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REINHARD SCHULTZ

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CORRECTION TO THE PAPER

"COMPACT FIBERINGS OF HOMOGENEOUS SPACES. I"

Reinhard Schultz

It was pointed out by R. Stong that the methods of [I] do not apply to the oriented Grassmann manifolds $G^+_{8,2}(\mathbb{R})$ and $G^+_{12,2}(\mathbb{R})$, and in fact $G^+_{8,2}(\mathbb{R})$ fibers over S^6 with fiber $\mathbb{C}P^3$; there is an analogous fibering of the unoriented Grassmann manifold $G_{8,2}(\mathbb{R})$ over $\mathbb{R}P^6$ with the same fiber. The fallacy in [I] is that condition (6.6) requires $n \neq 3,5$ (compare the statement of Bertrand's Hypothesis in [I, §5]). Upon reflection it is apparent that these fiberings are consistent with the principal conjecture in [I]; specifically, they come from the fact that Spin_7 acts transitively on the Stiefel manifold $V_{8,2}(\mathbb{R})$ via the spinor representation $\text{Spin}_7 \to SO_8$ (in fact, the induced action on $V_{8,3}(\mathbb{R})$ is also transitive). Stong has indicated a more direct description of this fibering.

In contrast, the manifold $G^+_{12,2}(\mathbb{R})$ is indeed connectedwise prime, and we shall verify this here. We adopt notation from [I] as needed. The proof of [I, Theorem 6.1] implies that the question reduces to considering compact fiberings $F \to G^+_{12,2}(\mathbb{R}) \to B$ with B a 1-connected $\mathbb{Z}\,[6^{-1}]$ cohomology 10-sphere. The idea is to construct an associated compact fibering of $V_{12,2}(\mathbb{R})$ over B. Specifically, if $\hat{F}$ is the principal SO_2-bundle over F classified by the composite $F \to G^+_{12,2}(\mathbb{R}) \to BSO_2$, then the sequence

$$(1) \qquad \hat{F} \longrightarrow V_{12,2}(\mathbb{R}) \longrightarrow B$$

is exact.

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The first step in providing $G^{+}_{12,2}(\mathbb{R})$ is connectedwise prime is to show that B *is actually a* $\mathbb{Z}_{(3)}$*-homology sphere*. Given this, it is not difficult to modify the argument for $n = 5$.

The analysis of B begins with the observation that the boundary homomorphism $\partial_3 : \pi_3(B) \to \pi_2(F)$ is zero by a result of S. Weingram [55, §3]. But ∂_3 is an isomorphism since $V_{12,2}(\mathbb{R})$ is highly connected, and therefore B and $\hat{F}$ are 2- and 3-connected respectively (by [I, 4.2] we already knew that they were 1- and 2-connected).

To shorten notation, set $V_i = H^i(\hat{F};\mathbb{Z}_3)$ and $W_j = H^j(B;\mathbb{Z}_3)$. Then $V_i \otimes W_j = E_2^{i,j}$ in the $\mathbb{Z}_3$ Serre spectral sequence for (1). Our connectivity assumptions and Poincaré duality yield the following information:

$$W_0 = V_0 = W_{10} = V_{11} = \mathbb{Z}_3,$$

$$V_1 = V_2 = V_9 = V_{10} = 0,$$

$$W_1 = W_2 = W_3 = W_7 = W_8 = W_9 = 0,$$

$$V_3 = V_8,\ V_4 = V_7,\ V_5 = V_6,$$

$$W_4 = W_6,\ \dim W_5 \equiv 0(2).$$

Thus there are only five unknown dimensions. From the connectivity conditions and the Serre spectral sequence we have $V_3 = W_4$, $V_4 = W_5$, $V_5 = W_6$. Further inspection of the Serre spectral sequence shows $V_6 = V_3 \otimes W_4$ and $V_7 = (V_3 \otimes W_5) \oplus (V_4 \oplus W_4)$; the latter requires an observation that $d_2^{4,4} = 0$ by the multiplicative properties of the Serre spectral sequence. If we combine all this information, we obtain the following equation:

$$\dim V_4 = 2 \dim V_4 \dim V_3. \tag{2}$$

This has an integral solution only if $0 = \dim V_4 = \dim W_5$. But B is a rational homology sphere. Therefore, if W_4 were the first nonzero $\mathbb{Z}_3$ cohomology group in positive degree (we know nothing lower is), Bockstein considerations would imply $W_5 \neq 0$ also. This means $0 = W_4 = W_5 = W_6$, or B is a $\mathbb{Z}_3$ (hence $\mathbb{Z}_{(3)}$) homology 10-sphere. From our formulas it also follows that $\hat{F}$ is a $\mathbb{Z}_{(3)}$ homology 11-sphere.

This brings us to the final step. Let $S^{11}_{(3)} \to E' \to S^{10}_{(3)}$ be the localization of (1) at 3. It is immediate from obstruction theory that this fibration has a cross section. Hence the localized fibration $F_{(3)} \to G^{+}_{12,2}(\mathbb{R})_{(3)} \to B_{(3)}$ also has a cross section. If one uses this 3-local cross section in place of the transfer and sets $p = 3$, then the argument in the last paragraph of the proof of [I, 6.1] goes through word for word. ∎

I am grateful to R. Stong for pointing out my mistake.

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