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ABELIAN CLOSURE IN SOLUBLE LIE RINGS

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In this paper we examine the question of what properties are inherited by soluble Lie rings from their abelian subrings.

More formally, suppose $\mathscr{X}$ is a class of Lie rings which is closed with respect to taking abelian subrings. We will say that $\mathscr{X}$ is *abelian-closed* if given any soluble Lie ring L all of whose abelian subrings are in $\mathscr{X}$ then L is also in $\mathscr{X}$.

Similar investigations have been made for groups by Mal'cev [6], Schmidt [7] and Čarin [2].

§1. Notation and Initial Observations

Most of the terminology we will use is already fairly standard in the theory of Lie algebras (see Jacobson [5] or Amayo and Stewart [1]). However some notions special to Lie rings, and due for the most part to the existence of torsion elements, need explanation.

The collection of all torsion elements of a Lie ring L is a characteristic ideal and is denoted by $T(L)$ (a characteristic ideal is invariant under any derivation of L).

If p is a prime then L_p denotes the collection of all $x \in L$ such that $p^k x = 0$ for some positive integer k. L_p is also a characteristic ideal, called the *p-component* of L and $T(L)$ is a direct sum of the various L_p as p ranges over all primes (cf. Fuchs [3] Ch XII).

Likewise the divisible subgroup of L is a characteristic ideal denoted by $D(L)$.

The *torsion free rank* $r_0(L)$, the *p-rank* $r_p(L)$ and the *total rank* $r(L)$ are just the corresponding ranks for the underlying abelian group of L (Fuchs [4] pg. 85 ff).

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We will use the following notation for the classes of Lie rings we consider:

$\mathscr{F}$	finite Lie rings
Max	Lie rings satisfying the ascending chain condition on subrings
Min	Lie rings satisfying the descending chain conditions on subrings
$\mathscr{G}$	finitely generated Lie rings
$\mathscr{G}_Z$	Lie rings of finite type i.e. having a finitely generated underlying abelian group
$\mathscr{C}$	one generator (i.e. cyclic) Lie rings
$\mathscr{A}$	abelian Lie rings
$\mathscr{A}_0$	abelian Lie rings with $r_0(L) < \infty$
$\mathscr{A}_1$	abelian Lie rings with $r_0(L) < \infty$ and $r_p(L) < \infty$ for all primes p.
$\mathscr{A}_2$	abelian Lie rings with $r(L) < \infty$
$\mathscr{A}_3$	abelian Lie rings with $r_0(L) < \infty$ and $T(L) \in \mathscr{F}$

Note that $\mathscr{C} < \mathscr{A}$ and also $\mathscr{A}_3 < \mathscr{A}_2 < \mathscr{A}_1 < \mathscr{A}_0 < \mathscr{A}$ since these relations are already true for abelian groups. $\mathscr{A}$, $\mathscr{A}_0$, $\mathscr{A}_1$, $\mathscr{A}_2$ and $\mathscr{A}_3$ are all closed under taking subrings but only $\mathscr{A}$, $\mathscr{A}_0$ and $\mathscr{A}_1$ are closed under quotients.

If $\mathscr{X}$ is a class of Lie rings then we write $\mathrm{E}\mathscr{X}$ for the class of Lie rings with a finite ideal series each of whose factors is in $\mathscr{X}$. We shall be interested in the classes $\mathrm{E}\mathscr{C}$ (polycyclic Lie rings), $\mathrm{E}\mathscr{A}$ (soluble Lie rings) and $\mathrm{E}\mathscr{A}_i$, $i = 0, \ldots, 3$. All these classes are closed under taking subrings.

Note that we will often consider abelian Lie rings simply as abelian groups, carrying over properties and terminology where convenient. Fuchs [4] can always be used as a reference. Similarly, properties of the underlying abelian group of a Lie ring L are often useful in determining the properties of L. We will frequently use the fact that the derivation ring, $\mathrm{Der}(L)$, of L is a Lie subring of the endomorphism ring of L considered as an abelian group and supplied with the usual commutator product.

§2. Abelian Closure and Chain Conditions

First we will consider the minimal condition on subrings. If $L \in$ Min then it contains a unique minimal ideal of finite index. If L is also soluble then we have:

LEMMA 1: *Let $L \in \mathrm{E}\mathscr{A} \cap \mathrm{Min}$, then L is a finite extension of a central, divisible, abelian ring (and consequently is countable).*

PROOF: L has an invariant abelian series each of whose factors satisfies the descending chain condition on subrings and hence each factor is a torsion abelian group. Hence L is a torsion ring.

Let N be the unique minimal ideal of finite index. Then N has no proper subrings of finite index. For suppose P were such a subring, then for some integer m

$$mL \leq P$$

Now mL is always a (characteristic) ideal of L and so we can form N/mL and by construction this has finite exponent.

By solubility there is a characteristic abelian series.

$$mL = L_0 < L_1 < \cdots < L_n = N$$

Then L_n/L_{n-1} is abelian of finite exponent and satisfies the minimal condition for subrings. Hence it is finite. But L_{n-1} is a characteristic ideal in N and so is an ideal in L (if I is an ideal of L and J is a characteristic ideal of I then J is always an ideal of L). But this is a contradiction, hence N has no proper subrings of finite index.

If $m > 0$ and $mN < N$ then as above there exists a characteristic abelian series from mN to N. Looking at the top factor again gives a contradiction and so $mN = N$ for all m and so N is divisible.

Finally, in a torsion Lie ring $D(L)$ is contained in the centre $Z(L)$. Indeed suppose $y \in L$ and $ny = 0$ for some integer $n > 0$. Let $x \in D(L)$. By divisibility we can find $z \in D(L)$ such that $x = nz$. Then

$$[x, y] = [nz, y] = [z, ny] = 0$$

and so $x \in Z(L)$.

Note that the initial conditions in this result can be weakened to allow L to have only the minimal condition on subideals. But then the result in any case forces L to satisfy the minimal condition on subrings.

LEMMA 2: *Let $L \in \mathrm{E}\mathscr{A} \cap \mathrm{Min}$ and let $\Gamma \leq \mathrm{Der}(L)$ be a torsion Lie ring of derivations of L. Then $\Gamma \in \mathscr{F}$*

PROOF: By Fuchs [3] p. 207 the endomorphism ring of a divisible abelian group is torsion free. Hence any Lie ring of derivations of a divisible Lie ring is torsion free.

By Lemma 1, $D = D(L)$ is torsion abelian, and $L/D \in \mathcal{F}$. Now Γ induces a Lie ring of derivations on the characteristic ideal D, and since Γ is torsion this action must be trivial by the above.

Now consider a derivation $d: L \to L$ inducing zero on L/D and killing D. Then d is fully determined by its action on a finite set X (of coset representatives for L/D), and it sends X to a subgroup Y of D, whose exponent is bounded (by the exponent of L/D). Hence Y is finite since D is divisible. Consequently, since d sends a finite set X to a finite set Y, only finitely many such d are possible. Further, since $L/D \in \mathcal{F}$, only finitely many derivations $L/D \to L/D$ are possible. These two facts taken together mean $\Gamma \in \mathcal{F}$.

THEOREM 3: Min *is abelian-closed.*

PROOF: Suppose L is soluble with each of its abelian subrings in Min. Let L have derived length d. We will use induction on d.

Let N be an ideal in L maximal with respect to $N^{(d-1)} = 0$ and $N \geq L^{(1)}$ ($L^{(n)}$ denotes the nth term in the derived series of L). By the induction hypothesis $N \in \text{Min}$.

Now consider

$$C = C_L(N) = \{y \in L \mid [N, y] = 0\}$$

By the maximality of N we have $C \leq N$. The cyclic subrings generated by each element of L are abelian and so are in Min. Hence L is a torsion ring.

Since N is an ideal of L, so is C and we can consider L/C to be a (torsion) Lie ring of derivations of N. Hence by lemma 2

$$L/C \in \mathcal{F} \leq \text{Min}$$

The result now follows since Min is closed under extensions.

We now consider what happens with the maximal condition. Once again we need information about derivations.

LEMMA 4: *Let L be a Lie ring and $\Gamma \leq \text{Der}(L)$. (i) If $L \in \mathcal{G} \cap \mathcal{A}$ then $\Gamma \in \mathcal{G}_Z < \mathcal{G}$. (ii) If $L \in \mathcal{G}_Z$ then $\Gamma \in \mathcal{G}_Z < \mathcal{G}$. (iii) If $L \in \text{E}\mathcal{C}$ then $\Gamma \in \mathcal{G}_Z < \mathcal{G}$. Further if $\Gamma \in \text{E}\mathcal{A}$ then $\Gamma \in \text{E}\mathcal{C}$.*

PROOF: (i) This follows from the fact that if L is a finitely generated abelian group then its ring of endomorphisms is finitely generated as an abelian group (Fuchs [3] p. 212 ff). (ii) By case (i). (iii) Since $\mathcal{G}_Z \cap \mathrm{E}\mathcal{A} = \mathrm{E}\mathcal{C}$.

THEOREM 5: *If $L \in \mathrm{E}\mathcal{A}$ and each of its abelian subideals is finitely generated then $L \in \mathrm{E}\mathcal{C}$.*

PROOF: This is clearly true when $L \in \mathcal{A}$ so assume $L \notin \mathcal{A}$. Let N be the last nontrivial term of the derived series of L. By hypothesis $N \in \mathcal{G}$.

Let H/N be an abelian subideal of L/N and let $C = C_H(N)$. Now $C \geq N$ and so $H/C \in \mathcal{A}$ and hence by Lemma 4(i), $H/C \in \mathcal{G}$.

Now $C^2 = [C, C] \leq N$ and so $[C^2, C] = 0$ and C is nilpotent (of length 2). Let M be a maximal abelian ideal of C. M is a subideal of L and so $M \in \mathcal{G}$. By the maximality of M we have

$$M = C_C(M)$$

Hence since $C^2 \leq Z(C) \leq M$ we have $C/M \in \mathcal{A}$. By lemma 4(i) again we have $C/M \in \mathcal{G}$. Hence $H \in \mathcal{G}$.

Thus $H/N \in \mathcal{G}$ and L/N satisfies the initial hypotheses of the theorem. By induction on the derived length the result now follows.

We can now restate this result in a number of forms:

COROLLARY: (*i*) $\mathrm{E}\mathcal{C}$ *is abelian-closed.* (*ii*) *If $L \in \mathrm{E}\mathcal{A}$ is such that all its abelian subrings are finitely-generated then $L \in \mathcal{G}$. (Note that the terminology of abelian-closure cannot be used here since $\mathcal{G}$ is not closed with respect to taking abelian subrings.)* (*iii*) Max *is abelian-closed.* (*iv*) $\mathcal{F}$ *is abelian-closed.*

PROOF: (i) and (ii) follow from $\mathrm{E}\mathcal{C} < \mathcal{G}$. (iii) Follows since $\mathrm{E}\mathcal{C} <$ Max. (iv) Let $L \in \mathrm{E}\mathcal{A}$ with all its abelian subrings finite. Now $\mathcal{F} < \mathcal{G} \cap \mathrm{Min}$. Hence by Theorem 3 and Theorem 5

$$L \in \mathrm{E}\mathcal{C} \cap \mathrm{Min}$$

Now if $L \in \mathrm{E}\mathcal{C}$ then $D(L) = 0$ and so by Lemma 1 we have $L \in \mathcal{F}$.

§3. Abelian Closure and Rank Conditions

LEMMA 7: Let *L be a torsion Lie ring, and suppose (by slight abuse of notation) that the underlying abelian group of L* is in $\mathcal{A}_1$. *Then every finite set of elements of L lies in a finite characteristic ideal of L.*

PROOF: Since L is torsion it is the direct sum of its p-components L_p, and the underlying abelian group of each L_p is a (group) direct sum of finitely many cyclic groups of order p^k for various k and finitely many Prufer $\mathscr{C}_{p^\infty}$ groups.

Let $x_1, \ldots, x_n \in L$ with each x_i of order m_i say. Let $m = m_1 m_2 \ldots m_n$. Clearly m involves only finitely many primes and so

$$L[m] = \{x \in L | mx = 0\}$$

is finite. But this is a characteristic ideal since it is a fully invariant subgroup of L considered as an abelian group.

LEMMA 8: *Suppose L is a Lie ring and H is an ideal of L with $H \in \mathrm{E}\mathscr{A}_1$ and $L/H \in \mathscr{A} \backslash \mathscr{A}_0$. Then L contains a free abelian subring of countable rank.*

PROOF: *Case* (*i*) $H = 0$. This follows immediately from the definition of $\mathscr{A}_0$.

In view of this case, since L/H will always contain a free abelian subring of countable rank we may assume without loss of generality that L/H is in fact such a ring.

Case (*ii*) $H \in \mathscr{A}_1$ and H is torsion free.

Let A be a maximal abelian subring of L with $A \geq H$. Let $r_0(H) = n$ say. Suppose $r_0(A) = m (\geq n)$. By the maximality of A we have $A = C_L(A)$ and since $L/H \in \mathscr{A}$ we have that A is an ideal of L. Hence L/A may be considered as a subring of $\mathrm{Der}(A)$.

Now we can consider A as being embedded in $V = Q \otimes_Z A$ a vector space over the rationals of dimension m. Now V has dimension m, hence the endomorphism ring of V has dimension m^2 (being isomorphic to the ring of $m \times m$ matrices over Q). Hence $\mathrm{Der}(A)$ has rank $\leq m^2$ and consequently so too does L/A. This means $L/A \in \mathscr{A}_1$ and in particular $r_0(L) < \infty$, which is a contradiction.

Case (*iii*) $H \in \mathscr{A}$ and H is torsion. Since we are assuming that L/H is free abelian of countable rank suppose

$$L/H \cong \bigoplus_{i \in Z} \langle x_i + H \rangle$$

We will construct a sequence of elements $y_1, y_2, \ldots$ such that $y_i = k_i x_i$ for nonzero integers k_i and

$$[y_i, y_j] = 0 \quad \text{for all } i \text{ and } j.$$

Since $L/H \in \mathfrak{A}$ we have $[x, y] \in \mathfrak{A}$ for all $x, y \in L$. Take $y_1 = x_1$ and suppose that $y_1, \ldots, y_n$ have been constructed. By Lemma 7 $[y_i, x_{n+1}]$ for $i = 1, \ldots, n$ all lie in a finite characteristic subring $F \leq H$. Since H is an ideal of L and F is characteristic, F is also an ideal of L. Suppose $|F| = m$, then for all $i = 1, \ldots, n$

$$[y_i, mx_{n+1}] = m[y_i, x_{n+1}] = 0$$

Put $y_{n+1} = mx_{n+1}$ and then y_{n+1} is as required.

Now let A be the subring generated by $y_1, y_2, \ldots$. The natural homomorphism

$$\xi : L \to L/H$$

maps A onto $(A + H)/H$. Now L/H is torsion free so $nx_i \notin H$ for all n, and since the $x_i + H$ generate L/H, ξ restricted to A is injective. Hence A is free abelian of countable rank.

Case (*iv*) $H \in \mathfrak{A}$.

Let $T = T(L)$ and use induction on $r_0(H/T) = n$.

If $n = 0$ then $T = H$ and case (ii) applies. If $n > 0$ choose an ideal K of L with $T \leq K \leq H$ and K of maximal rank subject to

$$r_0(K/T) < r_0(H/T)$$

We may assume L/K is torsion free (for otherwise we can just factor out the torsion ideal). Case (ii) now applies to show that L/K has a subring A/K which is free abelian of countable rank.

The induction hypothesis can now be applied to show that A has a free abelian subring of countable rank.

Case (*v*) The general case.

We now use induction on the derived length d of H.

If $d = 1$ use case (iv). Suppose $d > 1$. Then by induction $L/H^{(d-1)}$ has a free abelian subring of countable rank. But $H^{(d-1)} \in \mathfrak{A}$ so a further application of case (iv) finishes the argument.

LEMMA 9: *If* $L \in \mathrm{E}\mathfrak{A}$ *and* L *is torsion free then* L *has a finite characteristic series with factors which are torsion free abelian.*

PROOF: The proof will be by induction on the derived length d of L.

The result is clear when $d = 1$ so suppose $d > 1$. Then $L/L^{(d-1)} \in \mathrm{E}\mathfrak{A}$ and has derived length $d - 1$. Put

$$T/L^{(d-1)} = T(L/L^{(d-1)})$$

Now T is a characteristic ideal of L, L/T is torsion free and has derived length $\leq d-1$, hence by induction L/T has a finite series of the required type.

Let $C = C_T(L^{(d-1)})$. Then C is a characteristic ideal of L and as usual T/C may be considered a subring of $\mathrm{Der}(L^{(d-1)})$. Now $L^{(d-1)}$ is torsion free so considered as an abelian group its endomorphism ring is torsion free. Hence T/C is torsion free. But T/C is a quotient of $T/L^{(d-1)}$ which is a torsion ring. Hence $T = C$.

So $[T, L^{(d-1)}] = 0$ and $L^{(d-1)} \leq Z(T)$.

Now if L is a torsion free Lie ring then $L/Z(L)$ is also torsion free. Indeed let $x \in L$ be such that $nx \in Z(L)$ for some integer $n \neq 0$. Then for any $y \in L$

$$0 = [nx, y] = n[x, y]$$

Hence $[x, y] = 0$ since L is torsion free and $x \in Z(L)$.

Now $T/Z(T)$ is a quotient of $T/L^{(d-1)}$ and so is torsion, but by the above observation it is also torsion free. Hence $T = Z(T)$ and $T \in \mathfrak{A}$. The result now follows from the case $n = 1$.

Suppose $\mathfrak{X}$ is any class of torsion, abelian Lie rings. Define a class $\bar{\mathfrak{X}}$ by $L \in \bar{\mathfrak{X}}$ if and only if

(i) $L \in \mathfrak{A}$

(ii) $T(L) \in \mathfrak{X}$

and

(iii) $r_0(L/T(L)) < \infty$

THEOREM 10: *Let $\mathfrak{X}$ be a class of torsion, abelian Lie rings such that: (a) $\mathfrak{F} \cap \mathfrak{A} \leq \mathfrak{X} \leq \mathfrak{A}_1$; (b) $\mathfrak{X}$ is closed under the taking of subrings. Then if $\mathrm{E}\mathfrak{X}$ is abelian-closed, $\mathrm{E}\bar{\mathfrak{X}}$ is abelian-closed.*

PROOF: Let $L \in \mathrm{E}\mathfrak{A}$ and suppose that all its abelian subrings lie in $\bar{\mathfrak{X}}$. We will use induction on the derived length d of L.

If $d = 1$ then $L \in \bar{\mathfrak{X}}$. If $d > 1$ then by induction we may assume

$$L^2 \in \mathrm{E}\bar{\mathfrak{X}} \leq \mathrm{E}\mathfrak{A}_1$$

If $L/L^2 \notin \mathfrak{A}_0$ then by Lemma 8 L has a free abelian subring of countable rank which is a contradiction. Hence $L/L^2 \in \mathfrak{A}_0$ and so $L \in \mathfrak{A}_0$.

Put $T = T(L)$. Then by Lemma 9 L/T has a finite characteristic series with torsion free, abelian factors. Hence $L/T \in \mathrm{E}\bar{\mathscr{X}}$ (clearly each of the factors is in $\bar{\mathscr{X}}$).

By hypothesis $T \in \mathrm{E}\mathscr{X}$ and since $\mathscr{X}$ and $\bar{\mathscr{X}}$ coincide for torsion rings we have $T \in \mathrm{E}\bar{\mathscr{X}}$. Hence $L \in \mathrm{E}\bar{\mathscr{X}}$ as required.

We now obtain the required results as corrollaries to this theorem.

COROLLARY 11: *(i)* $\mathrm{E}\mathscr{A}_1$ *is abelian-closed. (ii)* $\mathrm{E}\mathscr{A}_2$ *is abelian-closed. (iii)* $\mathrm{E}\mathscr{A}_3$ *is abelian-closed.*

PROOF: (i) Take $\mathscr{X}$ to be the class of torsion Lie rings in $\mathscr{A}_1$. Let $L \in \mathrm{E}\mathscr{A}$ be a torsion ring and suppose that all its abelian subrings are in $\mathscr{A}_1$.

Now $L \in \mathrm{E}\mathscr{A}_1$ if and only if

$$L_p \in \mathrm{Min} \cap \mathrm{E}\mathscr{A}$$

for all primes p. (Use induction on the derived length d. Then $L_p^{(d-1)}$ is in $\mathscr{A}_1$ and hence, since it is torsion, in Min.)

Since L_p is a direct factor of L, the abelian subrings of L_p are precisely the abelian subrings of L intersected with L_p. Hence Theorem 3 together with Theorem 10 gives the result.

(ii) Take $\mathscr{X} = \mathscr{A} \cap \mathrm{Min}$ and use Theorems 3 and 10.

(iii) Take $\mathscr{X} = \mathscr{A} \cap \mathscr{F}$ and use Corollary 6(iv) and Theorem 10.

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