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# SPLITTING THEOREMS OF RIEMANNIAN MANIFOLDS

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(Dedicated to Professor I. Mogi on his 60th birthday)

## 1. Introduction

A function $F$ defined on a complete Riemannian manifold $M$ without boundary is said to be *convex* iff on each unit speed geodesic $F$ is a one variable convex function. In [7], [8], such functions are studied in detail. For example, if $F$ has no minimum, then $M$ is diffeomorphic to $N \times \mathbb{R}$, where $N$ is homeomorphic to a level of $F$, and if $F$ has a compact level, then all levels are compact and the diameter function of levels of $F$ is monotone nondecreasing as a function for values of $F$. Moreover, $M$ with a locally nonconstant convex function $F$ has at most two ends, and $M$ has one end if $F$ has a noncompact level. These facts will be used in §3.

In the present paper we study functions on $M$ which are affine functions on unit speed geodesics, and we apply them to prove splitting theorems of Riemannian manifolds.

Let $M$ be a complete Riemannian manifold without boundary. A function $F$ on $M$ is by definition *affine* if on each unit speed geodesic $\gamma:(-\infty,\infty) \to M$, $F \circ \gamma(st_1 + (1-s)t_2) = sF \circ \gamma(t_1) + (1-s)F \circ \gamma(t_2)$ for every $s \in (0,1)$ and for every $t_1, t_2 \in (-\infty, \infty)$. A function $F$ on a Riemannian product manifold $M := N \times \mathbb{R}$ is clearly affine on $M$ if $F(x,t) = t$ for each $(x,t) \in N \times \mathbb{R}$.

The main theorem of our investigation is:

MAIN THEOREM: *A complete Riemannian manifold $M$ without boundary admits a non-trivial affine function if and only if $M$ is isometric to a Riemannian product $N \times \mathbb{R}$.*

In fact:

THEOREM 1: *Let $M$ be a complete Riemannian manifold without bound-*

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*ary. If $M$ admits a non-trivial affine function $F$, then $F^{-1}(a)$, for every $a \in \mathbb{R}$, is a totally geodesic submanifold of $M$ without boundary, and furthermore there exists an isometric map $I$ of $F^{-1}(a) \times \mathbb{R}$ onto $M$ such that there is a constant $b$ such that $F \circ I(x,t) = bt + a$ for every $x \in F^{-1}(a)$ and for every $t \in \mathbb{R}$.*

Examples and applications of this theorem are as follows.

Let $V$ be the totality of all affine functions on $M$. Then $V$ is evidently a vector space containing all constant functions on $M$ and hence $\dim V$ is at least one. If $M$ is the $n$-dimensional Euclidean space, then $V$ is an $(n+1)$-dimensional vector space. Conversely, from the fact that $\operatorname{grad} F$ of an affine function $F$ is parallel on $M$ and by iterating Theorem 1, we have

THEOREM 2: *Let $M$ be an $n$-dimensional complete noncompact Riemannian manifold without boundary. Then $1 \leq \dim V \leq n+1$. If $\dim V = k+1$, then $M$ is isometric to the Riemannian product $N \times \mathbb{R}^k$, where $N$ admits no non-trivial affine function. In particular $M$ is the Euclidean space if and only if $\dim V = n+1$.*

Next we discuss when an affine function on $M$ exists.

A unit speed geodesic $\gamma: [0,\infty) \to M$ ($(-\infty,\infty) \to M$) is by definition a *ray* (a *straight line*) if $d(\gamma(t_1), \gamma(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [0,\infty)$ ($t_1, t_2 \in (-\infty,\infty)$).

THEOREM 3: *Let $M$ be a complete noncompact Riemannian manifold of nonnegative sectional curvature and without boundary. If there exist two rays $\gamma_1, \gamma_2: [0,\infty) \to M$ and a positive constant $a$ such that for every $t \in [0,\infty)$, $2t - d(\gamma_1(t), \gamma_2(t)) < a$, then the Busemann functions $f_{\gamma_i}(.)\colon = \lim_{t\to\infty}\{t - d(., \gamma_i(t))\}$, $i = 1, 2$, of $\gamma_i$ are non-trivial affine functions. In particular $M$ is isometrically a Riemannian product $N \times \mathbb{R}$.*

Using the Toponogov comparison theorem (see [3]), we know that the existence of $\gamma_1$ and $\gamma_2$ in the assumption is equivalent to the existence of a straight line. Thus we obtain a restatement of the Toponogov splitting theorem (see [4], [5], [10]).

But if $M$ is of nonpositive sectional curvature, the existence of a straight line does not imply the existence of a non-trivial affine function on $M$ (see Example 1 in §3). However the following holds in this case.

THEOREM 4: *Let $M$ be a complete noncompact Riemannian manifold of nonpositive sectional curvature and without boundary. Suppose there* *exists an isometry $\delta$ of $M$ such that it translates a straight line $\gamma:(-\infty,\infty)\to M$, i.e., there exists a constant $a\neq 0$ such that $\delta\circ\gamma(t)=\gamma(t+a)$ for every $t\in(-\infty,\infty)$, which connects different ends of $M$ (see §3) and it leaves all free homotopy classes of closed curves in $M$ invariant. Then the Busemann functions $f_{\gamma_\pm}(.):=\lim\{d(.,\gamma_\pm(t))-t\}$ are non-trivial affine functions and hence $M$ is isometrically a Riemannian product $N\times\mathbb{R}$, where $\gamma_\pm:[0,\infty)\to M$ is defined by $\gamma_+(t)=\gamma(t)$ and $\gamma_-(t)=\gamma(-t)$ for every $t\in[0,\infty)$.*

In §2 we prove Theorem 3. The proof is very simple and the idea is useful to that of Theorem 4. In §3 we deal with Theorem 4 and we shall see there another statement (Proposition 1) which explains satisfactorily the meaning of split. In §4 we give the proof of Theorem 1.

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Note: Busemann and Phadke (see [2]) have independently proved the analogous result to Theorem 2 in more general spaces ($G$-spaces).

## 2. Proof of Theorem 3

Let $M$ be a complete noncompact Riemannian manifold of nonnegative sectional curvature and without boundary. It is well known (see [5], [11]) that the Busemann function $f_\alpha(.)=\lim_{t\to\infty}\{t-d(.,\alpha(t))\}$ of every ray $\alpha$ in $M$ is convex on $M$. Hence $f_{\gamma_1}+f_{\gamma_2}$ is convex on $M$ where $f_{\gamma_1}$ and $f_{\gamma_2}$ are functions in the assumption of Theorem 3. Moreover $f_{\gamma_1}+f_{\gamma_2}$ is bounded above by $a$ on $M$, since $f_{\gamma_1}(p)+f_{\gamma_2}(p)=\lim\{t-d(p,\gamma_1(t))\}+\lim\{t-d(p,\gamma_2(t))\}=\lim\{2t-(d(p,\gamma_1(t))+d(p,\gamma_2(t)))\}\leq\lim\{2t-d(\gamma_1(t),\gamma_2(t))\}\leq a$ for all $p\in M$. Thus $f_{\gamma_1}+f_{\gamma_2}$ is constant on $M$, say, $c$, and therefore $f_{\gamma_i}$, $i=1,2$, is affine. In fact, for every unit speed geodesic $\gamma:(-\infty,\infty)\to M$

$$\begin{aligned}
f_{\gamma_1}\circ\gamma(st_1+(1-s)t_2) &\leq sf_{\gamma_1}\circ\gamma(t_1)+(1-s)f_{\gamma_1}\circ\gamma(t_2)\\
&= s(c-f_{\gamma_2}\circ\gamma(t_1))+(1-s)(c-f_{\gamma_2}\circ\gamma(t_2))\\
&= c-(sf_{\gamma_2}\circ\gamma(t_1)+(1-s)f_{\gamma_2}\circ\gamma(t_2))\\
&\leq c-f_{\gamma_2}\circ\gamma(st_1+(1-s)t_2)\\
&= f_{\gamma_1}\circ\gamma(st_1+(1-s)t_2)
\end{aligned}$$

for every $s\in(0,1)$ and for every $t_1,t_2\in(-\infty,\infty)$.

Hence $f_{\gamma_i}$ is affine and also $f_{\gamma_2}$ is. This completes the proof of Theorem 3.

## 3. Splitting theorems in the case of nonpositive curvature

Notions and notations in this section are due to P. Eberlein and B. O'Neill [6].

A Hadamard manifold $H$ is a complete, simply connected Riemannian manifold of dimension $n \geq 2$ having nonpositive sectional curvature. Geodesics $\alpha$ and $\beta$ in $H$ are *asymptotic* provided there exists a positive $c$ such that $d(\alpha(t), \beta(t)) \leq c$ for all $t \geq 0$. The asymptotic relation is an equivalence relation on the set of all geodesics in $H$. Let $H(\infty)$ be the set of all asymptotic classes of geodesics of $H$ and let $\bar{H} = H \cup H(\infty)$. $\bar{H}$ with the cone topology (see [6]) is homeomorphic to the closed unit $n$-ball. If $\alpha:(-\infty, \infty) \to H$ is a geodesic, let $\alpha(\infty)$ be the asymptote class of $\alpha$ and let $\alpha(-\infty)$ be the asymptote class of the reverse curve $t \mapsto \alpha(-t)$. If $\varphi$ is an isometry of $H$ and $x$ is a point in $H(\infty)$ we set $\varphi(x) = (\varphi \circ \alpha)(\infty)$, where $\alpha$ is any geodesic representing $x$. Thus we obtain a well-defined map $\varphi: \bar{H} \to H$ which is bijective and carries $H(\infty)$ into itself.

A complete manifold $M$ of dimension $n \geq 2$ and of nonpositive sectional curvature is precisely the quotient manifold $H/D$ where $D$ is a properly discontinuous group of isometries of $H$. A continuous curve $\alpha:[0, \infty) \to M$ is by definition *divergent* if for any compact set $K$ in $M$ there exists $t = t_K$ such that for $s \geq t$, $\alpha(s) \in M - K$. Divergent curves $\alpha$ and $\beta$ in $M$ will be called *cofinal*, if given any compact set $K$ in $M$ some final segments $\alpha([s, \infty))$ and $\beta([t, \infty))$ of $\alpha$ and $\beta$ lie in the same connected component of $M - K$. This is clearly an equivalence relation on the set of all divergent curves in $M$, and the resulting equivalence classes are the *ends* of $M$. A (unit speed) geodesic $\gamma$ in $M$ is *almost minimizing* if there is a positive $c$ such that $d(\gamma(0), \gamma(t)) \geq t - c$ for all $t \geq 0$. Further, $x \in H(\infty)$ is *almost D-minimizing* if for any geodesic $\gamma$ representing $x \in \pi(\gamma)$ is almost minimizing, where $\pi$ is the covering projection of $H$ onto $M = H/D$. P. Eberlein and B. O'Neill have proved in [6] that if there exists an $x \in H(\infty)$ such that it is almost $D$-minimizing and a common fixed point of $D$, then a Busemann function of any $\gamma \in x$ is invariant under $D$. Hence the Busemann function $f$ on $M$ of $\pi(\gamma)$ is induced from $f_\gamma$ and convex since every Busemann function on $H$ is convex, and hence $M$ is homeomorphic to a product manifold $N \times \mathbb{R}$ where $N$ is a level of $f$.

Now we consider the case that there are two points in $H(\infty)$ which are common fixed points of $D$ and almost $D$-minimizing.

PROPOSITION 1: *Let $M$ be a complete noncompact Riemannian manifold without boundary and of nonpositive sectional curvature and let $M = H/D$. If there exist distinct points $x$ and $y$ in $H(\infty)$ such that* (1) *they* *are common fixed points of $D$,* (2) *they are almost $D$-minimizing and* (3) *$\pi(x)$ and $\pi(y)$ are in different ends of $M$, then $M$ is isometric to a Riemannian product $N \times \mathbb{R}$.*

PROOF: (3) in the assumption implies in combination with (2) the existence of a straight line $\gamma:(-\infty, \infty) \to M$ such that it connects $\pi(x)$ and $\pi(y)$, more precisely, $\gamma \in \pi(x)$ and the reverse curve $t \mapsto \gamma(-t) \in \pi(y)$. In fact, since geodesics $\alpha \in \pi(x)$ and $\beta \in \pi(y)$ in $M$ are almost minimizing, and hence divergent, a sequence of distance minimizing geodesic segments from $\alpha(t)$ to $\beta(t)$ for $t \geq 0$ contains a subsequence which converges to the desired straight line.

Let $\gamma_\pm$ be two rays such that $\gamma_+(t) = \gamma(t)$ and $\gamma_-(t) = \gamma(-t)$ for $t \geq 0$. The Busemann functions $f_{\gamma_\pm}(.) = \lim_{t\to\infty}\{d(., \gamma_\pm(t)) - t\}$ are locally nonconstant convex functions without minimum by the preceding remark to Proposition 1. Moreover, from the facts at the beginning of §1, $M$ is topologically a cylinder $N_+ \times \mathbb{R}$ (or $N_- \times \mathbb{R}$), where $N_+$ (or $N_-$) is a level of $f_{\gamma_+}$ (or $f_{\gamma_-}$), and all levels are compact.

There exists a compact set $K$ of $M$ such that $f_{\gamma_\pm}^{-1}(0) \subset K$ and $p \in M - K$ implies that $f_{\gamma_+}(p) < 0$ or $f_{\gamma_-}(p) < 0$. In fact, otherwise there exists an unbounded sequence $\{p_i\}$ of points in $M$ such that $f_{\gamma_\pm}(p_i) \geq 0$ and hence there exists an $i_0$ such that $f_{\gamma_+}(p_{i_0}) > f_{\gamma_+}(p_1) \geq 0$ and $f_{\gamma_-}(p_{i_0}) > f_{\gamma_-}(p_1) \geq 0$. Let $\gamma_0:(-\infty, \infty) \to M$ be a geodesic such that $\gamma_0(0) = p_1$ and $\gamma_0(d(p_1, p_{i_0})) = p_{i_0}$. Then $f_{\gamma_\pm} \circ \gamma_0(t)$, $t \geq d(p_1, p_{i_0})$, are positive and monotone increasing from convexity of $f_{\gamma_\pm}$. Clearly $\gamma_0$ is divergent. We assert that $\gamma_0$ is not contained in the ends of $M$ containing $\gamma_+$ and $\gamma_-$, contradicting the fact that $M$ has at most two ends.

Suppose $\gamma_0$ and $\gamma_+$ ($\gamma_-$) are contained in the same end of $M$. If $K'$ is a compact set containing $f_{\gamma_+}^{-1}(0)$ and $f_{\gamma_-}^{-1}(0)$, then by the definition of ends there is an $s = t_{K'}$ such that $\gamma_0([s, \infty))$ and $\gamma_+([s, \infty))$ ($\gamma_-([s, \infty))$) are in the same component of $M - K'$. If $t' > s$ and if $\varepsilon$ is a curve in the component of $M - K'$ joining $\gamma_0(t')$ and $\gamma_+(t')$ ($\gamma_-(t')$), then continuity of $f_{\gamma_\pm}$ implies that $\varepsilon$ must meet $K'$, a contradiction, since $f_{\gamma_+} \circ \gamma_0 | [s, \infty) > 0$ ($f_{\gamma_-} \circ \gamma_0 | [s, \infty) > 0$) and $f_{\gamma_+} \circ \gamma_+ | [s, \infty) < 0$ ($f_{\gamma_-} \circ \gamma_- | [s, \infty) < 0$).

Next we assert that there is a positive $c$ such that for any $p \in M$ $d(p, \gamma(\mathbb{R})) < c$. In fact, if $f_{\gamma_+}(p) < 0$ (or $f_{\gamma_-}(p) < 0$), then by the fact stated at the beginning of §1, $d(p, \gamma(\mathbb{R})) \leq$ the diameter of $f_{\gamma_+}^{-1}(f_{\gamma_+}(p))$ (or $f_{\gamma_-}^{-1}(f_{\gamma_-}(p))$) $\leq$ the diameter of $f_{\gamma_+}^{-1}(0)$ (or $f_{\gamma_-}^{-1}(0)$) $\leq$ the diameter of $K$.

If we prove that $f_{\gamma_+} + f_{\gamma_-}$ is bounded above on $M$, then from convexity of $f_{\gamma_+} + f_{\gamma_-}$, it is constant and hence $f_{\gamma_+}$ and $f_{\gamma_-}$ are affine. For any point $p \in M$ let $t_1$ be such that $d(p, \gamma(\mathbb{R})) = d(p, \gamma(t_1))$. Then

$$f_{\gamma_+}(p) + f_{\gamma_-}(p) = \lim_{t\to\infty}\{d(p, \gamma_+(t)) - t + d(p, \gamma_-(t)) - t\}$$

$$
\begin{aligned}
&= \lim_{t\to\infty} \{d(p,\gamma(t)) - t + d(p,\gamma(-t)) - t\} \\
&= \lim_{t\to\infty} [\{d(p,\gamma(t)) - (t - t_1)\} + \{d(p,\gamma(-t)) - (t_1 + t)\}] \\
&\leq 2d(p,\gamma(\mathbb{R})) < 2c.
\end{aligned}
$$

This completes the proof of Proposition 1.

Now we can proceed to the proof of Theorem 4. It turns out that the aassumption of Theorem 4 satisfies what we suppose in Proposition 1, and the proof of Theorem 4 is achieved by Proposition 1.

PROOF OF THEOREM 4: Let $\tilde{\gamma}$ be any lift of $\gamma$ to $H$. First we will construct an isometry $\tilde{\delta}$ over $\delta$ such that $\tilde{\delta}\circ\varphi = \varphi\circ\tilde{\delta}$ for any $\varphi\in D$ and $\varphi\circ\tilde{\gamma}(t) = \tilde{\gamma}(t+a)$ for all $t\in(-\infty,\infty)$.

For points $q$ and $r$ in $H$ let $T(q,r)$ be a geodesic from $q$ to $r$ in $H$. For every point $p$ in $H$, define $\tilde{\delta}p$ by a point in $H$ which is the endpoint of the lift of $\delta\circ\pi(T(p,\tilde{\gamma}(0)))$ to $H$ starting at $\tilde{\gamma}(a)$. We know from (28.7), [1] p. 177, that $\tilde{\delta}$ is well-defined and an isometry of $H$ over $\delta$. From the construction of $\tilde{\delta}$, $\tilde{\delta}\circ\tilde{\gamma}(t) = \tilde{\gamma}(t+a)$ for all $t\in(-\infty,\infty)$.

Now we prove that $\tilde{\delta}\circ\varphi = \varphi\circ\tilde{\delta}$ for any $\varphi\in D$. Since all free homotopy classes of closed curves in $M$ are invariant under $\delta$, $\delta\circ\pi(T(p,\varphi p))$ corresponds to $\varphi$ for every point $p$ in $H$. And $\delta\circ\pi(T(p,\varphi p)) = \pi\circ\tilde{\delta}(T(p,\varphi p)) = \pi(T(\tilde{\delta}p,\tilde{\delta}\circ\varphi p))$. Thus $\tilde{\delta}\circ\varphi = \varphi\circ\tilde{\delta}$.

Let $x$ and $y$ be points in $H(\infty)$ which contain $\tilde{\gamma}$ and the reverse geodesic $\tilde{\gamma}_-: t\mapsto\tilde{\gamma}(-t)$ respectively. Since $d(\tilde{\gamma}(na),\ \varphi\circ\tilde{\gamma}(na)) = d(\tilde{\delta}^n\circ\tilde{\gamma}(0),\ \varphi\circ\tilde{\delta}^n\circ\tilde{\gamma}(0)) = d(\tilde{\delta}^n\circ\tilde{\gamma}(0),\ \tilde{\delta}^n\circ\varphi\circ\tilde{\gamma}(0)) = d(\tilde{\gamma}(0),\ \varphi\circ\tilde{\gamma}(0))$ for all integer $n$, $x$ (or $y$) contains $\varphi\circ\tilde{\gamma}$ (or $\varphi\circ\gamma_-$) and therefore $x$ and $y$ are common fixed points of $D$. Since $\gamma$ is a straight line, $x$ and $y$ are almost $D$-minimizing and hence the assumptions of Proposition 1 is satisfied. The proof is complete.

It is necessary for $\delta$ to leave all free homotopy classes of closed curves in $M$ invariant. In fact, there is an example of a surface $S$ in the Euclidean 4-space $E^4$ which is not isometric to a flat cylinder and on which a non-trivial isometry $\delta$ exists and translates a straight line on $S$ along itself but $\delta$ does not leave all free homotopy classes of closed curves invariant.

EXAMPLE 1: $S$ is constructed by a union of countably many congruent flat tori in $E^4$ with two plane disks removed and countably many congruent cylinders $S^1\times[0,1]$ which are joined along their boundary circles.

The construction is to put congruent flat tori, to which cylinders are attached, into a deliberate order in such a way that each torus contains 2 parallel plane squares along some line from which disks are removed, and the plane squares on the tori and the boundary circles of cylinders are all parallel along the line in $E^4$.

Let $(\alpha_1, \alpha_2)$ be an arc-length parametrized $C^\infty$-curve of $[0, \lambda]$ in $E^2$ with $(\alpha_1(0), \alpha_2(0)) = (\alpha_1(\lambda), \alpha_2(\lambda)) = (0, -1)$ such that (1) it contains two segments, $\{(x_1, 1);\ -1 \le x_1 \le 1\}$ and $\{(x_1, -1);\ -1 \le x_1 \le 1\}$, (2) it is contained in the strip $\{(x_1, x_2);\ -1 \le x_2 \le 1\}$ and (3) it is symmetric with respect to the origin of $E^2$, And let $X$ denote its image in $E^2$. Then we can consider canonically $Y' := X \times X$ as the figure in $E^4$.

$Y'$ has clearly 4 disjoint flat squared faces and we denote two of them, $\{(x_1, 1, x_3, 1);\ -1 \le x_1, x_3 \le 1\}$ and $\{(x_1, -1, x_3, 1);\ -1 \le x_1, x_3 \le 1\}$, by $A_1$ and $A_2$ respectively. We remove the disk $D_1$ (or $D_2$) from $A_1$ (or $A_2$) with center $(0, 1, 0, 1)$ (or $(0, -1, 0, 1)$) and radius 1/2. And we denote the resulting figure by $Y$. It should be noted that $\gamma_0 : [1/2, \lambda/2 - 1/2] \to Y$, $\gamma_0(t) = (\alpha_1(t), \alpha_2(t), 0, 1)$ is a distance minimizing geodesic segment from $\partial D_2$ to $\partial D_1$. Because in the universal covering space $H$ of $Y'$ any lift of $\gamma_0$ to $H$ is a distance minimizing geodesic segment from the lift of $\partial D_2$ to the lift of $\partial D_1$.

$Y$ is joined with a certain cylinder $S^1 \times [0, \mu]$ along their boundary circles, and this is done in the affine $(x_1, x_2, x_3, 1)$-subspace as follows. Let $c = (\beta_1, \beta_2, 0, 1)$ be an arc-length parametrized $C^\infty$-convex curve of $[0, \mu]$ in the $(x_1, x_2, 0, 1)$-space with $c(\mu) = (1/2, 2, 0, 1)$ such that (1) $c$ does not intersect the $x_2$-axis, (2) $c$ starts at $b := (1/2, 1, 0, 1) = (\alpha_1((\lambda - 1)/2, \alpha_2((\lambda - 1)/2), 0, 1)$, (3) $c$ contains the segment such that $c([0, 1/4]) = \{(x_1, 1, 0, 1);\ 1/4 \le x_1 \le 1/2\}$ and (4) $c$ is symmetric with respect to the line $\{(x_1, 3/2, 0, 1);\ -\infty < x_1 < \infty\}$ in the $(x_1, x_2, 0, 1)$-space. Revolving $c$ about the $x_2$-axis in the $(x_1, x_2, x_3, 1)$-space produces a surface $C$ with boundary $\partial D_1$ and $\partial D_1'$ which is congruent to $\partial D_1$ and hence to $\partial D_2$. Attaching $C$ to $Y$ along $\partial D_1$ we obtain a surface with boundary $\partial D_1' \subset C$ and $\partial D_2 \subset Y$, and they are on the parallel planes normal to the $x_2$-axis in the $(x_1, x_2, x_3, 1)$-space. We denote this surface by $W$.

For each $i = 0, \pm 1, \pm 2, \ldots$, let $\varphi_i$ be a translation along the $x_2$-axis in $E^4$ such that $\varphi_i(x_1, x_2, x_3, x_4) = (x_1, x_2 + 3i, x_3, x_4)$. Then $S := \cup \varphi_i W$ is the desired surface, namely, $S$ is of nonpositive curvature and has an isometry $\delta$ which satisfies the assumption of Theorem 4 except for invariancy of all free homotopy classes of closed curves under $\delta$.

The desired $\delta$ is obtained by putting $\delta = \varphi_1$. Clearly $\delta$ does not leave all free homotopy classes of closed curves in $S$ invariant. Now we find a straight line $\gamma$ which is translated by $\delta$. Let $\gamma_0$ be the distance minimizing geodesic segment in $W$ which is already realized, i.e., the endpoints are $(\alpha_1(1/2), \alpha_2(1/2), 0, 1) = (1/2, -1, 0, 1) \in \partial D_2$ and $(\alpha_1((\lambda - 1)/2), \alpha_2((\lambda$

$-1)/2),0,1) = (1/2,1,0,1) \in \partial D_1$. Then $\partial D_1$ and $\partial D_2$ are plane circles, so $\gamma_0$ is perpendicular to both $\partial D_1$ and $\partial D_2$, and therefore the extension of $\gamma_0$ into $C$ in $W$ is nothing but a profile curve, and the endpoint $(1/2,2,0,1)$ is identified with $\varphi_1(1/2,-1,0,1) = (1/2,2,0,1) \in \varphi_1 \partial D_2$. Hence by iterating this step $\gamma_0$ is smoothly extended in $S$ and the resulting geodesic is identified with $\cup \varphi_i \gamma_0 =: \gamma$. Because $\gamma_0$ is distance minimizing from $\partial D_2$ to $\partial D_1$ and hence so is $\varphi_i \gamma_0$ from $\varphi_i \partial D_2$ to $\varphi_i \partial D_1$, for every $i$, and because so is a profile curve of $\varphi_i C$, for each $i$, $\gamma$ is a straight line. And $\delta$ obviously translates $\gamma$ along itself.

## 4. Proof of Theorem 1

In this section we give the proof of Theorem 1.

Clearly for each geodesic $\gamma : (-\infty, \infty) \to M$ there are constants $m$ and $n \in \mathbb{R}$ such that $F \circ \gamma(t) = mt + n$ for all $t \in (-\infty, \infty)$ and hence $F$ has no minimum on $M$. It follows from this formula that all levels of $F$ are connected totally convex set (for definition see [5]) and hence totally geodesic embedded hypersurfaces without boundary (see [3], [5]). Because if a geodesic $\gamma$ passes through two points in a level of $F$, then $m = 0$.

We are going to show that the exponential map of the normal bundle of each level onto $M$ yields the desired isometric map. For a subset $A$ of $M$ and for a point $q$ in $M$, we call a point $f \in A$ a *foot* of $q$ on $A$ if $d(q,f) = d(q,A)$. We shall often use this notion. We take the following steps to complete the proof.

Assertion 1: For given $a \in \mathbb{R}$ and for each $q \notin F^{-1}(a)$, let $f$ be a foot of $q$ on $F^{-1}(a)$. Then a (distance minimizing geodesic) segment $T(q,f)$ from $q$ to $f$ satisfies the following properties;

(1) For every $c \in [a, F(q)]$ (or $[F(q), a]$ if $F(q) < a$), $T(q,f)$ intersects $F^{-1}(c)$ at exactly one point, say, $f_c$,

(2) $f_c$ is a foot of $q$ on $F^{-1}(c)$ and $f$ is a foot of $f_c$ on $F^{-1}(a)$.

Proof: (1) is evident since $F$ is a non-trivial affine function along $T(q,f)$. Concerning the second part of (2), see (20.6), [1] p. 120.

Suppose there is a point $f_c'$ in $F^{-1}(c)$ such that $d(q,f_c) > d(q,f_c')$. If $F(q) > c$ (if $F(q) < c$), then $(F(q)-c)/d(q,f_c) < (F(q)-c)/d(q,f_c')$ $((c - F(q))/d(q,f_c) < (c - F(q))/d(q,f_c'))$. Let $f'$ be a point in $F^{-1}(a)$ at which the extension of $T(q,f_c')$ meets $F^{-1}(a)$. Then the length of $T(q,f)$ is greater than the length of the extension of $T(q,f_c')$ up to a point $f'$. This contradicts the choice of $f$. Hence the first part of (2) is proved.

ASSERTION 2: Each point $q \notin F^{-1}(a)$ has a unique foot on $F^{-1}(a)$ and moreover there is a unique segment $T(q,f)$ from $q$ to $f$.

PROOF: From the existence of a strongly convex ball around $q$ (see [9]) and the total convexity of $F^{-1}(a)$ it follows that for sufficiently small positive $\varepsilon$ there is a unique foot of $q$ on $F^{-1}(F(q)-\varepsilon)$ (or $F^{-1}(F(q)+\varepsilon)$). If there are distinct feet $f$ and $f'$ of $q$ on $F^{-1}(a)$, then Assertion 1 implies that there are at least two feet of $q$ on $F^{-1}(F(q)-\varepsilon)$ (or $F^{-1}(F(q)+\varepsilon)$) through which $T(q,f)$ and $T(q,f')$ pass respectively, a contradiction.

The argument above is useful to prove the second part. If the uniqueness of the existence of the segment from $q$ to $f$ is false, then there exist at least two feet of $q$ on a level of $F$ between $F^{-1}(a)$ and $F^{-1}(F(q))$, a contradiction.

Let $f$ be a foot of $q$ on $F^{-1}(a)$ and let $\gamma:(-\infty,\infty)\to M$ be a geodesic determined by $\gamma(0)=f$ and $\gamma(d(q,f))=q$.

ASSERTION 3: $d(\gamma(t),f)=d(\gamma(t),F^{-1}(a))=|t|$ for all $t\in(-\infty,\infty)$.

PROOF: Let $t_0$ be the least upper bound of $t$ where $\gamma(t)$ has the foot $f$ on $F^{-1}(a)$. We must prove that $t_0=\infty$. Suppose $t_0<\infty$. Then for sufficiently small $\varepsilon>0$ $\gamma(t_0-\varepsilon)$ is evidently a foot of $\gamma(t_0+\varepsilon)$ on $F^{-1}(F(\gamma(t_0-\varepsilon)))$ since Assertion 1 implies that $\gamma$ is perpendicular to $F^{-1}(F(\gamma(t_0-\varepsilon)))$ at $\gamma(t_0-\varepsilon)$. However since $\gamma(t_0+\varepsilon)$ has a foot on $F^{-1}(a)$ different from $f$, $\gamma(t_0-\varepsilon)$ cannot be a foot of $\gamma(t_0+\varepsilon)$ on $F^{-1}(F(\gamma(t_0-\varepsilon)))$, a contradiction.

We can prove similarly on the nonpositive part of $\gamma$.

By the Assertion 1 to 3 we obtain the following.

ASSERTION 4: The exponential map of the normal bundle of $F^{-1}(a)$ onto $M$ is a diffeomorphism.

It turns out that this diffeomorphism is an isometric map by the following assertion and thus the proof is complete.

ASSERTION 5: Let $q$ and $q'$ be any points in $F^{-1}(c)$ and let $f$ and $f'$ be the feet of $q$ and $q'$ on $F^{-1}(a)$ respectively. Then $d(q,q')=d(f,f')$ and $d(q,f)=d(q',f')$.

The first part implies that all levels of $F$ are isometric to each other and the second part implies that there exists an isometric map $I$ of the Riemannian product $F^{-1}(a)\times\mathbb{R}$ onto $M$ such that $F\circ I(x,t)=bt+a$ for all $x\in F^{-1}(a)$ and for all $t\in\mathbb{R}$.

PROOF: First we consider the case where $q'$ is close to $q$, more precisely, the least upper bound of the set of all the lengths of $T(f_d, f'_d)$, $a \leq d \leq c$ (or $c \leq d \leq a$ if $c < a$), is smaller than the greatest lower bound of the set of all the convex radii of points in $T(q, f)$, where $f_d$ and $f'_d$ are points at which $T(q, f)$ and $T(q', f')$ intersect $F^{-1}(d)$ respectively. Since $T(q, f)$ and $T(q', f')$ are perpendicular to each level through which they pass, it follows from the first variation formula (see [9]) that $d(q, q') = d(f, f')$.

In general case, take a partition of a segment $T(q, q')$, $q = q_0, q_1, \ldots, q_n = q'$, in this order such that every pair of $q_i$ and $q_{i+1}$, $i = 0, 1, \ldots, n-1$, satisfies the condition above. And let $f_i$, $i = 0, 1, \ldots, n$, be a foot of $q_i$ on $F^{-1}(a)$. Then

$$d(q, q') = \sum_{i=0}^{n-1} d(q_i, q_{i+1}) = \sum_{i=0}^{n-1} d(f_i, f_{i+1}) \geq d(f, f').$$

From Assertion 1, (2), it follows that $f$ ($f'$) is the foot of $q$ ($q'$) on $F^{-1}(a)$ implies that $q$ ($q'$) is the foot of $f$ ($f'$) on $F^{-1}(c)$. Thus we obtain $d(f, f') \geq d(q, q')$. This proves $d(q, q') = d(f, f')$.

On the second part, we have only to consider the variation made of the totality of segments each of which joins a point of $T(q, q')$ and its foot on $F^{-1}(a)$. We complete the proof.

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