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REDUCIBILITY OF THE COMPACTIFIED JACOBIAN

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Let X be an integral, projective curve of arithmetic genus p over an algebraically closed ground field k. Denote by P the compactified jacobian of X, defined as the moduli space of torsion-free sheaves on X with rank 1 and Euler characteristic $1-p$. Altman, Iarrobino and Kleiman proved an irreducibility theorem [1, Theorem (9)]: P is irreducible if X lies on a smooth surface, or equivalently, if the embedding dimension at each point of X is at most two [3, Corollary (9)]. They also constructed an example [1, Example (13)] of an X which is a complete intersection in $\mathbb{P}^3$ and for which P is reducible. The example suggests that the converse of the theorem holds. In the present article, we prove the converse in the following form.

THEOREM (1): *If X does not lie on a smooth surface, then the compactified jacobian P is reducible.*

Rego [5] asserted Theorem (1) and offered a sketchy proof, which runs as follows. First he showed that $\mathrm{Hilb}^2(X)$ is reducible if X does not lie on a smooth surface. Then, if X is also Gorenstein, he concluded that P is reducible from the fact that the Abel map, $\mathrm{Hilb}^n(X) \to P$, is smooth for large n. This map is no longer smooth if X is not Gorenstein, and so Rego devised other methods to obtain reducibility in general.

However, Altman and Kleiman [2] developed a theory in which $\mathrm{Quot}^n(\omega/X)$, where ω is the dualizing sheaf on X, replaces $\mathrm{Hilb}^n(X)$ as the source of an Abel map,

$$A^n_\omega : \mathrm{Quot}^n(\omega/X) \to P.$$

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Whether or not X is Gorenstein, A_ω^n is smooth and its fibers are projective spaces for all $n \geq 2p-1$ [2, Theorem (8.4) (v), Lemma (5.17) (ii) and Theorem (4.2)]. Hence, P will be reducible if $\mathrm{Quot}^n(\omega/X)$ is reducible for large n.

This reducibility is proved below in two steps. First, we show that, if $\mathrm{Quot}^m(\omega/X)$ is reducible, then $\mathrm{Quot}^n(\omega/X)$ is reducible for $n \geq m$ (Proposition (3)). Secondly, we show that, if X does not lie on a smooth surface, then $\mathrm{Quot}^d(\omega/X)$ is reducible, for small d, in fact, for $d=2$ if X is Gorenstein, and for $d=1$ if X is not Gorenstein (Proposition (4)). Thus, by a natural adaptation of part of Rego's method, Theorem (1) is proved.

Fix a torsion-free, rank-1 sheaf $\mathcal{F}$ on X. Denote by U the open subscheme of X consisting of nonsingular points.

The open subscheme Q^nU of $\mathrm{Quot}^n(\mathcal{F}/X)$ parameterizing quotients of $\mathcal{F}$ with support contained in U is isomorphic to $\mathrm{Hilb}^n(U)$, because $\mathcal{F}$ restricted to U is invertible; so Q^nU is irreducible of dimension n [1, Lemma (1)]. Hence, $\mathrm{Quot}^n(\mathcal{F}/X)$ is irreducible if and only if Q^nU is dense in $\mathrm{Quot}^n(\mathcal{F}/X)$. Using the valuative criterion [4, Ch. II, Prop. 7.1.4 (i)], we therefore get Lemma (2) below.

LEMMA (2): $\mathrm{Quot}^n(\mathcal{F}/X)$ *is irreducible if and only if, for all quotients* F *of* $\mathcal{F}$ *of length* n, *there exists a scheme* $T = \mathrm{Spec}(A)$, *where* A *is a complete, discrete valuation ring, and a* T*-flat quotient* $\bar{F}$ *of* $\mathcal{F}_T$ *such that* $\bar{F}(t) \simeq F$ *and* $\mathrm{Supp}\,\bar{F}(g) \subseteq U_T(g)$, *where* t *and* g *denote the closed and generic points of* T.

PROPOSITION (3): *If* $\mathrm{Quot}^n(\mathcal{F}/X)$ *is irreducible, then* $\mathrm{Quot}^m(\mathcal{F}/X)$ *is irreducible for all* $m < n$.

PROOF: Let F be a quotient of $\mathcal{F}$ of length m. Let I denote the kernel of the natural map $\mathcal{F} \to F$ and let $x_1, \ldots, x_{n-m}$ be different nonsingular points on X such that $x_i \notin \mathrm{Supp}\, F$ for $i = 1, \ldots, n-m$. Then

$$F' = \mathcal{F}/M_1 \ldots M_{n-m}I,$$

where M_i denotes the ideal of x_i, is a quotient of $\mathcal{F}$ of length n. By Lemma (2) there exists a complete, discrete valuation ring A and a quotient $\bar{F}'$ of $\mathcal{F}_T$, $T = \mathrm{Spec}(A)$, with all the properties listed in that lemma and such that $\bar{F}'(t) \simeq F'$.

Let W be the closed subscheme of X_T defined by the annihilator of $\bar{F}'$. It is easy to see that we have an inclusion

$$\{x_1\} \cup \cdots \cup \{x_{n-m}\} \cup V \subseteq W(t),$$

where V is the closed subscheme of X defined by the annihilator of F. Hence, since A is a henselian ring [4, Ch. IV, Prop. 18.5.14], W may be written in the form,

$$W = W_1 \oplus \cdots \oplus W_{n-m} \oplus W',$$

where $\{x_i\} \subseteq W_i(t)$ and $V \subseteq W'(t)$ [4, Ch. IV, Théorème 18.5.11(c)].

Denote by i the inclusion $W' \subseteq X_T$ and put

$$\bar{F} = i_* i^* \bar{F}'.$$

Then $\bar{F}$ is a flat quotient of $\bar{F}'$ and $\bar{F}(t) \simeq F$. Hence, by Lemma (2) the proposition is proved.

PROPOSITION (4): *Let x be a point of X and denote by M the ideal defining x.*
(a) *If* $\dim_k(\omega/M\omega) \geq 2$, *then* $\mathrm{Quot}^1(\omega/X)$ *is reducible.*
(b) *If* $\dim_k(\omega/M\omega) = 1$ *and if* $\dim_k(M/M^2) \geq 3$, *then* $\mathrm{Quot}^2(\omega/X)$ *is reducible.*

PROOF: (a). Set $\omega_1 = \omega/M\omega$. Obviously the functors $\underline{\mathrm{Quot}}^1(\omega_1/X)$ and $\underline{\mathrm{Grass}}_1(\omega_1/k)$ are isomorphic. Since $\dim_k(\omega_1) \geq 2$, $\mathrm{Grass}_1(\omega_1/k)$ has dimension at least 1. Hence, since $\mathrm{Quot}^1(\omega_1/X)$ is a closed subscheme of $\mathrm{Quot}^1(\omega/X)$, we therefore get

$$\dim \mathrm{Quot}^1(\omega/X) \geq 1.$$

If equality holds, $\mathrm{Quot}^1(\omega/X)$ is reducible since $\mathrm{Quot}^1(\omega_1/X)$ is a closed 1-dimensional subscheme which is obviously different from $\mathrm{Quot}^1(\omega/X)$. If equality fails, then the closure of Q^1U is a component of dimension 1, and so $\mathrm{Quot}^1(\omega/X)$ is reducible.
(b) ω is torsion-free [2, 6.5], so ω is invertible at x because $\dim_k(\omega/M\omega) = 1$. Since $\dim_k(M/M^2) \geq 3$, we get that

$$\dim_k(M\omega/M^2\omega) \geq 3.$$

Set $\omega_2 = \omega/M^2\omega$. A vector subspace of $M\omega/M^2\omega$ of codimension 1 corresponds to a quotient of ω_2 of length 2. It is not hard to see that this correspondence extends to families of quotients and vector subspaces, so that $\underline{\mathrm{Grass}}_1(M\omega/M^2\omega)$ can be considered as a subfunctor of $\underline{\mathrm{Quot}}^2(\omega_2/X)$. Hence, since a proper monomorphism is a closed embedding [4, Ch. IV, Prop. 8.11.5], $\mathrm{Quot}^2(\omega_2/X)$ contains

$\mathrm{Grass}_1(M\omega/M^2\omega)$. Since the latter has dimension at least 2, reasoning as in the proof of (a) we conclude that $\mathrm{Quot}^2(\omega/X)$ is reducible.

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