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Minimal models for elliptic curves with complex multiplication

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Let $R$ be the ring of integers in an algebraic number field $F$. An abelian variety $A$ of dimension $g$ over $F$ determines an element $c_A$ in the ideal class group $R$ in the following manner. Let $N$ denote the Néron model of $A$ over $R$ [4]; the space $\varphi_{N/R}$ of invariant differentials on $N$ is a projective $R$-module of rank $g$. We may define $c_A$ to be the class of $\Lambda_{\varphi_{N/R}}$ in $\text{Pic}(R)$.

When $\dim A = 1$ Tate has given an alternate description of the class $c_A$ in terms of minimal Weierstrass models [5]. We use this formulation, and some classical results of Deuring [1] and Hasse, to calculate $c_A$ for some elliptic curves with complex multiplication.

§ 1. Minimal models of elliptic curves

Let $A$ be an elliptic curve over $F$, a number field with ring of integers $R$. The space $\varphi_{A/F} = H^0(A, \Omega^1/F)$ of invariant differentials is an $F$-vector space of dimension 1. Associated to any non-zero differential $\omega$ we have its discriminant $\Delta_\omega \in F^*$ [5]. If $\omega' = u^{-1}\omega$ then $\Delta_{\omega'} = u^{12}\Delta_\omega$; hence $A$ determines a coset $\Delta_A \in F^*/F^{*12}$.

For any discrete valuation $v$ of $F$, let $\omega_v$ and $\Delta_v = \Delta_{\omega_v}$ be the differential and discriminant of a minimal Weierstrass equation for $A$ at $v$ [5]. We define the discriminant ideal $\mathcal{D}_A$ by the formula:

\[ (1.1) \quad \mathcal{D}_A = \prod_v \mathcal{P}_v^{\nu(\mathcal{D}_v)}, \]

where $\mathcal{P}_v$ is a prime ideal at the place $v$. For any non-zero differential
ω on A over F we define the ideal δω by the formula:

\[
\delta_\omega = \prod_v \mathcal{O}_v^{\omega(v)}.
\]  

One then has the equality of ideals in R:

\[
(\Delta_\omega)\delta_\omega^{12} = \mathcal{D}_A.
\]

The class of the ideal δω in Pic(R) is independent of the choice of ω. We denote this class by δA; then A has a global differential ω with \((\Delta_\omega) = \mathcal{D}_A\) if and only if \(\delta_A \sim 1\) in Pic(R). In this case one can find a global minimal model for A: i.e., an equation for A over R which is simultaneously minimal at all places \(v\).

By (1.3) one has:

\[
\delta_A^{12} \sim \mathcal{D}_A \quad \text{in Pic}(R).
\]

Hence a necessary condition for the existence of a global minimal model is that the ideal \(\mathcal{D}_A\) be principal. By (1.4) this is also sufficient when the group Pic(R) has no 12-torsion.

It is not difficult to compare \(\delta_A\) with the class \(c_A\) of Néron differentials defined in the introduction. Let X be the minimal regular model for A over \(R_v\); \(X\) is a regular projective scheme over \(R_v\) which can be obtained by resolving the possible singularity on a minimal Weierstrass equation for A over \(R_v\) [4, pp. 94–101]. The Néron minimal model N is a smooth group scheme over \(R_v\); it is obtained by removing all fibres of multiplicity greater than one on \(X\) and all singular points in the remaining fibres. The pull-back of a minimal Weierstrass differential \(\omega_v\) on \(A/R_v\) is everywhere non-zero on N. Hence we find:

\[
\omega_{N/R_v} = \omega_v R_v \subset \omega_{A/F_v},
\]

so globally we have the identity:

\[
\omega_{N/R} = \omega \delta^{-1}_\omega \subset \omega_{A/F}.
\]

To sum up, we have the following
PROPOSITION 1.7:
(1) \( c_A \sim \delta_A^{-1} \) in \( \text{Pic}(R) \).
(2) The following statements are equivalent
(a) \( c_A \sim \delta_A \sim 1 \) in \( \text{Pic}(R) \).
(b) \( A \) has a global minimal Weierstrass model over \( R \).
(c) \( A \) has a non-zero differential \( \omega \) with \( (\Delta_\omega) = \mathcal{D}_A \).
(d) \( \omega_{N/R} \) is a free \( R \)-module of rank 1.

§2. Elliptic curves with complex multiplication

We now assume that \( A \) is an elliptic curve with complex multiplication by the ring of integers \( \mathcal{O} \) of an imaginary quadratic field \( K \). We assume further that the field \( F \) of definition for \( A \) is \( H \), the Hilbert class field of \( K \). Then all endomorphisms of \( A \) are defined over \( H \), and the curve \( A \) is determined up to isomorphism by its modular invariant \( j_A \) and the associated Hecke character \( \chi_A \) on the idèles \( I_H \) of \( H \) [2; 9.1.3].

PROPOSITION 2.1: Both the ideal \( \mathcal{D}_A \) and the class \( \delta_A \) depend only on the character \( \chi_A \), and not on the modular invariant \( j_A \).

PROOF: Let \( B \) be another elliptic curve over \( F \) with \( \chi_B = \chi_A \); then \( j_B = j_A^\sigma \) with \( \sigma \in \text{Aut}(H) \). The group \( \text{Hom}_H(B, A) \) is described in [2, 9.4.2]: for any integral ideal \( a \) of \( K \) such that \( \sigma = \sigma_a^{-1} \) in \( \text{Aut}(H) \) we have an isogeny \( \phi_a : B \to A \) with kernel isomorphic to \( \mathcal{O}/a \). More precisely, we may choose an embedding of \( H \) into \( C \) so that the following diagram commutes:

\[
\begin{array}{ccc}
B(C) & \xrightarrow{\phi_a} & A(C) \\
\downarrow \phi_{a*} & & \downarrow \omega \\
C/\Omega a & \xrightarrow{p} & C/\Omega \mathcal{O}
\end{array}
\]

where \( \omega \) is a non-zero differential on \( A, \Omega \in C^* \) is a fixed integral period of \( \omega \), and \( p \) is the natural projection.

Now let \( v \) be a fixed place of \( H \) and choose \( a \) with \( \sigma_a^{-1} = \sigma \) and \( Na \) prime to \( v \) (this is always possible). Then the induced map \( \phi_a^* : \omega_{B/R_v} \to \omega_{A/R_v} \) on the spaces of local Néron differentials is an isomorphism. Hence to show that \( \mathcal{D}_A = \mathcal{D}_B \) it suffices to show that \( v(\Delta_\omega) = v(\Delta_{\phi_a^* \omega}) \). But by (2.2), if we compute over \( C \),

\[
\Delta_\omega = \frac{\Delta(\mathcal{O})}{\Delta(a)} \Delta_{\phi_a^* \omega}.
\]
It is well-known that $\Delta(\mathfrak{C})/\Delta(a)$ is an algebraic integer in $H$ which generates the ideal $a^{12}$ [1, p. 33], [3, p. 165]. Since this is prime to $\nu$, the minimal discriminants have the same valuation.

Now let $\omega$ be any non-zero differential on $A$ over $H$ and put $\nu = \phi_{\frac{\omega}{\Delta}}(\omega)$. Then by (1.3) and the above paragraph:

$$\left(\Delta_{\omega}\right)^{12} = \mathfrak{D}_A = \mathfrak{D}_B = \left(\Delta_{\nu}\right)^{12}. $$

Since $\Delta_{\omega}/\Delta_{\nu} = \Delta(\mathfrak{C})/\Delta(a)$ by (2.3), we have

$$\left(\delta_{\nu}/\delta_{\omega}\right)^{12} = \left(\Delta(\mathfrak{C})/\Delta(a)\right) = a^{12}. $$

Hence $\delta_{\nu} = \delta_{\omega} \cdot a$ as ideals of $H$. But the ideal $a$ of $K$ capitulates in $H$; hence $\delta_A \sim \delta_B$ in $\text{Pic}(R)$.

Note: If we assume that the Hecke character $\chi_A: I_H \to K^*$ is $\text{Gal}(H/K)$-equivariant, then by Proposition 2.1 the ideal $\mathfrak{D}_A$ is fixed by $\text{Gal}(H/K)$. Since $H$ is unramified over $K$, any fixed ideal is represented by an ideal of $K$. But all ideals of $K$ capitulate in $H$, so $\mathfrak{D}_A \sim 1$ in $\text{Pic}(R)$. Is $\delta_A \sim 1$ in $\text{Pic}(R)$? We will show this is the case when $K$ has prime discriminant.

§3. A global minimal model for $A(p)$

We now specialize to the case where the multiplication field $K = \mathbb{Q}(\sqrt{-p})$ has prime discriminant.

Lemma 3.1: For any fractional ideal $a$ of $K$, the ratio $\Delta(\mathfrak{C})/\Delta(a)$ is a $12^{\text{th}}$ power in $H^*$.

Proof: By Deuring [1, p. 14, 41] the ratio $\Delta(\mathfrak{C})/\Delta(b^2)$ is a $24^{\text{th}}$ power in $H^*$ when $(6, b) = 1$. When $K$ has prime discriminant, its class group has odd order. Hence we may find an ideal $b$ prime to 6 such that $(\alpha)a = b^2$. Then

$$\Delta(\mathfrak{C})/\Delta(a) = a^{12} \cdot \Delta(\mathfrak{C})/\Delta(b^2) \equiv 1 \pmod{H^{12}}. $$

We can now answer affirmatively a question posed by D. Zagier. Assume that $p > 3$ and let $A(p)$ denote $\mathbb{Q}$-curve over the field $F = \mathbb{Q}(j_{A(p)})$ studied in chapter 5 of [2]. Recall that $A(p)$ has good reduction outside $p$ and has minimal discriminant ideal $\mathfrak{D}_{A(p)} = (-p^3)$. The
fact that this ideal is principal raises the possibility of a global minimal model.

**Proposition 3.2:** The curve $A(p)$ has a global minimal model over the field $F = \mathbb{Q}(j_{A(p)})$ with discriminant $\Delta = -p^3$. The associated differential $\omega(p)$ is determined up to sign.

**Proof:** In §23 of [2] we constructed a pair $(A, \omega)$ over $F$ with $j_A = j_{A(p)}$, $\Delta_\omega = -p^3$, and sign $c_6 = \left(\frac{2}{p}\right)$. Recall that $A$ is given by the equation

$$y^2 = x^3 + \frac{mp}{2^4 \cdot 3} x - \frac{np^2}{2^4 \cdot 3^3}$$

where

$$m^3 = j_{A(p)}$$

$$n^2 = (j_{A(p)} - 1728)/ -p, \quad \text{sign } n = \left(\frac{2}{p}\right),$$

The differential $\omega = dx/2y$ on $A$ has $\Delta_\omega = -p^3$. To prove Proposition 3.2 we will show that $A$ is isomorphic to $A(p)$ over $F$. We will then have a global minimal model by Proposition 1.7, as $(\Delta_\omega) = D_{A(p)}$. The differential $\omega = \omega(p)$ with $\Delta_\omega = -p^3$ is determined up to sign, as $\mu(F^*) = \langle \pm 1 \rangle$.

In summary, we are reduced to proving:

**Proposition 3.5:** The elliptic curve $A$ defined by equations (3.3–3.4) is a $\mathbb{Q}$-curve which is isomorphic over $F$ to the curve $A(p)$.

**Proof:** Consider the map

$$f_A: \text{Gal}(H/\mathbb{Q}) \to \text{Hom}(I_H, K^*)$$

$$\sigma \mapsto \chi_A^{\sigma^{-1}}$$

where all Homs refer to continuous homomorphisms of topological groups. Then $f_A$ is a 1-cocycle, which takes values in the group $\text{Hom}(I_H, H^*, K^*)$. Since $K^*$ is totally disconnected, this group may be identified with the group $\text{Hom}(\text{Gal}(\overline{H}/H), K^*)$ via the Artin homomorphism of global class field theory. Since $\text{Gal}(\overline{H}/H)$ is compact and $K^*$ is discrete, any continuous homomorphism takes values
in the finite group $\mu(K^*) = \langle \pm 1 \rangle$. Finally, we may identify

$$\text{Hom}(\text{Gal}(\overline{H}/H), \pm 1) = H^*/H^{*2},$$

by Kummer theory, and view $f_A$ as a map

$$(3.5) \quad f_A : \text{Gal}(H/\mathbb{Q}) \to H^*/H^{*2}.$$ 

To show $A$ is a $\mathbb{Q}$-curve is equivalent to showing that $f_A(\sigma) \equiv 1$ for all $\sigma \in \text{Gal}(H/\mathbb{Q})$. Since $A$ is defined over $F$ we have $f_A(\tau) \equiv 1$. Hence, it suffices to show $f_A(\sigma) = 1$ for all $\sigma \in \text{Gal}(H/K)$.

For this, we need a concrete description of $f_A(\sigma)$ in $H^*/H^{*2}$. Embed $F$ in $\mathbb{C}$ via its real place, and let $a$ be an integral ideal of $K$ with $\sigma = \sigma_a^{-1}$. There is an isogeny $\phi_a$ defined over $\widetilde{\mathbb{Q}}$ which makes the following diagram commutative:

$$
\begin{array}{c}
A^\sigma \\
\downarrow \phi_a \\
C/\Omega a \\
\end{array} \quad \begin{array}{c}
A \\
\downarrow \omega \\
C/\Omega \mathbb{C}.
\end{array}
$$

If we write $\phi_a^*(\omega) = h_a \cdot \omega^\sigma$ with $h_a \in \widetilde{\mathbb{Q}}^*$, then the isogeny $\phi_a$ is defined over the extension $H(h_a)$. The identities:

$$c_4(\mathcal{O})/c_4(a) = h_a^4 \cdot c_4^{1-\sigma}$$

$$c_6(\mathcal{O})/c_6(a) = h_a^6 \cdot c_6^{1-\sigma}$$

show that $h_a^2 \in H^*$ [3, p. 158]. In fact, we have the formula

$$(3.6) \quad f_A(\sigma) \equiv h_a^2 \pmod{H^{*2}}.$$ 

On the other hand, we have the identity:

$$\Delta(\mathcal{O})/\Delta(a) = h_a^{12} \cdot \Delta^{1-\sigma} = h_a^{12}$$

as $\Delta = -p^3$ is fixed by $\text{Gal}(H/\mathbb{Q})$. By Lemma 3.1, $h_a^{12}$ is a $12$th power in $H^*$. Since $h_a^2 \in H^*$, we must have $h_a \in H^*\mu_4$ and $f_A(\sigma) \equiv \pm 1 \pmod{H^{*2}}$. But $f_A$ is a cocycle and the order of $\text{Gal}(H/K)$ is odd. Hence $f_A(\sigma) = 1$ and $A$ is a $\mathbb{Q}$-curve.

Since $v_\mathcal{P}(\Delta_a) = 3$ we see $A = A(p)^d$ with $(p, d) = 1$ [2, 12.3.2]. But $\mathcal{O}_A = \mathcal{B}^{12}(-p^3)$ and $\mathcal{D}_{A(p)^d} = \mathcal{C}^{12}(-p^3d^2)$ where $\mathcal{B}$ and $\mathcal{C}$ are ideals of $H$. 

Hence \((d) = (b/c)^2\) is the square of an ideal of \(H\). Since \(H\) is unramified over \(K\) and \(d\) is a quadratic discriminant, there are only two possibilities: \(d = 1\) and \(d = -4\). But the curve \(A(p)^{-4}\) has the wrong sign of \(c_6\), so \(A = A(p)\).

§4. Global minimal models for \(K\)-curves

Let \(\omega(p)\) be one of the differentials on \(A(p)\) given by Proposition 3.2. For any integral ideal \(a\) of \(K\) we may define \(h_a\) in \(H^*/\pm 1\) by the formula:

\[
\phi_a^*(\omega(p)) = h_a \cdot \omega(p)^{\sigma_a^1}.
\]

The ambiguity in sign is caused by the ambiguity in the choice of isogeny \(\phi_a\); we will discuss a choice of the sign in §5. In \(H^*/\pm 1\) we have the cocycle relations

\[
\begin{align*}
h_{ab} &= h_{a^1}^{-1} \cdot h_b, \\
h_{a^r} &= h_a^r
\end{align*}
\]

We have seen in §3 that when \(F\) is embedded into \(\mathbb{C}\) via its real place we have the complex identity:

\[
h_a^{12} = \Delta(\mathcal{O})/\Delta(a).
\]

Hence \(h_a\) is integral in \(H\) and generates the ideal \(a\). The same is true for \(h_a^\sigma\) for any \(\sigma \in \text{Gal}(H/K)\).

**Lemma 4.1:** For all \(\sigma \in \text{Gal}(H/K)\), \(h_a^{\sigma^{-1}} \equiv 1 \pmod{H^*/2}\).

**Proof:** First note that this identity makes sense, independent of the choice of sign for \(h_a\). We have seen, in the proof of Lemma 3.1, that \(\Delta(\mathcal{O})/\Delta(b^2) = h_a^{12}\) is a 24th power in \(H^*\). Hence \(h_a^{12} \equiv \pm 1 \pmod{H^*/2}\). Since we may find \(b\) such that \(a = (\alpha)b^2\), we find from (4.2) that \(h_a \equiv \pm \alpha \pmod{H^*/2}\). Hence \(h_a^{\sigma^{-1}} \equiv (\mod H^*/2)\) for any \(\sigma \in \text{Gal}(H/K)\).

**Lemma 4.2:** Let \(K'\) be a quadratic extension of \(K\) with conductor \(\alpha\). Then we may choose the sign of \(h_a\) so that \(HK' = H(\sqrt{h_a})\).

**Proof:** Write \(K' = K(\sqrt{\alpha})\). Since \(\alpha\) is the discriminant ideal of \(K'/K\) and \(\alpha\) is the discriminant of the specific \(K\)-basis \((1, \sqrt{\alpha}/2)\) we
find \((\alpha) b^2 = a\) with \(b\) an ideal of \(K\). Raising this identity to the \(h\)th power and writing \((\beta) = b^h\) we find \((\alpha^h \beta^2) = a^h = (N_{H/K} h_a)\). Since \(h\) is odd and \(\mathcal{O}_K^h = (\pm 1)\), we may choose the sign of \(h_a\) so that \(\alpha = N_{H/K} h_a \pmod{K^*}\). Then \(K' = K(\sqrt{N_{H/K} h_a})\) and \(HK' = H(\sqrt{h_a})\).

By Lemma 4.1, \(h_a = h_a^h \pmod{H^*}\) so multiplying over the entire Galois group we find \(h_a^h = N_{H/K} h_a \pmod{H^*}\). Since \(h\) is odd, \(h_a = h_a^h = N_{H/K} h_a \pmod{H^*}\) and \(HK' = H(\sqrt{h_a})\) as claimed.

Now let \(A\) be an elliptic curve over \(H\) such that \(\chi_A\) is \(\text{Gal}(H/K)\) equivariant. By [2, 12.3.1] we may write \(A = A(p)^\psi\) with

\[\psi \in \text{Hom}(\text{Gal}(\overline{H}/H), \pm 1)_{\text{Gal}(H/K)} = \text{Hom}(\text{Gal}(\overline{K}/K), \pm 1).\]

Let \(a\) be the conductor of \(\psi\) and write the associated quadratic extension \(H' = H(\sqrt{h_a})\) as permitted by Lemma 4.2. For simplicity, assume that \(a\) is prime to \(p\). Let \(\rho\) be a generator of \(\text{Gal}(H'/H)\); we then have the identification

\[\omega_{A/H} = \{\omega \in \psi_{A(p)/H'} : \omega^\rho = -\omega\}.\]

Hence the differential \(\omega_A = (1/\sqrt{h_a}) \cdot \omega(p)\) descends to \(A\) over \(H\).

**Proposition 4.3:** Either \(\omega_A\) or \(2\omega_A\) is a global minimal differential on \(A/H\).

**Proof:** We clearly have \(\Delta_{\omega_A} = -p^3 h_a^6\) so \(\Delta_{\omega_A} = (-p^3) a^6\). This is equal to \(D_A\) except in the case when \((2/p) = -1\) and \(8 \mid a\) [2, 14.1.1]. In that case it is equal to \((2^{12}) D_A\).

**Corollary 4.4:** If \(K\) has prime discriminant and the Hecke character \(\chi_A\) of \(A\) is \(\text{Gal}(H/K)\) equivariant, then \(\delta_A - c_A \sim 1\) in \(\text{Pic}(R)\).

Indeed, the minimal differential given in Proposition 4.3 is determined up to sign.

§5. The sign of \(h_a\)

When the ideal \(a\) of \(K\) is prime to \((p)\), we may normalize the sign of \(h_a\) by insisting that \(N_{H/K} h_a\) is a square \(\pmod{\sqrt{p}}\). Then the following identities hold in \(H^*:\)

\[h_{ab} = h_a^{\sigma_1} h_b^\sigma,\]

(5.1)

\[h_{a^p} = h_a^\sigma,\]

\[h_{(a)} = \alpha \text{ if } \alpha \equiv 1 \pmod{\sqrt{p}}.\]
Hence there is a unique continuous 1-cocycle

\[ \phi : I_K \rightarrow H^* \]

which is the identity on principal idèles and satisfies \( \phi(a) = \prod_{v \mid p, \ell} h_{a_v} \) for all idèles which are trivial at \( \infty \) and congruent to 1 (mod \( \sqrt{-p} \)). (The group \( I_K \) acts on \( H^* \) via its quotient \( I_K / K^* \cdot (C^* \times \Pi_{v} \mathcal{O}_v^*) = \text{Gal}(H/K) \), and the cocycle \( \phi \) is \( \tau \)-equivariant.)

Recall the elements \( t_a \) in \( T^*/\pm 1 \) defined in [2, 15.2.5]. Again, when \( a \) is prime to \( (p) \) we may normalize the sign of \( t_a \) by insisting that \( t_a \) is a square (mod \( \sqrt{-p} \)). We then have the identities in \( T^* \):

\[
\begin{align*}
t_{ab} &= t_a t_b \\
t_a^* &= t_a^* \\
t_{(a)} &= \alpha \quad \text{if } \alpha \equiv 1 (\sqrt{-p}).
\end{align*}
\]

Since \( (t_a) = a \) we find:

**Proposition 5.3:** The elements \( u_a = t_a / h_{a^*} \) are units in the field \( HT \) which satisfy the identities

\[
\begin{align*}
u_{ab} &= u_a \cdot u_b^* \\
u_{a^*} &= u_a^* \\
u_{(a)} &= 1.
\end{align*}
\]

Since \( u_a \) depends only on the class of \( a \) in \( \text{Pic}(\mathcal{O}) \) it is convenient to write \( u_{a_\xi} \) for the unit \( u_a \). By Proposition 5.3 the assignment

\[
\begin{align*}
\sigma &\rightarrow u_\sigma \\
\tau &\rightarrow 1
\end{align*}
\]

gives a 1-cocycle \( f \) on \( \text{Gal}(HT/T^+) \approx \text{Gal}(H/\mathbb{Q}) \) with values in the units \( U \) of \( (HT)^* \).

**Question 5.4:** Is \( f \sim 1 \) in \( H^1(\text{Gal}(HT/T^+), U) \)?

As a stronger question, one can ask if \( \varepsilon = \sum_\sigma u_\sigma \) is a unit of \( HT \).
REFERENCES


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