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## AN $L_2$ -ISOLATION THEOREM FOR YANG–MILLS FIELDS

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### 1. Introduction

The Yang–Mills Lagrangian associated to a Lie group  $G$  is the square of the  $L_2$ -norm of the curvature of a  $G$ -connection  $\omega$  on a 4-dimensional riemannian manifold  $M$ . The critical points of this functional are called Yang–Mills fields or instantons. In certain cases, where the Pontrjagin number is the only topological constraint, the absolute minimum of the functional is attained by the so-called self-dual (resp. anti-self-dual) Yang–Mills fields. For the case of the sphere with the standard metric and  $G$  a simple Lie group there exist explicit algebraic constructions of such fields. It is still an open question whether in this case there are any other critical points which are irreducible connections. Bourguignon, Lawson and Simons [3], [4] have proved that for  $G = \text{SU}(2)$ ,  $\text{SU}(3)$ , or  $\text{U}(2)$  there exist no other weakly stable critical points, i.e. there are no other relative minima. They also proved that on  $S^4$  and for any compact Lie group  $G$ , the  $\pm$  self-dual fields are isolated from other critical points in the sense that there is a  $C^0$ -neighborhood of the curvature where no other critical points exist.

In this paper we obtain a stronger version of this last result by proving that the value of the functional itself, which is just the action, of any other critical point is bounded away from the absolute minimum by a constant depending only on the conformal structure of the base manifold  $M$ . Our result holds for any compact Lie group  $G$  and for any, not necessarily self-dual, 4-dimensional riemannian manifold satisfying a certain positivity condition on the curvature. Examples of such manifolds are, besides the standard spaces  $S^4$  and  $P^2(\mathbb{C})$ , the Kähler mani-

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folds of positive scalar curvature. In the special case of the standard  $S^4$  we give explicit bounds.

The proof is based on the well-known technique of Bochner and Lichnerowicz, which consists of deriving a Weitzenböck formula for the associated Laplacian and showing that it is positive definite under the given assumptions. The new ingredient in our proof, which leads to the  $L_2$ -bounds instead of the usual pointwise conditions, is the application of the Sobolev inequality to the term involving the integral of the square of the covariant derivative in the Weitzenböck formula. In dimension 4 this gives  $L_4$ -bounds which can now be used to balance the other terms. In higher dimensions we could use the same technique to obtain similar theorems with  $L_p$ -bounds, where  $p$  is half the dimension, but we will restrict ourselves to dimension 4 in this paper.

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## 2. Notations and results

Let  $M$  be a four dimensional oriented riemannian manifold,  $\Lambda^p$  the bundle of exterior  $p$ -forms on  $M$  and  $A^p = \Gamma(\Lambda^p)$  its space of smooth sections. The *Hodge star operator*  $*$ :  $\Lambda^p \rightarrow \Lambda^{4-p}$  is defined by  $\alpha \wedge * \beta = \langle \alpha, \beta \rangle \nu$  where  $\alpha, \beta \in \Lambda^p$ ,  $\langle, \rangle$  is the induced scalar product on  $p$ -forms and  $\nu$  is the volume form. On 2-forms  $*$  is an endomorphism satisfying  $*^2 = \text{id}$  and  $\Lambda^2$  splits as a direct sum:  $\Lambda^2 = \Lambda_+^2 \oplus \Lambda_-^2$ , where  $\Lambda_\pm^2$  are the  $\pm 1$ -eigenspaces of  $*$ . The bundles  $\Lambda_+^2$  and  $\Lambda_-^2$  are called the bundles of *self-dual* and *anti-self-dual* 2-forms respectively.

The riemannian curvature tensor is a self-adjoint transformation  $R: \Lambda^2 \rightarrow \Lambda^2$  and we write it as a block matrix as follows:

$$R = \begin{pmatrix} R_+ & S \\ S^t & R_- \end{pmatrix} \quad (2.1)$$

where  $S \in \text{Hom}(\Lambda_-^2, \Lambda_+^2)$ , and  $R_\pm \in \text{End}(\Lambda_\pm^2)$ .

As is well known [2], [8], the irreducible components of  $R$  are given by:

$$4\text{tr } R_+ = 4\text{tr } R_- = \kappa = \text{scalar curvature}$$

$$S = \text{traceless Ricci tensor}$$

$$R_\pm - \frac{1}{3}\text{tr } R_\pm = W_\pm$$

where  $W = W_+ + W_-$  is the conformally invariant *Weyl tensor*.

An oriented 4-dimensional riemannian manifold is called *self-dual* (resp. *anti-self-dual*) if  $W_- = 0$  (resp.  $W_+ = 0$ ). Since both the Weyl tensor and the Hodge star operator on 2-forms are conformally invariant, the property of being self-dual (or anti-self-dual) depends only on the underlying conformal structure and the choice of orientation on  $M$ .

Let  $\omega$  be a connection on a principal  $G$ -bundle  $P$  over  $M$  with curvature  $\Omega = d\omega + [\omega, \omega]$ ,  $G$  being a compact Lie group.

Let  $E = P \times_{Ad} \mathfrak{g}$  be the vector bundle associated to  $P$  via the adjoint representation of  $G$  on its Lie algebra  $\mathfrak{g}$ . The curvature form  $\Omega$  on  $P$  descends to a 2-form on  $M$  with values in  $E$ , i.e.,  $\Omega \in A^2(E)$ , where we denote by  $A^p(E)$  the space of smooth sections of the vector bundle  $A^p \otimes E$ .

The exterior covariant derivative induced by the connection  $\omega$  gives rise to a sequence of first order differential operators

$$\dots \rightarrow A^p(E) \xrightarrow{d^\omega} A^{p+1}(E) \xrightarrow{d^\omega} A^{p+2}(E) \rightarrow \dots \tag{2.2}$$

which is in general not a complex, i.e.,  $d^\omega \circ d^\omega \neq 0$ . Instead one has the following formula:

$$d^\omega d^\omega \alpha = \Omega \wedge \alpha \tag{2.3}$$

where  $\Omega \wedge \alpha$  is the exterior product defined by using the Lie bracket of  $\mathfrak{g}$ . Explicitly, this is given by

$$\begin{aligned} (\Omega \wedge \alpha)(X, Y, Z) = & [\Omega(X, Y), \alpha(Z)] + \\ & + [\Omega(Y, Z), \alpha(X)] + [\Omega(Z, X), \alpha(Y)]. \end{aligned} \tag{2.4}$$

$d^\omega \circ d^\omega$  is therefore an algebraic operator of order zero, which reflects the fact that the symbol sequence of (2.2) is exact. In fact, the principal symbol of  $d^\omega$  does not depend on the connection.

The *Euclidean Yang–Mills functional* is defined on the space of all  $G$ -connections by

$$\mathcal{YM}(\omega) = \frac{1}{2} \int_M |\Omega|^2 \nu \tag{2.5}$$

where we use a positive definite  $Ad_G$ -invariant inner product on  $\mathfrak{g}$ , satisfying the normalization

$$\max_{|X|=|Y|=1} |[X, Y]| = \sqrt{2}. \tag{2.6}$$

The *Euler–Lagrange equation* for this functional is

$$\delta^\omega \Omega = 0 \tag{2.7}$$

where  $\delta^\omega$  is the adjoint of  $d^\omega$  with respect to the metrics defined on  $M$  and  $\mathfrak{g}$ . Since the curvature form  $\Omega$  automatically satisfies the Bianchi identity:

$$d^\omega \Omega = 0, \tag{2.8}$$

the critical points of the functional, called *sourceless Yang–Mills fields*, are therefore the connection forms  $\omega$  whose curvature is harmonic.

In terms of the star operator  $\delta^\omega = *d^\omega*$ , and hence a self-dual or an anti-self-dual curvature form, i.e.,  $\Omega$  satisfying  $*\Omega = \pm \Omega$  is automatically harmonic. In fact, provided they exist, these give the absolute minima of the functional. This is because if one decomposes  $\Omega$  as  $\Omega_+ + \Omega_-$  with

$$\Omega_\pm \in A_\pm^2(E) = \Gamma(A_\pm^2 \otimes E),$$

then the first Pontrjagin number of  $E$  is given by

$$p_1(E) = \frac{1}{4\pi^2} \int_M |\Omega_+|^2 - |\Omega_-|^2 \tag{2.9}$$

and therefore, the self-dual (resp. anti-self-dual) connections with  $\Omega_- = 0$  (resp.  $\Omega_+ = 0$ ) give the minimum value of the functional, namely  $\pm p_1(E)$ .

The group of *gauge transformations*, which is by definition the group of automorphisms of the principal bundle  $P$  acting trivially on the base space  $M$ , leaves the Yang–Mills functional invariant and therefore in particular the set of all self-dual-connections. This set is in fact even invariant under the larger group of all automorphisms of  $P$  acting as conformal diffeomorphisms on the base space. The space of moduli of the irreducible self-dual connections under the gauge group is a finite dimensional manifold. For the case  $M = S^4$  with the flat conformal structure and  $G$  a simple Lie group, there exist explicit algebraic constructions of irreducible self-dual Yang–Mills fields [1], [5].

It is still unknown whether there exist irreducible Yang–Mills fields on  $S^4$  which are not self-dual or anti-self-dual for a simple Lie group  $G$ .

Our main result shows that the other critical points of the Yang–Mills functional have energies bounded away from the absolute minimum by a constant depending only on the conformal structure of the

base manifold  $M$ . In case of the sphere  $S^4$  with its flat conformal structure, which is the case of main interest in the theory, we give an explicit value of this constant.

In order to state the results we introduce the following invariants for a 4-dimensional riemannian manifold  $M$ :

$\kappa$  = scalar curvature of  $M$ .

$\mu_{\pm}$  = the largest eigenvalue of the Weyl conformal tensor  $W_{\pm}$  at each point.

( $\mu_{\pm}$  is non-negative since trace  $W_{\pm} = 0$  and  $\mu_{\pm} = 0$  if and only if  $M$  is  $\pm$  self-dual.)

$$k_{\pm} = \frac{\kappa}{12} - \frac{1}{2}\mu_{\pm}.$$

$V$  = volume of  $M$ .

$c_1$  = the isoperimetric constant of  $M$

$$= \inf_{N^3 \subset M^4} \frac{(\text{vol}(N))^4}{(\min \{ \text{vol}(M_1), \text{vol}(M_2) \})^3},$$

where the infimum is taken over all smooth hypersurfaces  $N$  in  $M$  dividing it into two connected pieces  $M_1$  and  $M_2$ .

Finally we assume in the following theorems that  $\omega$  is a connection over  $M$  with a Lie group  $G$  as structure group and that the  $Ad_G$ -invariant scalar product we use on the Lie algebra  $\mathfrak{g}$  to define our norms satisfies the normalization (2.6).

**THEOREM 1:** *Let  $\omega$  be a sourceless Yang–Mills field over  $M$ . If  $|\Omega_-| \leq \sqrt{3} k_-$  everywhere with strict inequality holding at one point, then  $\Omega_- \equiv 0$ , i.e.,  $\omega$  is self-dual. The analogous statement holds for  $\Omega_+$ .*

This theorem is due to J.P. Bourguignon and H.B. Lawson [4]. (See also C.-L. Shen [7] for a related result.)

In the next two theorems, which constitute our main result, we replace the pointwise inequality of the above Theorem by a bound on the  $L_2$ -norm of  $\Omega_-$ .

**THEOREM 2:** *Let  $\omega$  be sourceless Yang–Mills field over  $M$  and let  $M$  be such that  $k_- > 0$  everywhere. Then there exists a constant  $\varepsilon > 0$  depending only on  $V, c_1$  and a lower bound for  $k_-$  such that*

$$\int_M |\Omega_-|^2 < \varepsilon^2 \Rightarrow \Omega_- \equiv 0.$$

Since  $\int |\Omega_-|^2$  is invariant under conformal transformations we should choose  $\varepsilon$  to be the biggest possible constant in the conformal class of the metric. In this sense,  $\varepsilon$  depends only on the conformal structure, provided there is a metric in this conformal class satisfying  $k_- > 0$  everywhere.

For the special case of  $S^4$  with its unique flat conformal structure, which is the case of main interest, we have the following explicit bound:

**THEOREM 3:** *Let  $\omega$  be a sourceless Yang–Mills field on  $S^4$  with a conformally flat metric. Then*

$$\frac{1}{2} \int_M |\Omega|^2 \leq 2\pi^2 \left( |p_1(E)| + \frac{1}{128} \right) \Rightarrow \Omega \text{ is } \pm \text{ self-dual.}$$

In our last theorem we do not assume that  $\omega$  is a sourceless Yang–Mills field, i.e.  $\delta^\omega \Omega = 0$ , but instead give a bound for the  $L_2$ -norm of  $\Omega_-$  in terms of the  $L_2$ -norm of  $\delta^\omega \Omega$ .

Here we fix a metric of constant curvature on  $S^4$ , since  $\int |\delta^\omega \Omega|^2$  is not conformally invariant.

**THEOREM 4:** *Let  $\omega$  be a G-connection on the standard  $S^4$  with the metric of constant curvature 1. If  $\int |\Omega_-|^2 \leq \frac{1}{64} \pi^2$ , then the following estimate holds:*

$$\int |\Omega_-|^2 \leq \left( 8 - \frac{1}{\sqrt{2}} \right)^{-1} \int |\delta^\omega \Omega|^2.$$

This theorem might be of interest in studying the Yang–Mills equations with a small source term:  $d^\omega \Omega = 0$ ,  $\delta^\omega \Omega = J$ , where  $J \in A^1(E)$ .

### 3. The proof

The proof consists of deriving a Weitzenböck-formula for the associated Laplacian  $\Delta^\omega = d^\omega \delta^\omega + \delta^\omega d^\omega$  and showing that under the assumptions of the theorems,  $\Delta^\omega$  is positive definite on  $A_-^2$ , the  $E$ -valued anti-self-dual 2 forms, where  $E$  is the adjoint bundle.

We shall write  $d^\omega = d_+^\omega + d_-^\omega : A^1(E) \rightarrow A_+^2(E) \oplus A_-^2(E)$  and also  $\delta^\omega = \delta_+^\omega + \delta_-^\omega$  with  $\delta_\pm^\omega : A_\pm^2(E) \rightarrow A^1(E)$ . The Laplacian  $\Delta^\omega$  on 2-forms split accordingly as  $\Delta^\omega = \Delta_+^\omega + \Delta_-^\omega$  with  $\Delta_\pm^\omega : A_\pm^2(E) \rightarrow A_\pm^2(E)$ . Moreover, as a special feature for 4-dimensional manifolds we have  $*d\delta* = \delta d$  on 2-

forms and hence the formula:

$$\Delta_{\pm}^{\omega} = 2d_{\pm}^{\omega} \delta_{\pm}^{\omega} \quad (3.1)$$

As a consequence  $\beta = \beta_+ + \beta_- \in A_+^2(E) \oplus A_-^2(E)$  is harmonic if and only if  $\delta_+ \beta_+ = \delta_- \beta_- = 0$ .

To compute the operator  $\Delta^{\omega} = 2d^{\omega} \delta^{\omega}$  acting on a 2-form  $\beta_- \in A_-^2(E)$  explicitly we introduce the following notation. We define for any 2 anti-self-dual 2 forms  $\beta_-, \beta'_- \in A_-^2(E)$ , the bracket  $[\beta_-, \beta'_-]$  by the formula:

$$[\beta_-, \beta'_-](uv) = [\beta_-(u), \beta'_-(v)] - [\beta_-(v), \beta'_-(u)] \quad (3.2)$$

for all  $u, v \in A_-^2$ . This gives a section in the vector bundle  $A^2(A_-^2) \otimes E$ . We now identify the bundle of exterior 2-forms of the 3-dimensional vector bundle  $A_-^2$  with  $A_-^2$  itself via the Lie algebra structure on  $A_-^2$ . Using this identification we consider  $[\beta_-, \beta'_-]$  to be a section in  $A_-^2(E)$ . With this notation, we obtain the following Weitzenböck formula:

$$\Delta^{\omega} \beta_- = -\nabla^* \nabla \beta_- + \frac{\kappa}{6} \beta_- - \beta_- \circ W_- - [\Omega_-, \beta_-] \quad (3.3)$$

where  $\nabla^* \nabla$ , the so-called *rough Laplacian* is just the trace of the Hessian  $\nabla^2$ .

Taking pointwise the scalar product of (3.3) with  $\beta_-$  now gives

$$\begin{aligned} \langle \Delta^{\omega} \beta_-, \beta_- \rangle &= \frac{1}{2} \Delta |\beta_-|^2 + |\nabla \beta_-|^2 + \frac{\kappa}{6} |\beta_-|^2 \\ &\quad - \langle \beta_- \circ W_-, \beta_- \rangle - \langle \Omega_-, [\beta_-, \beta_-] \rangle \end{aligned} \quad (3.4)$$

where we have used the identity

$$\langle [\Omega_-, \beta_-], \beta_- \rangle = \langle \Omega_-, [\beta_-, \beta_-] \rangle$$

which follows from the fact that the scalar product in  $\mathfrak{g}$  is  $Ad$ -invariant. Since we use the normalization (2.6) we also have,

$$|[X, Y]| \leq \sqrt{2} |X| |Y| \text{ for all } X, Y \in \mathfrak{g}. \quad (3.5)$$

We now estimate the last term in (3.4) as follows:  
From the inequality

$$b_1 b_2 + b_2 b_3 + b_3 b_1 \leq \frac{1}{3} (b_1 + b_2 + b_3)^2 \quad (3.6)$$



which holds for any 3 real numbers  $b_1, b_2$  and  $b_3$  (with equality if and only if  $b_1 = b_2 = b_3$ ), it follows by taking  $b_i = |\beta_{-i}|^2, i = 1, 2, 3$  and using the inequality (3.5) that

$$|[\beta_{-}, \beta_{-}]| \leq \frac{2}{\sqrt{3}} |\beta_{-}|^2, \quad (3.7)$$

and therefore

$$|\langle \Omega_{-}, [\beta_{-}, \beta_{-}] \rangle| \leq \frac{2}{\sqrt{3}} |\Omega_{-}| |\beta_{-}|^2. \quad (3.8)$$

If  $\mu_{-}$  is the largest eigenvalue of  $W_{-}$  at each point, which is non-negative because  $\text{trace } W_{-} = 0$ , then we have

$$-\langle \beta_{-} \circ W_{-}, \beta_{-} \rangle \geq -\mu_{-} |\beta_{-}|^2. \quad (3.9)$$

Substituting these estimates into (3.4) and integrating over the compact manifold  $M$ , we obtain the inequality

$$\int \langle \Delta^{\omega} \beta_{-}, \beta_{-} \rangle \geq \int |\nabla \beta_{-}|^2 + \int \left( \frac{\kappa}{6} - \mu_{-} - \frac{2}{\sqrt{3}} |\Omega_{-}| \right) |\beta_{-}|^2. \quad (3.10)$$

So if  $|\Omega_{-}| \leq \sqrt{3} k_{-} = \frac{\kappa}{4\sqrt{3}} - \frac{\sqrt{3}}{2} \mu_{-}$  everywhere with strict inequality holding at one point, then  $\Delta^{\omega}$  is a positive definite operator which means that there are no non-trivial harmonic forms in  $A^2(E)$ , in particular, no harmonic anti-self-dual curvature forms. This proves Theorem 1.

To get the  $L_2$ -estimates in the remaining theorems we use the following Sobolev inequality due to P. Li [6, Lemma 2] for the case  $\dim M = 4$ :

$$9 \int |\nabla f|^2 \geq \left( \frac{c_1}{4} \right)^{1/2} \left( \int |f|^4 \right)^{1/2} - \left( \frac{c_1}{V} \right)^{1/2} \int |f|^2 \quad (3.11)$$

holding for all functions  $f \in H_{1,2}$ , the Sobolev space of functions having  $L_2$ -derivatives, where  $V$  is the volume of  $M$  and  $c_1$  is the isoperimetric constant as defined in the last section.

We now apply the Cauchy-Schwarz inequality to the integrand of the

first term and to the integral involving  $\Omega_-$  in the last term on the right hand side of (3.10) to get:

$$\int \langle \Delta^\omega \beta_-, \beta_- \rangle \geq \int |\nabla |\beta_-|^2| + 2 \int k_- |\beta_-|^2 - \frac{2}{\sqrt{3}} \|\Omega_-\|_2 \|\beta_-\|_4^2, \tag{3.12}$$

where  $\|\cdot\|_p$  denotes the  $L_p$ -norm.

The Sobolev inequality (3.11) applied to the first term now gives

$$\begin{aligned} \int \langle \Delta^\omega \beta_-, \beta_- \rangle &\geq \left( \frac{\sqrt{c_1}}{18} - \frac{2}{\sqrt{3}} \|\Omega_-\|_2 \right) \|\beta_-\|_4^2 + \\ &+ \left( 2c_2 - \frac{1}{9} \left( \frac{c_1}{V} \right)^{1/2} \right) \|\beta_-\|_2^2 \end{aligned} \tag{3.13}$$

where  $c_2 = \min_M k_-$ .

For  $M = S^4$  with the standard metric of constant curvature 1, the isoperimetric constant  $c_1 = \frac{27}{4}\pi^2$ ,  $V = \frac{8}{3}\pi^2$  and  $c_2 = 1$ . Therefore, in this case

$$\int \langle \Delta^\omega \beta_-, \beta_- \rangle \geq \left( \frac{\pi}{4\sqrt{3}} - \frac{2}{\sqrt{3}} \|\Omega_-\|_2 \right) \|\beta_-\|_4^2 + \left( 2 - \frac{\sqrt{2}}{8} \right) \|\beta_-\|_2^2 \tag{3.14}$$

and hence, if  $\|\Omega_-\|_2 \leq \frac{\pi}{8}$ , we have

$$\int |\beta_-|^2 \leq \left( 2 - \frac{\sqrt{2}}{8} \right)^{-1} \int |\delta^\omega \beta_-|^2. \tag{3.15}$$

From this Theorem 4 follows, since for  $\Omega_- = \beta_-$  we have the identity  $\delta^\omega \Omega_- = \delta_+^\omega \Omega_+ = \frac{1}{2} \delta^\omega \Omega$ , which is a consequence of the Bianchi identity  $d^\omega \Omega = 0$ .

Theorem 3 now follows from Theorem 4 and the fact that  $\int |\Omega_-|^2$  is conformally invariant.

For a general compact 4-dimensional manifold  $M$ , the inequality (3.13) proves Theorem 2 in case  $18c_2 \geq \left( \frac{c_1}{V} \right)^{1/2}$ . If this is not the case, we use (3.13) together with the following inequality which is obtained immediately from (3.12):

$$\int \langle \Delta^\omega \beta_-, \beta_- \rangle \geq 2c_2 \|\beta_-\|_4^2 - \frac{2}{\sqrt{3}} \|\Omega_-\|_2 \|\beta_-\|_4^2. \tag{3.16}$$

Since  $c_2$  is by assumption positive we can choose  $\|\Omega_-\|_2$  small enough, say  $< \frac{\sqrt{3}}{2} c_2 V^{1/2}$ , so that (3.13) and (3.16) together would imply that  $\Delta^\omega$  is positive definite. In fact, if  $\frac{2}{\sqrt{3}} \|\Omega_-\|_2 < c_2 V^{1/2}$ , then (3.16) implies

$$\int \langle \Delta^\omega \beta_-, \beta_- \rangle > c_2 \left( 2 \|\beta_-\|_2^2 - V^{1/2} \|\beta_-\|_4^2 \right)$$

which is positive if  $\|\beta_-\|_2^2 \geq \frac{1}{2} V^{1/2} \|\beta_-\|_4^2$ .

On the other hand, if  $\frac{2}{\sqrt{3}} \|\Omega_-\|_2 < c_2 V^{1/2}$ , then we have by (3.13)

$$\int \langle \Delta^\omega \beta_-, \beta_- \rangle > \left( \frac{1}{18} \sqrt{c_1} - c_2 V^{1/2} \right) \|\beta_-\|_4^2 + \left( 2c_2 - \frac{1}{9} \left( \frac{c_1}{V} \right)^{1/2} \right) \|\beta_-\|_2^2$$

which is positive if  $\|\beta_-\|_2^2 < \frac{1}{2} V^{1/2} \|\beta_-\|_4^2$ , since we are in the case where  $18c_2 < \left( \frac{c_1}{V} \right)^{1/2}$ . So Theorem 2 is proved if we take

$$\varepsilon = \frac{\sqrt{3}}{2} \min \left\{ \frac{1}{18} \sqrt{c_1}, c_2 \sqrt{V} \right\}.$$

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