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ESTIMATES OF THE FIRST EIGENVALUE OF A BIG CUP DOMAIN OF A 2-SPHERE

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Let $S^2 = (S^2, g_0)$ be the unit 2-sphere in the Euclidean 3-space E^3 and let $D(r)$ denote a cup domain of S^2 which is the geodesic disk of radius r centered at the south pole Q. Let $\lambda_1(r)$ denote the first eigenvalue of the Dirichlet problem for $D(r)$. For $\pi/2 \leq r < \pi$, we have $2 \geq \lambda_1(\mathrm{r}) > 0$. However nice estimates of $\lambda_1(r)$ seem to be unknown. For example, the value r for which $\lambda_1(r) = 1$ is related to the study of stability of minimal surfaces in E^4 (see [1], p. 27), and some estimate of r for which $\lambda_1(r) > 1$ was given in [1].

In this note we give estimates of $\lambda_1(r)$ for r which is near π (Theorem 1.1). If we let r tend to π, then

$$\lambda_1(r) \sim [-2\log(\pi - r)]^{-1}.$$

This agrees with a recent result by S. Ozawa [5] (cf. Remark ii in the section 2).

In the section 3, as an example, we calculate our estimates for $r = \pi - 10^{-50}$;

$$|\lambda_1(\pi - 10^{-50}) - 0.00438| < 0.00007.$$

Contrary to 2-dimensional case, the first eigenvalue of a cup domain of a unit 3-sphere S^3 is explicitly calculatable (see [4], p. 154, [2], p. 201).

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1. The first eigenvalue of a cup domain

Let (x, y, z) be the canonical coordinates of E^3. Then $S^2 = \{(x, y, z); x^2 + y^2 + z^2 = 1\}$. Put $P = (0, 0, 1)$ and $Q = (0, 0, -1)$. We denote the

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Laplacian on S^2 by Δ. Then it is known that

$$f_C(x, y, z) = f(z) = 2 - z \log \frac{1+z}{1-z} + Cz$$

satisfies $\Delta f + 2f = 0$ on $S^2 - \{P, Q\}$ for any constant C. $f(z) = 0$ has just two solutions for $-1 < z < 1$, and solutions depend on C. Let $r = r(x, y, z)$ denote the distance function from Q on S^2. Then

$$z = -\cos r.$$

We fix $r_0 (0 < r_0 < \pi)$ and consider the corresponding cup domain $D(r_0) = \{(x, y, z) \in S^2;\ r(x, y, z) \leq r_0\}$. Then we have some constant $C = C(r_0)$ such that the solutions $\gamma = \gamma(r_0)$, $\beta = \beta(r_0)$ of $f_C(z) = 0$ satisfy $\gamma < \beta = -\cos r_0$. Namely C is determined by β;

$$C = \log \frac{1+\beta}{1-\beta} - \frac{2}{\beta}. \tag{1.1}$$

Let $z = \alpha$ be the point where $f_C(z)$ takes its maximum. Then

$$C = \log \frac{1+\alpha}{1-\alpha} + \frac{2\alpha}{1-\alpha^2} \tag{1.2}$$

and the maximum value is equal to $2/(1-\alpha^2)$. We define a function F on $D(r_0)$ by

$$\begin{aligned} F(x, y, z) &= ((1-\alpha^2)/2) f_C(x, y, z) \qquad && \alpha \leq z \leq \beta \\ &= 1 && -1 \leq z \leq \alpha. \end{aligned}$$

F is a piecewise smooth C^1-function on $D(r_0)$ vanishing on the boundary of $D(r_0)$. Therefore the first eigenvalue $\lambda_1(r_0)$ of $D = D(r_0)$ satisfies

$$\lambda_1(r_0) < \int_D (dF, dF) / \int_D F^2.$$

Since the volume element of $D(r_0)$ is $\sin r\, dr \wedge d\theta$ (where θ denotes the rotation parameter around the z-axis), we get

$$\begin{aligned} \int_D F^2 &= 2\pi \int_{-1}^{\beta} F^2\, dz \\ &= 2\pi(1+\alpha) + 2\pi \int_{\alpha}^{\beta} F^2\, dz. \end{aligned}$$

Let $D = D' \cup D''$, where D' denotes the part of D for $z \leq \alpha$ and D'' corresponds to $\alpha \leq z \leq \beta$. Then

$$\int_D (\mathrm{d}F, \mathrm{d}F) = \int_{D''} (\mathrm{d}F, \mathrm{d}F) = \int_{D''} (-\Delta F, F)$$

$$= 2 \int_{D''} F^2 = 4\pi \int_\alpha^\beta F^2 \, \mathrm{d}z,$$

where the second equality follows from the fact that F vanishes on $z = \beta$ and the normal derivative of F vanishes on $z = \alpha$. Therefore we get

$$\lambda_1(r_0) < 2A/(1 + \alpha + A). \tag{1.3}$$

THEOREM 1.1: *Let $D(r_0)$ be a cup domain in S^2. Then the first eigenvalue $\lambda_1(r_0)$ is estimated by*

$$1/\log[2/(1 + \cos r_0)] < \lambda_1(r_0) < 2A/(1 + \alpha + A), \tag{1.4}$$

where β, C, α and A are determined by the following:

$$\beta = -\cos r_0, \tag{1.5}$$

$$C = \log \frac{1+\beta}{1-\beta} - \frac{2}{\beta} = \log \frac{1+\alpha}{1-\alpha} + \frac{2\alpha}{1-\alpha^2}, \tag{1.6}$$

$$A = \frac{(1-\alpha^2)^2}{4} \int_\alpha^\beta \left\{2 - z \log \frac{1+z}{1-z} + Cz\right\}^2 \mathrm{d}z. \tag{1.7}$$

PROOF: The first inequality of (1.4) follows from a calculation applying $\Theta(t) = \sin t$ to Theorem 3 in [3]. The second inequality of (1.4) is (1.3). q.e.d.

REMARK i: Our inequalities (1.4) are most effective for $r_0 \doteqdot \pi$. One merit of (1.4) is that it enables us to calculate approximated values of $\lambda_1(r_0)$, $r_0 \doteqdot \pi$.

2. Calculation of $A = A(r_0)$

Here we calculate

$$I = \frac{2A}{(1-\alpha^2)^2} = \frac{1}{2} \int_\alpha^\beta \left\{2 - z \log \frac{1+z}{1-z} + Cz\right\}^2 \mathrm{d}z.$$

We put $u=(1+z)/(1-z)$, $a=(1+\alpha)/(1-\alpha)$ and $b=(1+\beta)/(1-\beta)$. Then, $\mathrm{d}z = 2\,\mathrm{d}u/(u+1)^2$ and

$$I=\left[-\frac{C^2+4C+4}{u+1}+\frac{2C^2+4C}{(u+1)^2}-\frac{4C^2}{3(u+1)^3}\right]_a^b$$
$$+\left[\left(-\frac{1}{u+1}+\frac{2}{(u+1)^2}-\frac{4}{3(u+1)^3}\right)(\log u)^2\right]_a^b$$
$$+\int_a^b\left\{-\frac{2C+4}{(u+1)^2}+\frac{8C+8}{(u+1)^3}-\frac{8C}{(u+1)^4}\right\}\log u\,\mathrm{d}u$$
$$+\int_a^b\left\{\frac{1}{u+1}-\frac{2}{(u+1)^2}+\frac{4}{3(u+1)^3}\right\}\frac{2}{u}\log u\,\mathrm{d}u.$$

Continuing calculations and applying

$$\int_a^b\frac{\log u}{u+1}\,\mathrm{d}u=\frac{1}{2}[(\log(1-z))^2]_\alpha^\beta-[\log(1+z)\log(1-z)]_\alpha^\beta$$
$$+\int_\alpha^\beta\frac{\log(1-z)}{1+z}\,\mathrm{d}z,$$

we obtain the following

PROPOSITION 2.1: For $a=(1+\alpha)/(1-\alpha)$ and $b=(1+\beta)/(1-\beta)$,

$$\frac{2A}{(1-\alpha^2)^2}=\left[-\frac{C^2+8C/3+4/3}{u+1}+\frac{2C^2+8C/3}{(u+1)^2}-\frac{4C^2}{3(u+1)^3}\right.$$
$$+\left(-\frac{1}{u+1}+\frac{2}{(u+1)^2}-\frac{4}{3(u+1)^3}\right)(\log u)^2$$
$$\text{(2.1)}\qquad\left.+\left(\frac{2C+8/3}{u+1}-\frac{4C+8/3}{(u+1)^2}+\frac{8C}{3(u+1)^3}\right)\log u+\frac{2C}{3}\log\frac{u+1}{u}\right]_a^b$$
$$+\frac{1}{3}[(\log(1+z))^2]_\alpha^\beta-\frac{2}{3}\int\frac{\log(1-z)}{1+z}\,\mathrm{d}z.$$

The last term of (2.1) is estimated by the following

PROPOSITION 2.2: For $\alpha>0$ we have

$$-\int_\alpha^\beta\frac{\log(1-z)}{1+z}\,\mathrm{d}z<\frac{1}{1+\alpha}[(1-z)\log(1-z)]_\alpha^\beta+\frac{\beta-\alpha}{1+\alpha},$$
$$-\int_\alpha^\beta\frac{\log(1-z)}{1+z}\,\mathrm{d}z>\frac{1}{1+\beta}[(1-z)\log(1-z)]_\alpha^\beta+\frac{\beta-\alpha}{1+\beta}.$$

Next we study the behavior of $A(r_0)$ as $r_0 \to \pi$. We put $\delta = 1 - \alpha$ and $\rho = 1 - \beta$. Then we get

$$C = -\log \rho - 2 + \log 2 + \cdots,$$
$$\delta \log \rho = -1 + \delta \log \delta - 3\delta/2 + \cdots.$$

etc. By Propositions 2.1 and 2.2 we obtain

$$\lim_{r_0 \to \pi} \log(1-\beta)\, A(r_0) = -1,$$

and hence for $r_0 \to \pi$,

$$2A/(1+\alpha+A) \sim [-\log(1+\cos r_0)]^{-1}.$$

Therefore Theorem 1.3 implies

$$\lambda_1(r_0) \sim [-\log(1+\cos r_0)]^{-1}. \tag{2.2}$$

Corollary 2.3: *Let $D_\epsilon(P)$ be an ϵ-neighborhood of a point P in S^2. Then the first eigenvalue $\lambda_{1,\epsilon}$ of $S^2 - D_\epsilon(P)$ satisfies*

$$\lambda_{1,\epsilon} \sim [-2\log \epsilon]^{-1} \qquad (\epsilon \downarrow 0). \tag{2.3}$$

Proof: We put $\epsilon = \pi - r_0$. Then $1 + \cos r_0 = \epsilon^2/2 + \cdots$, and $\lambda_{1,\epsilon} = \lambda_1(r_0)$. Thus, (2.3) follows from (2.2). q.e.d.

Remark ii. A result of S. Ozawa [5] is stated as follows; Let (N, g) be a compact 2-dimensional Riemannian manifold and let $D_\epsilon(p)$ be an ϵ-neighborhood of a point p of N. Then the first eigenvalue $\lambda_{1,\epsilon}$ of $(N, g) - D_\epsilon(p)$ satisfies

$$\lambda_{1,\epsilon} = 2\pi[-\mathrm{Vol}(N,g)\log \epsilon]^{-1} + 0((\log \epsilon)^{-2}),$$

where $\mathrm{Vol}(N, g)$ denotes the volume of (N, g). His proof depends on abstract analysis, and it is difficult to get the geometric shape of an eigenfunction corresponding to $\lambda_{1,\epsilon}$. To get an approximated shape of an eigenfunction the following observation is useful. Any 2-dimensional Reimannian manifold (N, g) is locally conformal to (S^2, g_0). Let U be a neighborhood of p in N and let ϕ be a conformal map from U into $U^* = \phi U$ in S^2. We put $P = \phi(p)$. If ϵ_0 is sufficiently small, for any $\epsilon < \epsilon_0$ we have some $\mu = \mu(\epsilon)$ such that the boundary of an

ϵ-neighborhood $D_\epsilon(p)$ of p in N is mapped by ϕ into $D_{\epsilon+\mu}(P)-D_{\epsilon-\mu}(P)$ in S^2. The converse is also true. In the 2-dimensional case we have another special property that, for conformally related metrics g_0 and $g^*=hg_0$ (h: positive function) the Laplacians Δ and Δ^* corresponding to g_0 and g^* satisfy $\Delta^* f=(1/h)\Delta f$ for each function f. Therefore, in an infinitesimal domain, h looks like a constant, and a function similar to F in the section 1 can be constructed in N.

3. An example; $r_0 = \pi - 10^{-50}$

As an example we calculate our estimate of $\lambda_1(r_0)$ for $r_0=\pi-10^{-50}$.

$$229.64480 < C = \log b - 2/\beta < 229.64481$$

$$0.9955365 < \alpha < 0.9955366$$

$$0.00444348 < A < 0.00444382$$

$$0.0043169 < \lambda_1(r_0) < 0.0044439$$

Estimate in the introduction corresponds to

$$0.00431 < \lambda_1(r_0) < 0.00445.$$

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