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RELATIONS BETWEEN HOLOMORPHIC QUADRATIC DIFFERENTIALS II

Hershel M. Farkas*

Introduction

Many years ago in a paper entitled Relations Between Quadratic Differentials [F_2] we showed how the Schottky–Jung proportionalities enable one to write down relations among the holomorphic quadratic differentials on a compact Riemann surface. In this paper, which is total independent of the preceding one, we show how the singular point condition gives us a similar result.

Let S be a compact Riemann surface of genus $g \geq 4$ and (Γ, Δ) $\Gamma = \gamma_1, \ldots, \gamma_g$ $\Delta = \delta_1, \ldots, \delta_g$ a canonical homology basis on S. Let $\{\varphi_i\}$ $i = 1, \ldots, g$ be the basis for the vector space of holomorphic differentials on S dual to (Γ, Δ) so that $\int_{\gamma_j} \varphi_i = \delta_{ij}$. It is well known that the matrix $\Pi = (\pi_{ij})$ with $\pi_{ij} = \int_{\delta_j} \varphi_i$ is symmetric and has positive definite imaginary part. Thus $\pi \in \mathfrak{S}_g$, the Siegel upper half plane of degree g and we can consider the g-dimensional theta function $\theta(z, \pi) = \sum_{N \in Z^g} \exp 2\pi i[\frac{1}{2}{}^tN\Pi N + {}^tNZ]$ where $N = \begin{pmatrix} n_1 \\ \vdots \\ n_g \end{pmatrix} \in Z^g$, $Z = \begin{pmatrix} z_1 \\ \vdots \\ z_g \end{pmatrix} \in C^g$.

The points $e \in C^g$ for which $\theta(e, \pi) = 0 = \frac{\partial \theta}{\partial z_n}(e, \pi)$ $n = 1, \ldots, g$ are called the singular points of the theta divisor. The relations between the singular points of the theta divisor and linear relations among the holomorphic quadratic differentials has been discussed in detail in [F_1]. For our purposes here we only need the fact that each singular point e gives us a relation $\sum_{i,j=1}^{g} \frac{\partial^2 \theta}{\partial z_i \partial z_j}(e, \pi)\, \varphi_i \cdot \varphi_j \equiv 0$. This relation may be trivial since it is possible that $\frac{\partial^2 \theta}{\partial z_i \partial z_j}(e, \pi) = 0$ for all $i, j = 1, \ldots, g$.

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When this is not the case and it usually is not then the rank of the matrix $H = \left(\frac{\partial^2\theta}{\partial z_i \partial z_j}(e, \pi)\right)$ is either four or three. The usual case is rank four and the rank 3 situations are "special".

On a non-hyperelliptic surface the set $\{\varphi_i \cdot \varphi_j\}$ $i \leq j = 1 \ldots g$ spans the linear space of holomorphic quadratic differentials. There are $g(g+1)/2$ elements in this set and the dimension of the space of holomorphic quadratic differentials is $3g-3$. Thus there are $\frac{(g-3)(g-2)}{2}$ independent relations of the form $\sum_{i,j=1}^{g} a_{ij}\ \varphi_i \cdot \varphi_j = 0$. One of the main objectives of this paper is to obtain explicitly these relations.

If we focus our attention on the collection of $\pi \in \mathfrak{S}_g$ which come from compact Riemann surfaces of genus g together with a canonical homology basis then it is clear that $(S, (\Gamma, \Delta))$ and $(S, (-\Gamma, -\Delta))$ both give the same matrix $\pi \in \mathfrak{S}_g$. If we introduce the equivalence relation $(S_1, (\Gamma, \Delta)) \equiv (S_2, (\Gamma', \Delta'))$ iff there is a conformal map $f: S_1 \to S_2$ such that the induced map on homology takes the basis (Γ, Δ) to the basis (Γ', Δ') the set of equivalence classes may be identified with the Torelli space T^g of genus g, which is a $3g-3$ dimensional complex analytic manifold. The map from T^G into $\mathfrak{S}_g$ given by the construction of π is an analytic two to one map branched over the hyperelliptic locus since the hyperelliptic involution h induces the map on homomogy $(\Gamma, \Delta) \to (-\Gamma, \Delta)$. Since our results will only be concerned with neighborhoods of a non-hyperelliptic surface or local results within the hyperelliptic locus we can think of everything as occurring in $\mathfrak{S}_g$.

1. The first result which we require is the characterization of singular points given by the Riemann vanishing theorem [RF]. If φ_i $i = 1, \ldots, g$ are a basis for the space of holomorphic differentials on S dual to (Γ, Δ) then the map $u: p \to \int_{p_0}^{p} \phi$ where $\phi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_g \end{pmatrix}$ for a given point $p_0 \in S$ is an analytic injection of S into C^g/G. Here G is the group of translation of C^g generated by $z \to z + e^{(i)}$ $i = 1, \ldots, g$ $z \to z + \pi^{(i)}$ $i = 1, \ldots, g$ where $e^{(i)}$ is the ith column of the $g \times g$ identity matrix and $\pi^{(i)}$ is the ith column of the matrix $\pi = (\pi_{ij})$. Since C^g/G is a commutative group the map $u: p \to C^g/G$ may be extended to divisors $P_1^{\alpha_1} \ldots P_r^{\alpha_r}$ where P_i are points on S and α_i are integers. $u(P_1^{\alpha_1} \ldots P_r^{\alpha_r}) = \alpha_1 u(P_1) + \cdots + \alpha_r u(P_r)$. There is a map $\deg: D \to Z$ where D is the group of all divisors and Z is the group of integers

defined by $\deg(P_1^{\alpha_1} \dots P_r^{\alpha_r}) = \sum_{i=1}^{r} \alpha_i$ which is clearly a group' homomorphism with kernel the subgroup of divisors of degree zero. There is also a subgroup of the divisors of degree zero, the principal divisors, which are those divisors which are the divisors of meromorphic functions on S. If we denote the principal divisors by P and the divisors of degree zero by D_0 then Abel's theorem says that $C^g/G \cong D_0/F$ and in fact the isomorphism is given by u. In D_0/P there is a set of integral divisors of degree n. An example of a divisor in this set is the divisor $P_1^{\alpha_1} \dots P_r^{\alpha_r} P_0^{-n}$ with $\alpha_i > 0$, $\Sigma\, \alpha_i = n$ and P_0 the base point of the map u. The Riemann vanishing theorem asserts that the zero divisor of the theta function is precisely a translate of this set of integral divisors for $n = g - 1$ and the order of vanishing of the theta function at the point in question is precisely $i(P_1^{\alpha_1} \dots P_r^{\alpha_r})$ where $i(P_1^{\alpha_1} \dots P_r^{\alpha_n})$ is the dimension of the vector space of holomorphic differentials on S with zero at P_i of order at least α_i. Since we shall not have any use for the group structure in this presentation we may simply view the zero set as the image under u of unordered $(g-1)$ tuples of points of S translated by a fixed quantity. Thus $e \in C^g$ is a singular point if and only if $e = u(P_1 \dots P_{g-1}) + K$ and $i(P_1 \dots P_{g-1}) \geq 2$. Furthermore the set of unordered $(g-1)$ tuples of S is in a natural way a complex analytic manifold of dimension $g-1$ obtained by considering the cartesian product of S with itself $g-1$ times and then factoring by the symmetric group on $g-1$ letters. Local coordinates on this space may be chosen to be the elementary symmetric functions of the local coordinates on the cartesian product and in these coordinates the map from the space of integral divisors of degree $g-1$ to C^g/G is analytic and its differential $\mathrm{d}u$ at the point $P_1^{\alpha_1} \dots P_r^{\alpha_r}$ is precisely the matrix

$$\mathrm{d}u = \begin{pmatrix} \varphi_1(P_1)\varphi_1'(P_1) \dots \varphi_1^{(\alpha_1-1)}(P_1)\varphi_1(P_2) \dots \varphi_1(P_r) \dots \varphi_1^{(\alpha_r-1)}(P_r) \\ \varphi_g(P_1)\varphi_g'(P_1) \dots \varphi_g^{(\alpha_1-1)}(P_1)\varphi_g(P_2) \dots \varphi_g(P_r) \dots \varphi_g^{(a_r-1)}(P_r) \end{pmatrix}.$$

up to multiplication by a non-singular diagonal matrix. The rank of $\mathrm{d}u$ is equal to $g - i(P_1^{\alpha_1} \dots P_r^{\alpha_r})$. The details can be found in [FK]. The space is called the $g-1$st symmetric product of S with itself denoted S_{g-1}.

It is an immediate consequence of the Riemann–Roch theorem than an integral divisor of degree $g-1$, ζ, $i(\zeta) \geq 2$ has the property that there is an integral divisor ω of degree $(g-1)$ such that $\zeta\omega^{-1}$ is a principal divisor or that $u(\zeta) - u(\omega) = 0 \in C^g/G$. In other words if

$e \in C^g$ is a singular point there are a pair of integral divisors of degree $g-1$, ζ and ω and a set of $2g$ integers such that

$$u(\zeta) - u(\omega) + \sum_{j=1}^{g} n_j e^{(j)} + m_j \pi^{(j)} = 0. \tag{1}$$

Equation (1) above is a vector equation which we can write as g scalar equations

$$u_i(\zeta) - u_i(\omega) + n_i + \sum_{j=1}^{g} m_j \pi_{ij} = 0 \quad i = 1, \ldots, g. \tag{2}$$

We may view these equations as being analytic equations on $S_{g-1} \times S_{g-1}$. If we now write with $3 \leq h \leq g-1$ $\zeta = P_1 \ldots P_h P_{h+1} \ldots P_{g-1}$ $\omega = Q_1 \ldots Q_h P_{h+1} \ldots P_{g-1}$ with $P_i \neq P_j$ for $i \neq j$ and $Q_i \neq P_j$ for any i, j then it is clear that the Jacobian matrix of this system has rank $g - i(P_1 \ldots P_h P_{h+1} \ldots P_{g-1} Q_1 \ldots Q_h)$. The case we shall be ultimately interested in is when $i(P_1 \ldots P_h P_{h+1} \ldots P_{g-1} Q_1 \ldots Q_h) = 0$ so that in particular the matrix

$$\begin{pmatrix} \varphi_1(P_1) \ldots \varphi_1(P_h)\varphi_1(P_{h+1}) \ldots \varphi_1(P_{g-1})\varphi_1(Q_1) \ldots \varphi_1(Q_h) \\ \varphi_g(P_1) \ldots \varphi_g(P_h)\varphi_g(P_{h+1}) \ldots \varphi_g(P_{g-1})\varphi_g(Q_1) \ldots \varphi_g(Q_h) \end{pmatrix}$$

has maximal rank g and thus we can choose g linearly independent columns from the $g - 1 + h$ columns. If we in fact knew that the first h columns were dependent, the first $h-1$ columns were independent and that the first $h-1$ columns with columns $h+1$ to $g-1$ were independent it would follow that the matrix

$$\begin{pmatrix} \varphi_1(P_1) \ldots \varphi_1(P_{h-1})\varphi_1(P_{h+1}) \ldots \varphi_1(P_{g-1})\varphi_1(Q_k)\varphi(Q_\ell) \\ \varphi_g(P_1) \ldots \varphi_g(P_{h-1})\varphi_g(P_{h+1}) \ldots \varphi_g(P_{g-1})\varphi_g(Q_k)\varphi_g(Q_\ell) \end{pmatrix}$$

for some $k \neq \ell$ has maximal rank g. By a renumbering we may assume that $k = 1$ $\ell = 2$. Hence if we now denote the coordinates on one copy of S_{g-1} by $t_1, \ldots, t_{g-1}$ and on the other copy by $\omega_1, \ldots, \omega_{g-1}$ then we have in fact shown that the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial t_1} \ldots \frac{\partial f_1}{\partial t_{h-1}} & \frac{\partial f_1}{\partial t_{h+1}} \ldots \frac{\partial f_1}{\partial t_{g-1}} & \frac{\partial f_1}{\partial \omega_1} & \frac{\partial f_1}{\partial \omega_2} \\ \frac{\partial f_g}{\partial t_1} \ldots \frac{\partial f_g}{\partial t_{h-1}} & \frac{\partial f_g}{\partial t_{h+1}} \ldots \frac{\partial f_g}{\partial t_{h-1}} & \frac{\partial f_g}{\partial \omega_1} & \frac{\partial f_g}{\partial \omega_2} \end{pmatrix}$$

has maximal rank at the point in question and thus by the implicit function theorem we can solve for $t_1, \ldots, t_{h-1}, t_{h+1}, \ldots, t_{g-1}, \omega_1, \omega_2$ as analytic functions of $t_h, \omega_3 \ldots \omega_{g-1}$ in such a way that

$$
\begin{aligned}
(3) \qquad & f_i(t_1(t_h, \omega_3 \ldots \omega_{g-1}), \ldots, t_{h-1}(t_h, \omega_3, \ldots, \omega_{g-1}), \\
& t_h, t_{h+1}(t_h, \omega_3 \ldots \omega_{g-1}), \ldots, \\
& \ldots, t_{g-1}(t_h, \omega_3 \ldots \omega_{g-1}) \omega_1(t_h, \omega_3, \ldots, \omega_{g-1}), \\
& \omega_2(t_h, \omega_3 \ldots \omega_{g-1}), \omega_3, \ldots, \omega_{g-1}) \equiv 0
\end{aligned}
$$

where

$$
\begin{aligned}
& f_i(t_1 \ldots t_{g-1}, \omega_1, \ldots, \omega_{g-1}) = u_i(t_1, \ldots, t_{g-1}) \\
& - u_i(\omega_1, \ldots, \omega_{g-1}) + n_i + \sum m_j \pi_{ij}.
\end{aligned}
$$

In particular, the set of integral divisors of degree $g-1$ $\omega_1(t_h, \omega_3 \ldots \omega_{g-1})$, $\omega_2(t_h, \omega_3 \ldots \omega_{g-1})\omega_3, \ldots, \omega_{g-1}$ defined by these equations all have the property that their index i is greater than or equal to two and thus their image translated by a fixed quantity corresponds to singular points of the theta function. The set of divisors we have defined is clearly $g-3$ dimensional but one dimension is lost by mapping into C^g/G via u.

The above shows that with the assumptions made above on the rank, the singular point is always contained in a $g-4$ dimensional set. This result is not new but its proof can be easily extended to now show the following result which is central to the ideas of this paper.

In addition we remark that although the argument used pretended that the points $P_i Q_j$ in the above discussion were distinct this was totally unnecessary. If the points were not distinct then the matrix is the one previously indicated where for multiple points we have columns of derivatives rather than the differentials themselves evaluated at the points.

We consider the following situation. A fiber space F with base manifold being the Torelli space T^g and with fiber over each point $t \in T^g$ being the singular set of the associated Riemann theta function. Since our result will only be local the point $t \in T^g$ is represented by $\pi \in \mathfrak{S}_g$ and the singular set by $\Theta_{\text{sing}}(\pi)$. We are interested in the question of the existence of a local analytic cross section. In other words, a map on T^g which chooses a singular point for the associated Riemann theta function.

LEMMA 1: *Let* $e \in \Theta_{\text{sing}}(\pi)$ *with* $e = u(P_1 \ldots P_{g-1}) + K$

$i(P_1 \dots P_{g-1}) = 2$ and $i(P_1^2 \dots P_h^2 P_{h+1} \dots P_{g-1}) = 0$ where $P_1 \dots P_h$ is the polar divisor of a function in $L\left(\frac{1}{P_1 \dots P_{g-1}}\right)$. Then there is a local analytic section through $(\pi, e(\pi))$, and Rank $\left(\frac{\partial^2 \theta}{\partial z_i \partial z_j}(e, \pi)\right)$ $i, j =$ $1, \dots, g$ equals 4.

PROOF: The condition $i(P_1^2 \dots P_h^2 P_{h+1} \dots P_{g-1}) = 0$ implies that the quadric relation determined by e is of rank 4. This is Lemma 2 of [F₁] Theorem 2 of [F₁] now gives us the last statement of the lemma. To obtain the existence of a local analytic section we once more invoke Abel's theorem as before and write

$$u(P_1 \dots P_h P_{h+1} \dots P_{g-1}) - u(Q_1 \dots Q_h P_{h+1} \dots P_{g-1}) - m_i e^{(i)} - n_i \pi^{(i)} = 0 \tag{4}$$

The Torelli space for $g \geq 2$ is a $3g-3$ dimensional complex analytic manifold and one can choose local coordinates $\epsilon_1, \dots, \epsilon_{3g-3}$ such that the π_{ij} are analytic functions of these coordinates and the functions u_i are also analytic functions of these coordinates. Hence this time we can interpret the equations of Abel's theorem as being equations on $S_{g-1} \times S_{g-1} \times T^g$. Since $P_1 \dots P_h$ is equivalent to $Q_1 \dots Q_h$ the condition $i(P_1^2 \dots P_h^2 P_{h+1} \dots P_{g-1}) = 0$ gives us $i(P_1 \dots P_h P_{h+1} \dots P_{g-1} Q_1 \dots Q_h) = 0$ and the argument we used before goes through. Hence employing the implicit function theorem once again we get that we can solve for $t_1, \dots, t_{h-1}, t_{h+1}, \dots, t_{g-1}, \omega_1, \omega_2$ as analytic functions of $t_h, \omega_3, \dots, \omega_{g-1}, \epsilon_1, \dots, \epsilon_{3g-3}$. If we fix t_h, $\omega_3, \dots, \omega_{g-1}$ to be constant we have produced the required analytic section. We note for future reference that other choices are possible and will return to this point.

II. We now will use the existence of the local analytic section. We shall as before think of T^g as sitting in $\mathfrak{S}_g$, and denote the coordinates of T^g by $\epsilon_1, \dots, \epsilon_{3g-3} = \epsilon$. Hence the existence of a local analytic section through the point (π_0, z_0) means that we have $\pi(\epsilon), z(\epsilon)$ such that

$$\theta(z(\epsilon), \pi(\epsilon)) \equiv 0 \text{ as a function of } \epsilon, \tag{5}$$

$$\frac{\partial \theta}{\partial z_n}(z(\epsilon), \pi(\epsilon)) \equiv \text{ as a function of } \epsilon \quad n = 1, \dots, g, \tag{6}$$

$$\text{Rank}\left(\frac{\partial^2 \theta}{\partial z_i \partial z_j}(z(\epsilon), \pi(\epsilon)\right) \leq 4 \quad i, j = 1, \dots, g. \tag{7}$$

The equations in 7 are $\frac{(g-4)(g-3)}{2}$ in number and simply express the vanishing of all 5×5 subdeterminants. In order to facilitate the computations we have to make we shall consider only one dimensional variations of structure or what is the same thing one dimensional subvarieties of T^g. We shall still use the symbol $(z(\epsilon), \pi(\epsilon))$ but shall now think of ϵ as a complex variable in a neighborhood of the origin. If we now differentiate these expressions with respect to ϵ and evaluate at $\epsilon = 0$ we get

(8) $$\sum_{i\neq 1}^{g} \frac{\partial\theta}{\partial z_i}(z_0, \pi_0) z_i'(0) + \sum_{i\leq j=1}^{g} \frac{\partial\theta}{\partial\pi_{ij}}(z_0, \pi_0)\pi_{ij}'(0) = 0$$

(9) $$\sum_{i=1}^{g} \frac{\partial^2\theta}{\partial z_i\partial z_n}(z_0, \pi_0) z_i'(0) + \sum_{i\leq j=1}^{g} \frac{\partial^2\theta}{\partial\pi_{ij}\partial z_n}(z_0, \pi_0)\pi_{ij}'(0) = 0$$

$n = 1, \ldots, g,$

(10) $\Delta_n'(z_0, \pi_0) = 0 \quad n = 1, \ldots, (g-4)(g-3)/2$ and here each equation will be of the form

(10′) $$\sum_{i=1}^{g} c_{ni} z_i'(0) + \sum_{i\leq j=1}^{g} d_{nij}\pi_{ij}'(0) = 0, \quad n = 1, \ldots, \frac{(g-4)(g-3)}{2}.$$

We can now write this all as one big matrix equation

(11) $$AZ'(0) = B\begin{pmatrix} \pi_{11}'(0) \\ \vdots \\ \pi_{gg}'(0) \end{pmatrix} \quad \text{where} \quad A =$$

$$\begin{pmatrix} 0 & 0 & \cdots & 0 \\ & & \frac{\partial^2\theta}{\partial Z_i\partial Z_j}(Z_0, \pi_0) & \\ C_{11} & & & C_{1G} \\ C_{\frac{(g-4)(g-3)}{2}1} & & & C_{\frac{(g-4)(g-3)}{2}g} \end{pmatrix}$$

The question we need now ask is what is the rank of A? Our claim is that Rank $A = 4$ and thus the left null space of A is $\frac{(g-3)(g-2)}{2}$ dimensional.

LEMMA 2: *Rank* $A = 4$.

PROOF: We already know that $\Theta_{\text{sing}}(\pi_0)$ is $g-4$ dimensional and we can assume that Z_0 has been chosen as a regular point of $\Theta_{\text{sing}}(\pi_0)$. Let $(t_1, \ldots, t_{g-4})$ be local coordinates near Z_0. It is then clear that we have

$$\theta(Z(t_1 \ldots t_{g-4}), \pi_0) \equiv 0 \quad \text{in } t_1 \ldots t_{g-4}$$

$$\frac{\partial \theta}{\partial Z_n}(Z(t_1 \ldots t_{g-4}), \pi_0) \equiv 0 \quad \text{in } t_1 \ldots t_{g-4} \quad n = 1, \ldots, g,$$

$$\text{Rank} \frac{\partial^2 \theta}{\partial Z_i \partial Z_j}(Z(t_i, \ldots, t_{g-4}), \pi_0) \leq 4 \quad \text{in } t_1, \ldots, t_{g-4}.$$

The fact that Z_0 is a regular point implies that the $(g-4)$ vectors $\frac{\partial Z_0}{\partial t_i}(0), \ldots, \frac{\partial Z_0}{\partial t_{g-4}}(0)$ are linearly independent and differentiating successively with respect to t_i $i = 1, \ldots, g-4$, we find the system of equations $AZ' = 0$ with the vectors $\frac{\partial Z_0}{\partial t_i}(0)$ $i = 1, \ldots, g-4$ as linearly independent solutions. This implies that the right null space of A is at least $g-4$ dimensional or that rank $A \leq 4$. It cannot be less than 4 since by hypothesis Rank $\left(\frac{\partial^2 \theta}{\partial Z_i \partial Z_j}(Z_0, \pi_0)\right) = 4$.

We have established that Rank $A = 4$ and thus that the left null space of A has dimension $\frac{(g-3)(g-2)}{2}$. Let V_i $i = 1, \ldots, \frac{(g-3)(g-2)}{2}$ be a basis for the left null space of A. It thus follows from (11) that

(12) $$V_i B \begin{pmatrix} \pi'_{11}(0) \\ \pi_{gg}(0) \end{pmatrix} = 0 \quad \text{for each } i.$$

It is perfectly clear that each equation in (12) gives rise to a linear relation among the quantities $\pi'_{ij}(0)$ however what is not clear unfortunately is whether the linear equations are non-trivial in the sense that for some or all i it is conceivable that the V_i are in the left null space of B. Even if some relations are non-trivial we would like to know that they are linearly independent. A sufficient condition for the above to hold is that the matrix B have maximal rank. In this case the equations (12) represent $(g-3)(g-2)/2$ linearly independent linear relations among the quantities $\pi'_{ij}(0)$ $i, j = 1, \ldots, g$.

We now wish to explicitly compute the vectors V_i in terms of the entries of the matrix A.

In order to facilitate the writing down of a basis of the left null space of A we need to establish some notation. The matrix A is given by

$$
\begin{matrix}
0\ \ 0\ \ 0 & \cdots & 0 \\
\frac{\partial^2\theta}{\partial Z_1^2}(Z_0,\pi_0) & \frac{\partial^2\theta}{\partial Z_1\partial Z_4}(Z_0,\pi_g) \ \cdots & \frac{\partial^2\theta}{\partial Z_1\partial Z_g} \\
\frac{\partial^2\theta}{\partial Z_n\partial Z_1}(Z_0,\pi_0) & \frac{\partial^2\theta}{\partial Z_n^2}(Z_0,\pi_0) \ \cdots & \frac{\partial^2\theta}{\partial Z_n\partial Z_g} \\
\frac{\partial^2\theta}{\partial Z_g\partial Z_1}(Z_0,\pi_0) & & \frac{\partial^2\theta}{\partial Z_g^2}(Z_0,\pi_0) \\
C_{11} & & C_{1g} \\
C_{21} & & C_{2g} \\
C_{m1} & & C_{mg}
\end{matrix}
$$

with $m = \frac{(g-3)(g-4)}{2}$. We assume further that the matrix $H = \left(\frac{\partial^2\theta}{\partial Z_i\partial Z_j}\right)$ $i, j = 1, \ldots, 4$ actually has rank 4. We will let A_{ij} represent then 5×5 matrix formed by adjoining to H the row $(a_{il}, a_{i2}, a_{i3}, a_{i4}, a_{ij})$ and column

$$
\begin{pmatrix} a_{1j} \\ a_{2j} \\ a_{3j} \\ a_{4j} \\ a_{ij} \end{pmatrix}.
$$

In this notation the elements of the matrix A less its first row are denoted by a_{ij}. Last, we shall denote by A_{ij}^k the minor of the kth element of the fifth column of the matrix A_{ij}.

In terms of the notation just established we can now write down explicitly the equations Rank $\left(\frac{\partial^2\theta}{\partial Z_i\partial Z_j}\right) \leq 4$.

(13) $$\det(A_{ij}) = 0 \quad i, j = 5, \ldots, g \quad i \leq j.$$

From the fact that Rank $A = 4 =$ Rank H it is quite clear that the vectors

$$V_1' = (A_{55}^1, A_{55}^2, \ldots A_{55}^4, A_{55}^5, 0 \ldots 0)$$

$$V_2' = (A_{65}^1, A_{65}^2, \ldots A_{65}^4\ 0\ A_{65}^5\ 0 \ldots 0)$$

$$V'\frac{(g-4)(g-1)}{2} = \left(A'g + \frac{(g-4)(g-3)}{2}5, \ldots, A^4 g + \frac{(g-4)(g-3)}{2}5,\right.$$
$$\left. 0 \ldots 0, A^5 g + \frac{(g-4)(g-3)}{2}5\right)$$

are a basis for the left null space of A less its first row and adjoining $V_0 = (1, 0 \ldots 0)$ gives us a basis of the left null space of A after replacing V_i' by $V_i = (0, V_i')$.

THEOREM 1: *Under the hypothesis rank B maximal we have* $\Sigma_{i\leq j=1}^{g} q_{ij}^{k} \varphi_i \varphi_j \equiv 0$ *with* $\{\varphi_i\}$ *a basis for the holomorphic differentials on S dual to the homology basis which gives rise to* π and $\{q_{ij}^{k}\} = \{\vec{V}_k B\}$.

PROOF: The proof depends simply on the well known fact that we can choose our variation of structure in S or what is the same thing our curve in the Torelli space so that $\pi_{ij}'(0) = \varphi_i \varphi_j(Po)$ with Po arbitrary. Hence the equations $V_k B \begin{pmatrix} \pi_{11}'(0) \\ v_{gg}'(0) \end{pmatrix}$ are exactly the same as the ones we wrote down.

Theorem 1 should be viewed as an alternate result to the theorem of Andreotti and Mayer [AM] which says that one can generally choose $(g-3)(g-2)/2$ singular points such that $\dfrac{\partial^2 \theta}{\partial Z_1 \partial Z_1}(Z_\ell, \pi)\varphi_i \varphi_j \equiv$ $\ell = 1, \ldots, \dfrac{(g-3)(g-2)}{2}$. The Andreotti Mayer version has the advantage of the quadric relations all being of rank ≤ 4. In our case, with the exception of the first relation with $k = 0$ which corresponds exactly to the Andreotti Mayer result and is in fact classical, we do not know what the rank is.

The main idea used so far was the existence of a local cross section. It is quite clear that there is nothing "unique" about the section we chose and in fact the result obtained is independent of the section chosen.

Similar ideas allow us to write down explicit equations in $\mathfrak{S}_g$ locally describing the subset of $\mathfrak{S}_g$ whose theta functions have $g-4$ dimensional singular sets. This, together with the work of Beauville [B], allows us to obtain a local description of the space of Jacobians for $g = 4$ and $g = 5$.

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