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UNIRATIONALITY OF ENRIQUES SURFACES IN CHARACTERISTIC TWO

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§0. Introduction

The aim of this note is to give necessary and sufficient conditions for an Enriques surface over an algebraically closed field of characteristic two to be unirational. We show that such a surface is unirational if and only if it is either classical or supersingular in the sense of Bombieri and Mumford ([1], p. 197).

The method of proof is the following. Following the fundamental classification paper [1] we consider for every Enriques surface a double cover which is cohomologically 'K3 like'. We show that if the Enriques surface is either classical or supersingular then the smooth model of that double covering is either rational or a supersingular K3 surface. Then using a result of the beautiful paper of Rudakov and Šafarevič [9] we conclude that such Enriques surfaces are unirational.

For the remaining type of Enriques surfaces in characteristic two, namely singular Enriques surfaces, the non-unirationality has been shown by R. Crew in his 1981 Princeton thesis. R. Crew is a student of N. Katz. (Previously, T. Katsura proved that result for surfaces defined over a finite field.) Thus we simply quote R. Crew's result.

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§1. Notation and preliminaries

Let k be an algebraically closed field of characteristic $p > 0$. For any smooth and projective surface V over k we denote by the following: $b_i(V) = \dim H^i_{et}(V, Q_l)$, $\rho(V) =$ rank of Pic V/numerical equivalence,

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$\lambda(V)=b_2(V)-\rho(V)=$ Lefschetz number. We have $\lambda(V)\geq 0$ (Igusa's inequality). V is called supersingular iff $\lambda(V)=0$. Alb V denotes the Albanese variety of V. We recall that $\dim \operatorname{Alb} V=\frac{1}{2}b_1(V)$. Following [1], p. 197–216, we call V an Enriques surface iff V has Kodaira dimension zero and $b_1(V)=0$, $b_2(V)=10$, $\chi(\mathcal{O}_V)=1$. In characteristic two there are three types of Enriques surfaces, namely

(i) classical, characterized by the property that $\dim H^1(V,\mathcal{O}_V)=0$,

(ii) supersingular, characterized by the properties that $\dim H^1(V,\mathcal{O}_V)=1$ and the Frobenius map is zero on $H^1(V,\mathcal{O}_V)$,

(iii) singular, characterized by $\dim H^1(V,\mathcal{O}_V)=1$ and the Frobenius map is bijective on $H^1(V,\mathcal{O}_V)$.

V is called a *Zariski surface* if there exists a generically surjective, purely inseparable rational map $g\colon P^2_k\to V$ of degree p where P^2_k is the projective plane over k. For any projective Cohen-Macauley scheme Y of equidimension n over k we denote by ω_Y the dualizing sheaf on Y. (See [6], p. 242.) We will also use an alternative description of ω_Y in terms of rational differential forms. We refer the reader to Kunz's papers [3], [4] for the details of this description. Everywhere in this paper we will assume that the characteristic of k is $p=2$ except in Lemma 1 and in Corollary 1.1 where the characteristic $p>0$ is arbitrary. We begin with a simple lemma for which we could not find a ready reference.

LEMMA 1: *Let $g\colon W\to Z$ be a generically surjective purely inseparable rational map of non-singular surfaces. Then*

(i) $\lambda(W)=\lambda(Z)$,

(ii) $\dim \operatorname{Alb}(W)=\dim \operatorname{Alb}(Z)$.

PROOF OF (i): Shioda has shown that $\lambda(Z)\leq\lambda(W)$ (see [10], p. 234). To prove the opposite inequality, consider the schemes $(W,\mathcal{O}_W^{p^i})=W_i$, $i>0$. First, let us take the map $\alpha_i\colon W\to W_i$ corresponding to the inclusion $\mathcal{O}_W^{p^i}\subseteq\mathcal{O}_W$. Now α_i is a map of smooth surfaces over k. It is finite and radicial, therefore $b_i(W)=b_i(W_i)$ by [8], VIII, 1.2. On the other hand, if $i\gg 0$ the rational map α_i factors

$$\begin{tikzcd} & Z \arrow[dr, "\gamma"] & \\ W \arrow[ur, "g"] \arrow[rr, "\alpha_i"] & & W_i \end{tikzcd} \tag{*}$$

where γ is some dominant rational map over k. Thus, by Shioda's result quoted above, we have $\lambda(W)\geq\lambda(Z)\geq\lambda(W_i)$. To complete the proof we only need to show that $\rho(W) = \rho(W_i)$. For this we use another map, β_i: $W_i \to W$, which corresponds to the p^i-th power map $\mathcal{O}_W \to \mathcal{O}_W^{p^i}$. Although β_i is *not* a map of k-schemes, still β_i is an isomorphism of *abstract schemes* and as such it induces an isomorphism of the abstract groups Pic W_i with Pic W. It is not hard to see that this isomorphism preserves the intersection numbers. Hence $\rho(W) = \rho(W_i)$.

PROOF OF (ii): The diagram (*) shows that $\frac{1}{2}b_1(W) = \dim \mathrm{Alb}(W) \geq \dim \mathrm{Alb}(Z) \geq \dim \mathrm{Alb}(W_i)$, but we also have $\dim \mathrm{Alb}(W) = \frac{1}{2}b_1(W) = \frac{1}{2}b_1(W_i) = \dim \mathrm{Alb}(W_i)$. q.e.d.

COROLLARY 1.1: *In the assumptions of Lemma* 1, *if Z is an Enriques surface, then W is supersingular and* Alb W *is trivial.*

PROOF: It is shown in [1] that $\lambda(Z) = 0$ and we also have $b_1(Z) = 0$ from the definition of Enriques surface. q.e.d.

REMARK 1.2: In the assumptions of Lemma 1, if Z is simply connected, then so is W. (The proof is standard and we omit it.)

Also see [1] and [7].

§2. Unirationality

Our main result is the following theorem.

THEOREM 2: *An Enriques surface over an algebraically closed field of characteristic two is unirational if and only if it is either classical or supersingular.*

PROOF: First of all, R. Crew has shown that a singular Enriques surface is never unirational (see [2]). Therefore, we only have to prove that all supersingular and classical Enriques surfaces in characteristic two are unirational. For the remainder of the proof let X be a classical or a supersingular Enriques surface. Let $\pi: \tilde{X} \to X$ be the purely inseparable covering of degree two constructed in [1], p. 220. By Proposition 9, page 221, $\tilde{X}$ is 'K3 like', namely

$$\dim H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = \begin{cases} 1 & i = 0 \\ 0 & i = 1 \\ 1 & i = 2 \end{cases}$$

and $\omega_{\tilde{X}} \approx \mathcal{O}_{\tilde{X}}$ (where $\omega_{\tilde{X}}$ denotes the dualizing sheaf on $\tilde{X}$). Also $\tilde{X}$ is locally of codimension one in a smooth threefold, so that it is Cohen-Macauley and Gorenstein (i.e., $\omega_{\tilde{X}}$ is locally free). Thus $\tilde{X}$ is normal iff it is nonsingular in codimension one. We consider two cases.

CASE 1: $\tilde{X}$ is normal. Let $X_1 \xrightarrow{\rho} \tilde{X} \xrightarrow{\pi} X$ be a minimal desingularization of $\tilde{X}$. We have the injective map j: $\rho_*\omega_{X_1} \to \omega_{\tilde{X}} \approx \mathcal{O}_{\tilde{X}}$. If all the singularities of X are rational, then j is an isomorphism. Thus $\rho_*\omega_{X_1}$ and also ω_{X_1} has a nowhere vanishing section. Therefore, X_1 is a minimal model and it has Kodaira dimension zero. Also $H^i(X_1, \mathcal{O}_{X_1}) \approx H^i(\tilde{X}, \mathcal{O}_{\tilde{X}})$ for all i. From the table in [1], page 197, it follows that X_1 is a supersingular K3 surface. Now Šaferevič and Rudakov have shown in [9] (Corollary, page 151) that any supersingular K3 surface in characteristic two is unirational; in fact, it is a Zariski surface. Thus X_1 and also X are unirational. We still have to consider the possibility that X has an isolated singularity which is not rational. Since $\omega_{\tilde{X}} \approx \mathcal{O}_{\tilde{X}}$ let us take σ to be a nowhere vanishing section of $\omega_{\tilde{X}}$. Because $\omega_{\tilde{X}}$ is isomorphic to the sheaf of rational differential two-forms on $\tilde{X}$ with no polar curves on $\tilde{X}$, we can think of σ as a rational differential two-form. Now it is well-known that σ has a polar curve on X_1 because of the nonrational singularity. Let K_{X_1} be the divisor of σ on X_1. Let us show that $|nK_{X_1}| = \emptyset$. If not, there is an $f \in k(X_1) = k(\tilde{X})$, $f \neq 0$, such that $(f) + nK_{X_1} \geq 0$ on X_1. But then $f \geq (0)$ on $\tilde{X}$ because K_{X_1} is entirely supported on curves which are contracted to singular points of $\tilde{X}$. Therefore, f must be a constant so that $nK_{X_1} \geq 0$ which contradicts the fact that σ has a polar curve on X_1. We conclude that $P_n(X_1) = 0$ for all $n \geq 1$ so that X_1 is ruled by [1], and therefore rational by Corollary 1.1. Thus X is unirational; in fact, it is a Zariski surface in this situation.

CASE 2: $\tilde{X}$ is not normal. Following Kunz [3], [4], we identify $\omega_{\tilde{X}}$ with a certain sheaf of rational differential two-forms. Since $\omega_{\tilde{X}} \approx 0_{\tilde{X}}$ let σ be the differential form in $\omega_{\tilde{X}}(\tilde{X})$ which corresponds to 1 in $0_{\tilde{X}}$. Let $\tilde{X}_N \xrightarrow{\rho} \tilde{X}$ be the normalization. Let L be the common function field of $\tilde{X}$ and $\tilde{X}_N$. We wish to study the divisor of σ on $\tilde{X}_N$ to be denoted $(\sigma)_N$ (for the definition of the divisor of a differential, see Zariski [12], page 31).

LEMMA 2: $(\sigma)_N < 0$.

PROOF: Since σ corresponds to 1 in $0_{\tilde{X}}$, the divisor $(\sigma)_N$ is supported only on such curves in $\tilde{X}_N$ which map onto a multiple curve of $\tilde{X}$. Let D_1 be any irreducible curve on $\tilde{X}_N$ whose image $C = \rho(D_1)$ is an irreducible multiple curve of $\tilde{X}$ (there exists at least one such curve). Let v_1 be the discrete valuation of L which corresponds to D_1. It is enough to show that $v_1(\sigma) < 0$. Assume the contrary, i.e., $v_1(\sigma) \geq 0$. Let $D_1, D_2, \ldots, D_n$ be all the irreducible curves on $\tilde{X}_N$ which map *onto* C. Let $v_1, v_2, \ldots, v_n$ be the corresponding discrete valuations of L. Now we have assumed that $v_1(\sigma) \geq 0$. Let $v_k(\sigma) = m_k$ for $k \neq 1$. There exists a function f in L such that $v_1(f) = 0$ and $v_k(f) = -m_k$ for $k \neq 1$ by [13], Theorem 18, p. 45. It follows from this that the differential $f\sigma$ has no polar curves among the curves D_i. Now we apply Zariski's theory of subadjoints, [12], page 85, which applies without any essential changes since $\tilde{X}$ is a local complete intersection, to conclude that f belongs to the conductor ideal of the local ring of the curve C on $\tilde{X}$. (See also Kunz [4], pages 69–70.) But in our case that conductor ideal is a proper ideal. Thus f also belongs to the maximal ideal of the local ring of the curve D_1 on $\tilde{X}_N$. Therefore, $v_1(f) > 0$ contrary to our choice of f. This contradiction shows that $v_1(\sigma) < 0$ and thus proves the Lemma.

Let $g: X_1 \rightarrow \tilde{X}_N$ be a desingularization. Using Lemma 1 we now show that $P_n(X_1) = 0$ for $n \geq 1$. Let K be the divisor of σ on X_1. Then $K = \tau + E$ where E is supported on curves which g contracts to points and τ is the strict transform of $(\sigma)_N$ so that $\tau < 0$ by Lemma 2. Since K is a canonical divisor on X_1 it is enough to show that $|nK| = \emptyset$. Suppose $f \in L$, $f \neq 0$, and $(f) + nK \geq 0$ on X_1 but then on $\tilde{X}_N$ we have $(f)_N + (\sigma)_N \geq 0$ where $(f)_N$ is the divisor of f on $\tilde{X}_N$ so that $(f)_N \geq -(\sigma)_N > 0$ which is impossible. Thus $P_n(X_1) = 0$ and we conclude that X_1 is rational by Corollary 1.1. Therefore X is *unirational*, in fact a Zariski surface in this case. q.e.d.

REMARK 2.1: Unirationality of certain, but not all, supersingular Enriques surfaces in characteristic two follows also directly from [1], Proposition 15, and [9].

REMARK 2.2: The proof of Theorem 2 and Remark 1.2 shows that all supersingular and classical Enriques surfaces in characteristic two are simply connected. This is also well-known by other methods.

COROLLARY 2.3: *Shioda's conjecture that supersingular and simply connected surfaces are unirational (see* [11], *p.* 167) *is now established for all surfaces in characteristic two whose Kodaira dimension is* ≤ 0.

PROOF: It follows from [1] that every such surface in characteristic two is either rational, or K3, or a supersingular or classical Enriques surface. Thus the corollary follows from [9] and from our theorem.

OPEN PROBLEM 1. Are all supersingular and classical Enriques surfaces surfaces in characteristic two Zariski surfaces?

Our proof does not show this in the case when the purely inseparable cover X has rational singularities only.

OPEN PROBLEM 2. To determine all the unirational Enriques surfaces over an algebraically closed field of characteristic $p > 2$.

We recall that Shioda [11], p. 161, gave examples of both unirational and non-unirational (classical) Enriques surfaces in every characteristic $p > 2$.

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