V. MIQUEL

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THE VOLUMES OF SMALL GEODESIC BALLS FOR
A METRIC CONNECTION

V. Miquel

§1. Introduction

Let $M$ be a real-analytic Riemannian manifold of dimension $n$. Let $V^\nabla_m(r)$ denote the volume of the geodesic ball with center $m \in M$ and radius $r$, where $\nabla$ denotes the Levi-Civita connection. Then $V^\nabla_m(r)$ can be expanded in a power series in $r$. In 1848 Bertrand–Diguet–Puiseux [3] computed the first two terms for surfaces in $\mathbb{R}^3$. Vermeil [14] in 1917 and Hotelling [11] in 1939 generalized it to arbitrary Riemannian manifolds. Recently, the third and fourth term have been computed by A. Gray [5] and by A. Gray and L. Vanhecke [6], respectively.

To obtain that expansion, it is necessary to discuss general power expansions of tensor fields in normal coordinates as used for example for harmonic spaces (see [13]).

The volumes of tubes about submanifolds of $\mathbb{R}^n$, $\mathbb{C}^n$, $S^n$, $CP^n$ have been computed by H. Weyl [15], R.A. Wolf [17], F. J. Flaherty [4], P. A. Griffiths [9]. The expansions of volumes of tubes about submanifolds of arbitrary Riemannian manifolds are given in [11], [7], [8].

In this note we consider a metric connection $D$ on $M$. Let $V^D_m(r)$ denote the volume of the $D$-geodesic ball $\tilde{B}^D_r(m)$ of center $m$ and radius $r$. Then $\tilde{B}^D_r(m) \subseteq \tilde{B}_r^\nabla(m)$ (see §2). We compute the first non-trivial term $C^D_1$ of the expansion of $V^D_m(r)$. This is our main theorem 5.4. If $M$ is $C^\infty$, we can compute the Taylor expansion of $V^D_m(r)$, since it is the same as in the analytic case, although it may not be convergent.

We shall show that the difference $C^D_1 - C^\nabla_1$ with the case $D = \nabla$ has constant sign and it vanishes only if $\nabla$ and $D$ have the same geodesics.
(Corollary 5.5). On the total volume function, this result \( V_D^m(r) \leq V_v^m(r) \) is a consequence of the inclusion of the \( D \)-balls in the \( v \)-balls. This fact also implies that the volumes coincide only if \( v \) and \( D \) have the same geodesics. The Corollary 5.5 shows that it is also true with the weaker hypothesis \( C^P = C^v \).

These results were announced in [12].

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§2. Geodesic balls for a metric connection

Let \( \langle , \rangle \) be the metric tensor of \( M \), \( \chi(M) \) the algebra of vector fields over \( M \) and \( M_m \) the tangent space to \( M \) at the point \( m \in M \).

A metric connection \( D \) over \( M \) is a linear connection which satisfies

\[
X \langle Y, Z \rangle = \langle D_X Y, Z \rangle + \langle Y, D_X Z \rangle \text{ for every } X, Y, Z \in \chi(M).
\]

By a normal coordinate system \((U; x^1, \ldots, x^n)\) at \( m \) with respect to \( D \) we take a normal coordinate system in the sense of [10] such that the local vector fields \( X_i = \partial/\partial x^i \) are orthonormal at \( m \). Then, if \( \exp_m : B_r(0) \rightarrow U \) is the exponential map associated to \( D \), the normal coordinates are given by \( x^i(\exp_m(\Sigma_{j=1}^{r} a^j e_j)) = a^i \), where \( \{e_1, \ldots, e_n\} \) is an orthonormal basis of \( M_m \).

In this paper we always work in the domain \( U \) of a normal coordinate system.

The injectivity radius \( r_D \) of \((M, D)\) at \( m \) is the supremum of the positive real numbers \( r \) such that \( \exp_m \) is a diffeomorphism of \( B_r(0) \) onto its image.

Let \( \mathcal{U} \) be the open set \( \mathcal{U} = \exp_m B_{r_D}(0) \). For any \( p \) in \( \mathcal{U} \), there exists a unique \( D \)-geodesic arc joining \( m \) and \( p \). Then, we define \( \delta_D^P(m, p) \) as the length of this geodesic arc. Then, since the velocity vector of a geodesic for a metric connection has constant length,

\[
\delta_D^P(m, p) = \|\exp_m^{-1}(p)\|.
\]

Let \( r \) be a positive real number such that \( r < r_D \). We call a
D-geodesic ball of center \( m \) and radius \( r \), the set \( \overline{B}^D_r(m) = \{ p \in U/ D(m, p) \leq r \} \). By (2.2) we have \( \overline{B}^D_r(m) = \exp_m(\overline{B}_r(0)) \).

Now, we examine the inclusion relation between \( \overline{B}^D_r(m) \) and \( \overline{B}^\gamma_r(m) \).

It is well known that, if \( d \) is the standard distance function for the Riemannian manifold \( M \), and \( d(m, p) = r < r_\gamma \), there exists a unique arc of \( \nabla \)-geodesic \( \sigma \) from \( m \) to \( p \) of length \( r \). Moreover, if \( \alpha \) is another arc of curve from \( m \) to \( p \), then the length of \( \alpha \) is greater or equal than \( r \).

Let \( r \) be a real number such that \( r < \min(r_D, r_\gamma) \). If \( p \in \overline{B}^D_r(m) \), there exists an arc of \( D \)-geodesic \( \alpha \) joining \( m \) and \( p \), and another arc of \( \nabla \)-geodesic \( \sigma \) from \( m \) to \( p \). As we have indicated above, we have \( d(m, p) = \) length of \( \sigma \leq \) length of \( \alpha = \delta^D(m, p) \leq r \). Then \( p \in \overline{B}^\gamma_r(m) \) and, consequently, \( \overline{B}^D_r(m) \subseteq \overline{B}^\gamma_r(m) \). It implies \( V^D_m(r) \leq V^\gamma_m(r) \).

We are going to obtain an integral formula for \( V^D_m(r) \), the volume of \( \overline{B}^D_r(m) \). In [6], it was done for the Levi–Civita connection \( \nabla \), by using the Gauss lemma. This approach fails for a general metric connection, and we require the use of polar coordinates as defined in [1] and [2] for a new proof of this formula.

2.1. PROPOSITION: Let \( M \) be orientable, \( \omega \), the standard volume form on \( M \) and \( \omega_1 \ldots n = \omega(X_1, \ldots, X_n) \). For any \( r_0 < r_D \) we have

\[
V^D_m(r_0) = \int_0^{r_0} r^{n-1} \left( \int_{S^{n-1}} \omega_1 \ldots n(\exp_m(ru))\sigma \right) dr,
\]

where \( \sigma \) is the standard volume form on \( S^{n-1} \).

PROOF: The definition of \( V^D_m(r_0) \) gives

\[
V^D_m(r) = \int_{\overline{B}^D_r(m)} \omega = \int_{\overline{B}^\gamma_r(0)} \exp_m^* \omega = \int_{B^\gamma_r(0)} (\omega_1 \ldots n \circ \exp_m) \theta,
\]

where \( \theta \) is the standard volume form on \( M_m \).

Let be \( f : S^{n-1} \times ]0, r_\theta[ \to B^\gamma_r(0) - \{0\} \) the map defining the polar coordinates \((u, r)\). It is well known [2] that \( f^* \theta = r^{-1} dr \wedge \sigma \), so

\[
V^D_m(r) = \int_{S^{n-1} \times ]0, r_\theta[} (\omega_1 \ldots n \circ \exp_m(ru)) r^{-1} dr \wedge \sigma.
\]

From this, (2.3) follows immediately. \( \Box \)
§3. Power expansions in normal coordinates of a r-covariant tensor

Let $S$ be the curvature operator of $D$ given by

$$S_{XY} = D_{[X,Y]} - [D_X, D_Y], \quad S_{XYZW} = \langle S_{XY}, W \rangle.$$  

We denote by $T$ the torsion of $D$, and

$$D_{X_1 \ldots X_p} Y = D_X (D_{X_2} \ldots (D_{X_p} Y) \ldots).$$

We say that $X \in \chi(M)$ is a coordinate vector field at $m$ if there exists constants $a^1, \ldots, a^n$ such that, in $\mathcal{U}$, $X = \sum_{i=1}^n a^i X_i$. From now on $X, Y, Z, \ldots$ will denote coordinate vector fields and $a, b, c, \ldots$ their corresponding integral curves with initial conditions $a(0) = b(0) = c(0) = \cdots = m$. Thus, $a, b, c, \ldots$ are geodesics starting at $m$, and $a'(t) = X_{a(t)}$ wherever $a(t)$ is defined. Moreover we have $S_{XY} = -D_X^2 Y + D_Y^2 X$, $T_X Y = D_X Y - D_Y X$, and $T_{XYZ} = \langle T_X Y, Z \rangle$.

Then, we have the following results, whose proofs follow closely the ones given in [5] for the corresponding ones.

3.1. Lemma:

(3.1.1) \( (D_{X \ldots X} Y)_{a(t)} = 0 \) \quad \( p = 1, 2, \ldots \)

(3.1.2) \( (D_X Y)_m = \frac{1}{2} (T_X Y)_m. \)

3.2. Lemma:

(3.2.1) \( (D_{X \ldots X} Y)_m + \sum_{k=1}^p (D_{X \ldots X} Y)_m = 0. \)

(3.2.2) \( \sum_{k=1}^p (D_{X \ldots X} Y)_m + \sum_{k \neq 1}^p (D_{X \ldots X} Y)_m = 0. \)

(3.2.p-1) \( \sum_{k=1}^p (D_{X \ldots X} Y)_m + \sum_{k=1}^p (D_{X \ldots X} Y)_m = 0. \)

(3.2.p) \( \sum_{k=1}^p (D_{X \ldots X} Y)_m + (D_{X \ldots X} Y)_m = 0. \)
3.3. LEMMA: At \( m \), we have

\[
(p + 1)D^p_{X\ldots Y} - pD^{p-1}_{X\ldots x}(TxY) + (p - 1)D^{p-2}_{X\ldots x}(SxyX) = 0.
\]

From (3.1.1), (3.1.2) and (3.3) we have

\[
(D^3_{XX}Y)_m = \left\{ -\frac{1}{3} S_{XYX} + \frac{2}{3} D_X(T)XY + \frac{1}{3} TxTyY \right\}_m.
\]

The same method works for \( p \geq 3 \) to get \( D^p_{X\ldots x}Y \).

From now on, we assume that the manifold \( M \) and any mathematical object defined on \( M \) are real-analytic. (The expansions are the same for the \( C^\infty \) case).

Let \( W \) be a \( r \)-covariant tensor field on a neighbourhood of \( m \). We denote \( W(X_{a_1}, \ldots, X_{a_r}) \) by \( W_{a_1\ldots a_r} \) and \( D_{X_i} \) by \( D_i \). The power series expansion of \( W_{a_1\ldots a_r} \) is then

\[
(W_{a_1\ldots a_r})_x = \sum_{k=0}^{\infty} \sum_{i_1,\ldots,i_k=1}^{n} \frac{1}{k!} (X_{i_1} \ldots X_{i_k} W_{a_1\ldots a_r})_{mx} x_1 \ldots x_k,
\]

where \( x_1, \ldots, x^n \) are the coordinates of the point \( x \in M \).

Notice that

\[
(X^p W_{a_1\ldots a_r})_m = \sum_{\nu_1+\ldots+\nu_{r+1}=p} \frac{p!}{\nu_1! \ldots \nu_{r+1}!} \times D^\nu_{X_1\ldots x}(W) (D^\nu_{X_1\ldots x}X_{a_1}, \ldots, D^\nu_{X_1\ldots x}X_{a_r})_m.
\]

Then, it is possible to determine (3.5) as a function of \( S, T \) and their covariant derivatives. We can also determine the coefficients of the power series expansion of \( W_{a_1\ldots a_r} \) by linearizing the left hand side of (3.5).

3.4. THEOREM: For any point \( x \) in \( U \) we have the following expansion:

\[
W_{a_1\ldots a_r}(x) = W_{a_1\ldots a_r}(m) + \sum_{i=1}^{r} \left\{ D_i(W)_{a_1\ldots a_r}
\right.
\]

\[
+ \frac{1}{2} \sum_{s=1}^{r} \sum_{q=1}^{n} T_{aq} W_{a_1\ldots a_{s-1} q a_{s+1} \ldots a_r}(m)
\]

\[
\times x^i + \frac{1}{2} \sum_{i,j=1}^{n} \left\{ D_{ij}^r(W)_{a_1\ldots a_r}
\right\}
\]

\[
\times x^i x^j.
\]
PROOF: From (3.1.2), (3.4) and (3.5) we get

\[ + \sum_{s=1}^{r} \sum_{q=1}^{n} T_{ia;q}D_{i}(W)_{a_{1} \ldots a_{1-q}a_{1+1} \ldots a_{r}} + \frac{1}{3} \sum_{s=1}^{r} \sum_{q=1}^{n} \left( -S_{ia,jq} + 2D_{i}(T)_{ia,q} + \sum_{\beta=1}^{n} T_{\beta q}T_{ia,\beta} \right) \]

\[ \times W_{a_{1} \ldots a_{1-q}a_{1+1} \ldots a_{r}} + \frac{1}{4} \sum_{s \neq t=1}^{r} \sum_{q,h=1}^{n} T_{ia,q}T_{ja,h} \]

\[ \times W_{a_{1} \ldots a_{1-q}a_{1+1} \ldots a_{r}-1q_{h+1} \ldots a_{r}} \]  

\[ \{ m \} x^{i} x^{i}. \]

We apply this expansion to the metric tensor. Let \( g_{ij} = \langle X_{i}, X_{j} \rangle \), then \( g_{ij}(m) = \delta_{ij} \) and, since \( D(\ , \ ) = 0 \), we get

3.5. PROPOSITION: For any \( x \) in \( U \) and \( A, B = 1, \ldots, n \), we have

\[
g_{AB}(x) = \delta_{AB} + \frac{1}{2} \sum_{i=1}^{n} (T_{iAB} + T_{IBA})_{m} x^{i} + \frac{1}{6} \sum_{l,j=1}^{n} \left\{ -(S_{lAB} + S_{BjA}) + 2(D_{i}(T)_{lAB} + D_{i}(T)_{jBA} \right. \]

\[ + \sum_{\beta=1}^{n} (T_{\beta lA}T_{j\beta} + T_{l\beta B}T_{jA\beta}) \quad + \frac{3}{4} \sum_{\beta=1}^{n} (T_{lA\beta}T_{j\beta} + T_{lB\beta}T_{jA\beta}) \right\}_{m} x^{i} x^{j} + \ldots.\]
In the remainder of this paper we assume that $M$ is orientable. This is not a real restriction, since we are always working locally.

We choose the normal coordinates in such a way that $\{X_1, \ldots, X_n\}$ is a positively-oriented local frame. As $X_1, \ldots, X_n$ are orthonormal at $m$, we have $\omega_1 \ldots n(m) = 1$. Clearly then $D\omega = 0$.

Let $\rho$ be the Ricci tensor of the connection $D$. Then, for any local orthonormal frame $\{E_1, \ldots, E_n\}$, $\rho(X, Y) = \sum_{i=1}^n S_{XE_iYE_i}$.

3.6. **Proposition:** Applying 3.4 to $\omega$, we get, for any $x$ in $U$,

$$\omega_1 \ldots n(x) = 1 + \frac{1}{2} \sum_{i=1}^n \left( \sum_{\beta=1}^n T_{i\beta\beta} \right) x^i + \frac{1}{6} \sum_{i,j=1}^n \left( -\rho_{ij} + 2 \sum_{\beta=1}^n D_i(T)_{i\beta\beta} + \frac{1}{4} \sum_{\beta,\delta=1}^n T_{i\beta\beta} T_{i\delta\delta} \right) x^i x^j + \cdots.$$ 

§4. **Relationship between $T$ and $B$**

Let $B$ be the difference tensor of the connections $D$ and $\nabla$, $B_{XY} = D_{XY} - \nabla_{XY}$. We define $B_{XYZ} = \langle B_{XY}, Z \rangle$. Then

$$T_{XYZ} = B_{XYZ} - B_{YYZ}.$$ 

It is well known (see [10]) that the connections $D$ and $\nabla$ have the same geodesics if and only if $B_{XY} = -B_{YX}$. Moreover [18], a connection $D$ is a metric connection if and only if $B_{XYY} = -B_{XYX}$. Then the connections $D$ and $\nabla$ have the same geodesics if and only if $T_{XYZ} = -T_{XZY}$. In fact, $D$ and $\nabla$ have the same geodesics if and only if $B_{XY} = -B_{YX}$, i.e., $\langle T_X Y, Y \rangle = \langle B_{XY} - B_{YX}, Y \rangle = 2\langle B_X Y, Y \rangle = 0$. In this case the torsion $T$ and the tensor $B$ belong to the irreducible subspace $\Lambda^3 V^* \otimes \Lambda^2 V^*$ and $\Lambda^2 V^* \otimes \Lambda^3 V^*$, respectively, where $V = M_m$.

It is also useful to have an expression of $B$ in terms of $T$. Yano [18] proved

$$B_{XYZ} = \frac{1}{2} (T_{XYZ} + T_{ZXY} - T_{YZX}).$$

Here, we give another proof of this formula by using elementary representation theory. Later we shall use the same method to obtain a good formula for the expansion of $V^D_m(r)$.
The tensor $T_m$ belongs to $\Lambda^2 V^* \otimes V^*$ and $B_m$ to $V^* \otimes \Lambda^2 V^*$. Since the map $\alpha : V^* \otimes \Lambda^2 V^* \to \Lambda^2 V^* \otimes V^*$ given by $\alpha(B_m) = T_m$ is $\text{Gl}(n)$-invariant, it is a multiple of the intertwining operator between the $\text{Gl}(n)$-irreducible subspaces of $V^* \otimes \Lambda^2 V^*$ and those of $\Lambda^2 V^* \otimes V^*$.

The space $W = \Lambda^2 V^* \otimes V^* \cong V^* \otimes \Lambda^2 V^*$ decomposes in the form

$$W = \Lambda^3 V^* \otimes Y_1^3,$$

where $Y_1^3$ is the irreducible representation of $\text{Gl}(n)$ with Young diagram $\square$. The projection $\beta$ on $\Lambda^3 V^*$ is given by

$$\beta(A)_{XYZ} = \frac{1}{3} (A_{XYZ} + A_{ZXY} + A_{YXZ})$$

and $\ker \beta$ identifies itself with $Y_1^3$.

To obtain the inverse of $\alpha$, we observe that the restriction to $\Lambda^3 V^*$ is given by

$$\alpha(A)_{XYZ} = A_{XYZ} - A_{YXZ} = 2A_{XYZ},$$

and, the restriction to $Y_1^3 = \ker \beta$ is

$$\alpha(A)_{XYZ} = A_{XYZ} - A_{YXZ} = A_{XYZ} + A_{ZXY} + A_{XZY} = -A_{ZXY}.$$

Then

$$B_{XYZ} = \alpha^{-1}(T)_{XYZ} = \frac{1}{2} \beta(T)_{XYZ} - (T - \beta(T))_{YXZ}$$

$$= \frac{1}{2} (T_{XYZ} + T_{ZXY} - T_{YXZ}).$$

§5. Power series expansion of the volume function of a $D$-geodesic ball

5.1. PROPOSITION: For any $r$ such that $0 < r < r_D$, we have:

$$V^D_n(r) = \left( \frac{\pi r^2}{(n/2)!} \right) \left\{ 1 + \frac{1}{n+2} \left( -\frac{1}{6} \tau_D - \frac{1}{3} \partial \bar{T} + \frac{1}{24} \bar{T} + \frac{1}{8} \| \bar{T} \|^2 \right) r^2 + O(r^4) \right\}$$
where

$$\tau_D = \sum_{i,j=1}^{n} S_{ij}(m) \quad (\text{scalar curvature of } D \text{ at } m),$$

$$-\delta \bar{T} = \sum_{i,j=1}^{n} D_i(T)_{ij}(m), \quad \|T\|^2 = \sum_{i,j=1}^{n} T_{ij}^2(m)$$

$$\tilde{T} = \sum_{i,j,k=1}^{n} T_{ijk} T_{ikj}(m), \quad \|\tilde{T}\|^2 = \sum_{i=1}^{n} \tilde{T}^2_i(m),$$

$\tilde{T}$ being the one-form defined by $\tilde{T}_X = \sum_{j=1}^{n} T_{X E_j E_j}$ for any local orthonormal frame $\{E_1, \ldots, E_n\}$.

The proof – which makes use of 2.1 and 3.6 – follows closely the one given in [6] for the Levi–Civita connection.

Some geometric formulas will prove useful to eliminate $\delta \tilde{T}$ in 5.1. For this, we need the following lemmas:

5.2. **Lemma:** If $R$ is the curvature tensor of $\nabla$, at $m$, we have:

$$S_{xyxy} = R_{xyxy} - \frac{1}{2}(\|B_X Y\|^2 + \|B_Y X\|^2) + \langle B_X X, B_Y Y \rangle + \|T_X Y\|^2$$

$$+ \frac{1}{2}(\langle T_X T_X Y, Y \rangle + \langle T_Y T_Y X, X \rangle) + \langle D_X (T)_X Y, Y \rangle$$

$$+ \langle D_Y (T)_Y X, X \rangle.$$

**Proof:**

$$S_{xyxy} = -\langle D_X^2 X, Y \rangle + \langle D_Y^2 X, Y \rangle$$

$$= R_{xyxy} - \langle B_X \nabla Y X, Y \rangle - \langle \nabla X (B_Y X), Y \rangle$$

$$- \langle B_X B_Y X, Y \rangle + \langle B_Y \nabla X X, Y \rangle + \langle \nabla Y (B_X X), Y \rangle + \langle B_Y B_X X, Y \rangle.$$

But, at $m$, by (3.1.2)

$$\nabla_Y X = D_Y X - B_Y X = \frac{1}{2} T_Y X - B_Y X = -\frac{1}{2} (B_X Y + B_Y X).$$

Then, using (4.1) and (4.2) we have

$$S_{xyxy} = R_{xyxy} - \frac{1}{2} \langle B_X Y, B_Y X + B_X Y \rangle - X \langle T_Y X, Y \rangle$$

$$- \frac{1}{2} \langle B_Y X, B_X Y + B_Y X \rangle$$

$$+ \langle B_X Y, B_Y X \rangle + \langle B_Y Y, B_X X \rangle + Y \langle T_Y X, X \rangle;$$

and the lemma follows by a direct computation, using (3.1.2).
5.3. LEMMA: If \( \tau_V \) is the scalar curvature of \( V \), at \( m \), we have

\[
\tau_D = \tau_V + \frac{1}{4} \|T\|^2 + \frac{1}{2} \|\tilde{T}\|^2 - 2 \partial \tilde{T}.
\]

PROOF: From 5.2 we see that

\[
(5.1) \quad \tau_D = \tau_V - \|B\|^2 + \|\tilde{B}\|^2 + \|T\|^2 + \tilde{T} - 2 \partial \tilde{T},
\]

where \( \|B\|^2 = \sum_{i,j,k=1} B^2_{ijk}(m) \) and \( \|\tilde{B}\|^2 = \sum_{i=1} (\sum_{j=1} B_{ij})^2(m) \). The result is then immediate, since from (4.2) we have

\[
(5.2) \quad \|B\|^2 = \frac{3}{4} \|T\|^2 + \frac{1}{2} \tilde{T},
\]

\[
(5.3) \quad \|\tilde{B}\|^2 = \|\tilde{T}\|^2. \quad \square
\]

Now, we consider the decomposition of \( \Lambda^2 V^* \otimes V^* \) into \( O(n) \)-irreducible subspaces. \( \Lambda^3 V^* \) is already \( O(n) \)-irreducible, but \( Y^2_i \) decomposes into two subspaces, namely \( \tilde{Y}^2_i = \{ A \in Y^2_i / A_X = 0 \text{ for any } X \in V \} \) and

\[
(5.4) \quad \tilde{T} = \frac{1}{n-1} \left( -\langle X, Z \rangle A_Y + \langle Y, Z \rangle A_X \right)
\]

(\( A_X = \sum_{i=1} A_X e_i \), \( \{ e_i \} \) being an orthonormal basis of \( V \)). (cfr [16]).

If we split \( T = T^1 + T^2 + T^3 \), with \( T^1 \) belonging to \( \Lambda^2 V^* \), \( T^2 \) to \( \tilde{Y}^2_i \) and \( T^3 \) to \( \tilde{Y}^2_i \), obviously \( \|T\|^2 = \|T^1\|^2 + \|T^2\|^2 + \|T^3\|^2 \).

If \( \tilde{\alpha} : \Lambda^2 V^* \otimes V^* \to \Lambda^3 V^* \otimes V^* \) is the map given by \( \tilde{\alpha}(A)_{XYZ} = A_{YX} - A_{XY} \), then \( \tilde{\alpha}|_{Y^2_i} = -2I \) and \( \tilde{\alpha}|_{Y^2_i} = I \) (here \( I \) is the identity map). Moreover, \( \tilde{\alpha} \) is \( \text{Gl}(n) \)-invariant, and \( \tilde{T} = (1/2)(T, \tilde{\alpha}(T)) \). Then

\[
(5.5) \quad \|\tilde{T}\|^2 = \|T\|^2 = \frac{n-1}{2} \|T\|^2.
\]

Now 5.1 can be reformulated as follows:
5.4. **THEOREM:** For any $r$ such that $0 < r < r_D$ we have

$$V_m^D(r) = \frac{(\pi r^2)^{(n/2)}}{(n/2)!} \left( 1 - \frac{1}{6(n+2)} \left( \tau_\nu + \frac{3}{8} \|T\|^2 \right) + \frac{n+2}{8} \|T\|^2 \right) r^2 + O(r^4).$$

If $V_0 = \frac{\pi (n/2)}{(n/2)!}$ (the volume of the unit ball in $\mathbb{R}^n$), 5.4 can be rewritten

$$V_m^D(r) = V_0 r^n \{ 1 + C_1^D r^2 + C_2^D r^4 + \cdots + C_n^D r^{2n} + \cdots \}$$

and we can state the following corollaries:

5.5. **COROLLARY:** $D$ and $\nabla$ have the same geodesics if and only if $C_1^D = C_1^\nabla$ for any $m$ in $M$.

**PROOF:** $C_1^D = C_1^\nabla$ implies $T^2 = T^3 = 0$, so $T = T^1$, i.e., $T$ lies on $\Lambda^1 V^*$ and, as we have indicated in §4, $D$ and $\nabla$ have the same geodesics. \(\Box\)

5.6. **COROLLARY:** If $M$ has non-negative Ricci curvature $\rho_\nabla$ and $C_1^D = C_2^D = 0$ for any $m$ in $M$, then $M$ is locally flat ($R = 0$).

**PROOF:** $\rho_\nabla(X, X) \geq 0$ gives $\tau_\nabla \geq 0$. Since $C_1^D = 0$, from 5.4 we have $\tau_\nabla = 0$ and $T = T^1$. Then $D$ and $\nabla$ have the same geodesics, $V_m^D(r) = V_m^\nabla(r)$ and $C_1^D = C_2^D = C_1^\nabla = C_2^\nabla = 0$. But in [6] it is proved that if $\rho_\nabla(X, X) \geq 0$ and $C_1^\nabla = C_2^\nabla = 0$, then $R = 0$. \(\Box\)

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Departamento de Geometria y Topologia
Facultad de Ciencias Matematicas
Burjasot, Valencia
Spain