITIVE FUNCTIONS

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Let $A$ be a set of positive integers and let for $N \geq 2$

$$A(N) = \sum_{n \in A, n \leq N} 1.$$ 

Let $N_0$ be the smallest element of $A$.

Let $f$ be an arithmetic function and define for $N \geq N_0$,

$$F_{N,A}(t) = \frac{1}{A(N)} \sum_{n \in N, n \in A \atop f(n) \leq t} 1.$$ 

When $A$ is fixed, or $A = N^*$, the set of positive integers, we write simply $F_N(t)$.

We say that $f$ has a limiting distribution of the set $A$ if there exists a non-decreasing function $F$ satisfying $\lim_{t \to -\infty} F(t) = 0$, $\lim_{t \to +\infty} F(t) = 1$ and, at every continuity point, $t$, of $F$, $F_N(t)$ tends, when $N \to +\infty$, to $F(t)$.

Erdős and Wintner [2] showed that an additive function $f$ (i.e. $f(m \cdot n) = f(m) + f(n)$ for every coprime $m$ and $n$) has a limiting distribution on the set of all integers if and only if the following series converge

$$\sum_{|f(p)| \leq 1} \frac{f(p)}{p}, \quad \sum_{|f(p)| \leq 1} \frac{|f(p)|^2}{p}, \quad \sum_{|f(p)| > 1} \frac{1}{p}.$$ 

Kátai [3] proved that if these series converge, then the additive function $f$ has a limiting distribution on the set $\{p + 1 \mid p \text{ is prime}\}$. 

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Elliott [1] proved that if $f(p) \geq 0$ for every prime $p$ and $f(p^r) = f(p)$ if $r \geq 1$ and if $f$ has a limiting distribution on the set $\{p + 1\}$ these series converge. We shall be concerned in the following with non negative additive functions (i.e. $f(p^r) \geq 0$ for every prime power $p^r$).

**Theorem:** Let $A$ be a set of positive integers satisfying
(i) for every $d \in \mathbb{N}^*$,

$$\lim_{N \to \infty} \frac{1}{A(N)} \sum_{d|n, n \leq N} \frac{1}{n}$$

exists and is equal to $\omega(d)/d$.

(ii) $\omega$ is a multiplicative function satisfying

$$\sum_p \sum_{r \geq 2} \frac{\omega(p^r)}{p^r} < \infty.$$

Let $f$ be a non negative additive function satisfying

$$\sum_{0 \leq f(p) \leq 1} \frac{f(p)\omega(p)}{p} + \sum_{1 < f(p)} \frac{\omega(p)}{p} = +\infty$$

then $f$ has not a limiting distribution on the set $A$ and more precisely

$$\lim_{N \to \infty} F_N(t) = 0 \quad \text{for every } t.$$

**Corollary 1:** If $A = \{p + 1\}$ and if $f$ is a non negative additive function satisfying

$$\sum_{0 \leq f(p) \leq 1} \frac{f(p)}{p} = +\infty \quad \text{or} \quad \sum_{1 < f(p)} \frac{1}{p} = \infty$$

then $f$ has not a limiting distribution on $A$ and $\lim_{N \to \infty} F_N(t) = 0$ for every $t$.

This corollary contains Elliott’s result.

**Corollary 2:** Let $A$ be a set of positive integers such that

$$\lim \inf \frac{A(N)}{N} > 0.$$
If $f$ is a non negative additive function and if

$$\sum_{0 \leq f(p) \leq 1} \frac{f(p)}{p} = +\infty \quad \text{or} \quad \sum_{1 \leq f(p)} \frac{1}{p} = +\infty$$

then $f$ has not a limiting distribution on $A$ and $\lim_{N \to \infty} F_{N,A}(t) = 0$.

**Proof of the Corollaries:** Corollary 1 is immediate by remarking that for every

$$d \in \mathbb{N}^*, \frac{1}{A(N)} \sum_{p \leq N \atop d \mid p + 1} 1 \text{ tends to } \frac{1}{\varphi(d)} \text{ that is } \omega(d) = \frac{d}{\varphi(d)}$$

where $\varphi$ is Euler’s function.

For the proof of Corollary 2, let

$$F_{N,A}(t) = \frac{1}{A(N)} \sum_{n \leq N \atop f(n) > t} 1$$

and

$$F_N(t) = \frac{1}{N} \sum_{n \leq N} 1.$$

From the theorem with $A = \mathbb{N}^*$ we see that $\lim_{N \to \infty} F_N(t) = 0$. As

$$F_{N,A}(t) \leq \frac{N}{A(N)} F_N(t)$$

we get $\lim F_{N,A}(t) = 0$ for every $t$.

**Proof of the Theorem:** We first remark that

$$\sum_{p} \omega(p)(1 - e^{-f(p)}) = +\infty.$$

This is easily deduced from the following inequalities:

$$(1 - 1/e)t \leq 1 - e^{-t} \leq t \quad \text{if } 0 \leq t \leq 1$$

$$(1 - 1/e) \leq 1 - e^{-t} \leq 1 \quad \text{if } t > 1$$

and the hypothese on $f$.

For $y \geq 2$, define the non negative additive function $f$, by
let \( f_y(p^r) = \begin{cases} 1 & \text{if } p^r \leq y \\ 0 & \text{if } p^r > y \end{cases} \) for every prime power \( p^r \).

Let \( g_y = e^{-f_y} \ast \mu \) where \( \mu \) is the Möbius function. Clearly \( g_y \) is multiplicative, \( g_y(p^r) = e^{-f_y(p^r)} - e^{-f_y(p^{r-1})} \) which shows that \( |g_y(p^r)| \leq 2 \) and that \( g_y(n) \equiv 0 \) except on a finite set of integers \( S_y \), say.

Let \( \Pi_y = \sum_{n \leq y} \frac{g_y(n)}{n} \omega(n) \). Then one sees easily that

\[
\Pi_y = \prod_{p \leq y} \left( 1 - \frac{(1 - e^{-f_y(p)})\omega(p)}{p} + \sum_{r \geq 2} \frac{g_y(p^r)\omega(p^r)}{p^r} \right).
\]

As \( \sum_p \frac{(1 - e^{-f_y(p)})\omega(p)}{p} = +\infty \) and \( \sum_p \sum_{r \geq 2} \frac{\omega(p^r)}{p^r} < \infty \) we get \( \lim_{y \to \infty} \Pi_y = 0 \).

Now, as \( e^{-f_y(n)} = \sum_{d \mid n} g_y(d) \) we have

\[
\frac{1}{A(N)} \sum_{n \leq N} e^{-f_y(n)} = \sum_{d \in S_y} g_y(d) \frac{1}{A(N)} \sum_{n \leq N, n \mid d} 1,
\]

and so

\[
\lim_{N \to \infty} \frac{1}{A(N)} \sum_{n \leq N} e^{-f_y(n)} = \sum_{d \in S_y} g_y(d)\omega(d) = \Pi_y.
\]

Remarking that

\[
e^{-f_y(n)} = \prod_{p^r \mid n} e^{-f_y(p^r)} \leq \prod_{p^r \mid n} e^{-f_y(p^{r-1})} = e^{-f_y(n)}
\]

we obtain

\[
\lim_{N \to \infty} \frac{1}{A(N)} \sum_{n \leq N} e^{-f_y(n)} \leq \Pi_y
\]

and by taking the limit when \( y \to \infty \) we get

\[
\lim_{N \to \infty} \frac{1}{A(N)} \sum_{n \leq N} e^{-f_y(n)} = 0.
\]

As

\[
\frac{1}{A(N)} \sum_{n \leq N} e^{-f_y(n)} = \int_0^\infty e^{-x} \ dF_N(x)
\]
and for every $t$

$$\int_0^\infty e^{-x} \, dF_N(x) \geq e^{-t}F_N(t)$$

we obtain

$$\lim_{N \to \infty} F_N(t) = 0$$

and so the theorem.

REFERENCES


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