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Vector bundles on the cone over a curve

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Let $X \subset \mathbb{P}^n_k$ be a projectively normal curve over an algebraically closed field $k$, and let $C(X) \subset \mathbb{A}^{n+1}$ be the affine cone over $X$. The problem studied in this paper is to determine whether $K_0(C(X)) = \mathbb{Z}$, where $K_0$ denotes the Grothendieck group of vector bundles on $C(X)$ (see [2] for definitions). This is an important special case of a question raised by Murthy, as to whether $K_0(A) = \mathbb{Z}$ for any normal graded ring $A$.

Spencer Bloch recently showed that $K_0(A) \neq \mathbb{Z}$ for $A = \mathbb{C}[X, Y, Z]/(Z^2 - X^3 - Y^7)$ giving a counterexample to Murthy's question. However, one still suspected that the result would be true for cones. Partial positive results were known (see Varley [3]).

It turns out that the problem has a very different flavour in characteristic 0 than in positive characteristics. First consider the case of characteristic $p > 0$. We have

**THEOREM 1**: Let $A = \bigoplus \limits_{n \geq 0} A_n$ be a normal graded ring, finitely generated over $A_0 = k$, where $k$ is a field (see Bass [1]).

Here $A_0(\text{Spec } A)$ denotes the subgroup of $K_0(A)$ generated by the classes of smooth points of $\text{Spec } A$. Now $K_0(A)$ is generated by the class of the trivial line bundle, and classes of sub-varieties not meeting the singular locus (see [3]).
COROLLARY (1.3): Let $A$ be as in Theorem 1. Suppose that $\dim A = 2$. Then $K_0(A) = \mathbb{Z}$.

This answers Murthy's question affirmatively in the two dimensional case, and thus includes the result on cones over curves.

Next, we have a partial positive result in characteristic 0.

THEOREM 2: Let $X \subset \mathbb{P}^n_k$ be a projectively normal curve, where $k$ is an algebraically closed field of characteristic 0. Assume that $X$ is not contained in a hyperplane, and that $\deg X \leq 2n - 1$. Then $A_0(C(X)) = 0$, where $C(X) \subseteq \mathbb{A}^{n+1}$ is the affine cone over $X$.

As a consequence, we obtain

THEOREM 2': Let $X/k$ be a curve of genus $g$, and $D$ a divisor on $X$ such that $\deg D \geq 2g + 1$. Then $A_0(C(X)) = 0$, where $C(X)$ is the cone over $X$ in the embedding $X \subset [\mathbb{P}^n_D]$ given by the complete linear system $|D|$.

Using the cancellation theorem of Murthy and Swan, we can formulate the above theorems as follows.

THEOREM: Let $k$ be an algebraically closed field, and let $A = \bigoplus_{n \geq 0} A_n$ be a finitely generated graded $k$-algebra with $A_0 = k$. Then every projective module over $A$ is free, in each of the following cases:

i) $\text{char } k = p > 0$, and $A$ is normal of dimension 2.

ii) $\text{char } k = 0$, and $\text{Spec } A$ is the cone over a projectively normal curve $X$ properly contained in $\mathbb{P}^n$, and satisfying $\deg X \leq 2n - 1$.

iii) $\text{char } k = 0$, and $A = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))$ where $X/k$ is a smooth curve of genus $g$, and $D$ a divisor on $X$ satisfying $\deg D \geq 2g + 1$.

Finally, we construct an infinite family of examples of cones over $\mathbb{C}$ which admit non-trivial vector bundles. Let $L$ denote the field of algebraic numbers.

THEOREM 3: Let $X \subset \mathbb{P}^n_L$ be a projectively normal curve such that $H^1(X, \mathcal{O}_X(1)) \neq 0$. Then if $C(X_c)$ denotes the cone over the corresponding complex curve, we have $K_0(C(X_c)) \neq \mathbb{Z}$. (In fact, a slight modification of our argument will show that $K_0(C(X_c))$ is uncountable).

One remarkable fact about theorem 3 is the following. For a curve $X \subset \mathbb{P}^n_L$, let $Y \subset \mathbb{P}^{n+1}_L$ denote the projective cone over $X$. Let $Z \rightarrow Y$ be the blow up of $Y$ at the vertex. Then $Y \cong \mathbb{P}(\mathcal{O}_X \oplus \mathcal{O}_X(1))$. The Leray spectral sequence applied to the map $\pi$ yields an exact sequence

$$0 \rightarrow H^1(Y, \mathcal{O}_Y) \rightarrow H^1(Z, \mathcal{O}_Z) \rightarrow \Gamma(Y, R^1\pi_*\mathcal{O}_Z) \rightarrow$$

$$\rightarrow H^2(Y, \mathcal{O}_Y) \rightarrow H^2(Z, \mathcal{O}_Z) \rightarrow 0.$$
Since $Z$ is a ruled surface, $p_g(Z) = 0$, and $q = g(X)$, the genus of $X$. (In fact, all elements of $H^1(Z,\mathcal{O}_Z)$ are pulled back from $H^1(X,\mathcal{O}_X)$). Now $R^1\pi_*\mathcal{O}_Z$ is a torsion sheaf on $Y$ supported only at the vertex of the cone. From the formal function theorem (see [7]), $\Gamma(R^1\pi_*\mathcal{O}_Z)$ has a filtration whose associated graded module is $\bigoplus_{m \geq 0} H^1(E, I^m/I^{m+1})$ where $E$ is the exceptional set, and $I$ is its sheaf of ideals on $Z$. Now $E$ is a section of the fibration $Z \to X$, hence $E \cong X$. One easily checks that $I/I^2 \cong \mathcal{O}_X(1)$, and thus $I^m/I^{m+1} \cong \mathcal{O}_X(m)$. Also, the map $H^1(Z,\mathcal{O}_Z) \to \Gamma(R^1\pi_*\mathcal{O}_Z)$ maps the former isomorphically onto $H^1(E,\mathcal{O}_E)$. Hence $h^2(Y,\mathcal{O}_Y)$ vanishes precisely when $H^1(X,\mathcal{O}_X(1)) = 0$. Thus, the curves $X \subset \mathbb{P}^n_C$ with $H^1(X,\mathcal{O}_X(1)) \neq 0$ correspond precisely to the cones $Y$ with “geometric genus” (i.e. $h^2(\mathcal{O}) > 0$). Hence, Theorem 3 may be regarded as an analogue for cones of a famous result of Mumford on the infinite dimensionality of the Chow group of zero cycles on a surface with $p_g > 0$ (see [5]). In fact, one might conjecture that at least for cones, $A_0(C(X)) = 0 \iff p_g(Y) = 0$ (where $p_g$ stands for $h^2(\mathcal{O})$); this is the analogue of a conjecture of Bloch for smooth surfaces with $p_g = 0$ (see [13], ch. 1 for motivation and further references for that conjecture).

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§1) Results in characteristic $p > 0$

In this section we prove theorem 1. In this paper, the Chow group of zero cycles will always be the subgroup of the Grothendieck group $K_0$ generated by the classes of smooth points. In particular, it is not the same (in general) as the Chow group of Fulton [6] when the variety we are dealing with is singular. The proof of the theorem is based on two lemmas:

**Lemma (1.1):** Let $Y$ be an affine normal variety with isolated singularities over an algebraically closed field (of arbitrary characteristic). If $U \subset Y$ is an open (dense) set, then $A_0(Y)$ is generated by the classes of smooth points of $U$. $A_0(Y)$ is a divisible group.
PROOF: If \( Y \) is a curve, the result holds from the theory of Jacobians. In general, if \( P \in Y \) is a smooth point, we can find a curve \( C \subset Y \) such that \( P \in C, \ C \cap U \neq \emptyset \), and \( C \) misses the singular locus of \( Y \); we may take \( C \) to be smooth. Then there is a natural map \( \text{Pic} \ C \to A_0(Y) \), and the class of \( P \) is in the image of this map. Hence the result follows from the previous case.

**LEMMA (1.2):** Let \( A = \bigoplus_{n \geq 0} A_n \) be a graded normal ring of dimension 1, where \( A_0 = k \), and \( A \) is finitely generated over \( A_0 \). Then \( A \cong k[t] \), where \( t \) is homogeneous (perhaps of degree \( d > 1 \)).

**PROOF:** It is amusing to give two proofs. First, an algebraic one. Let \( M = \bigoplus_{n > 0} A_n \). Then \( A_M \) is a P.I.D. as \( A \) is normal. Since \( MA_M \) is generated by one element, but also has a set of homogeneous generators, it is generated by one homogeneous element (Nakayama’s Lemma). Let \( MA_M = fA_M \), with \( f \in A \) homogeneous, and let \( g \in A \) be any homogeneous element of positive degree. Since \( g \in f \cdot A_M \), \( g = u \cdot f^n = \frac{u_1}{u_2} \cdot f^n \), where \( u_1, u_2 \in A - M \). Comparing homogeneous terms of lowest degree on both sides of \( u_2 g = u_3 f^n \), we see that we may assume \( u_1, u_2 \) to be homogeneous. Since \( u_i \notin M, u_i \in A_0 = k \). Thus \( A = k[t] \) (since every element of \( A \) is a finite sum of homogeneous elements).

The second proof is geometric – since \( \text{Spec} \ A \) is an affine curve over \( k \) with a non-trivial \( G_m \)-action, it is a rational curve. Since it is normal, and has no units (because \( A \) is graded) apart from \( k^* \), it must be \( \mathbb{A}^1_k \). The group of automorphisms of \( \mathbb{A}^1_k \) fixing a point is \( G_m \); hence the grading on the coordinate ring of \( \mathbb{A}^1_k \) induced from \( A \) must be the usual one.

**PROOF OF THEOREM 1:** We first give a simple proof in the case when \( A \) is the homogeneous coordinate ring of a plane curve.

Let \( X = \text{Proj} \ A \subset \mathbb{P}^2_k \) be a smooth plane curve, and let \( C(X) \subset \mathbb{A}^3 \) be the cone over \( X \) (so that \( C(X) = \text{Spec} \ A \)). Let \( 0 \in C(X) \) be the vertex, and let \( \pi: C(X) - \{0\} \to X \) be the projection. Let \( P \in C(X) \) be a smooth point and \( \pi(P) = \tilde{P} \). Choose a line \( l \subset \mathbb{P}^2 \) such that \( l \cap X = \{P_1, \ldots, P_n\} \), where \( P_1 = \tilde{P} \), and \( n = \deg X \), and the \( P_i \) are distinct. Then \( \pi^{-1}(l) \cup \{0\} = \mathbb{S}_1 \cap C(X) \), where \( \mathbb{S}_1 \subset \mathbb{A}^3 \) is a plane (the cone over the line \( l \)). Thus \( \mathbb{S}_1 \cap C(X) = l_1 \cup \ldots \cup l_n \), where \( \pi(l_i - \{0\}) = P_i \), and \( l_i \) are lines on \( \mathbb{S}_1 \) which concur at \( 0 \). We can choose coordinates \( x \) and \( y \) on \( \mathbb{S}_1 \) so that \( l_i \) is the \( y \)-axis, \( P_i \in l_1 \) is the point \( (0, 1) \), and the lines \( l_2, \ldots, l_n \) respectively have slopes \( \lambda_2, \ldots, \lambda_n \). Thus

\[
\mathbb{S}_1 \cap C(X) = \text{Spec} \ k[x, y]/(x \cdot \prod_{i=2}^n (y - \lambda_i x))
\]
Consider the function \( f = 1 - (y - \lambda_2 \lambda_1)^r \prod_{i=3}^n (y - \lambda_i x) \) (where \( r > 0 \) will be chosen in a moment). Then \( f \) is identically equal to 1 on \( l_2 \cup \ldots \cup l_n \). On \( l_1, x = 0 \), so that \( f|_{l_1} = 1 - y^{r+n-2} \). Choose \( r > 0 \) to be the smallest integer such that \( r + n - 2 = p^* \), where \( p = \text{char } k \). Then \( f|_{l_1} = (1 - y)^{p^*} \). Thus, the zero cycle \( (P) \) represents a \( p^* \)-torsion element of \( \text{Pic}(\mathbb{S}_1 \cap C(X)) \), and hence a \( p^* \)-torsion element of \( A_0(C(X)) \). Since \( A_0(C(X)) \) is generated by the classes of smooth points \( P \in C(X) \), we have \( p^* \cdot A_0(C(X)) = 0 \). Now the divisibility of \( A_0(C(X)) \) forces \( A_0(C(X)) = 0 \).

Now we give the proof in the general case, based on the same idea. Let \( \mathcal{O}(1) \) denote the sheaf on \( \text{Proj } A \) associated to the graded module \( \bigoplus A_i \) (see [7], ch. II). Fix a large integer \( m > 0 \) such that \( \mathcal{O}(m) \) embeds into \( \text{Proj } A \) is some projective space, and let \( \text{dim } \text{Proj } A = r \) (so that \( \text{dim } A = r + 1 \)). Let \( 0 \in \text{Spec } A \) be the vertex, and let \( \pi : \text{Spec } A - \{0\} \to \text{Proj } A \) be the projection. Let \( U \subset \text{Spec } A - \{0\} \) be the inverse image of the locus of smooth points of \( \text{Proj } A \), and let \( P \in U \). Let \( \pi(P) = \bar{P} \). Choose \( r \) general hyperplane sections of \( \text{Proj } A \) through \( \bar{P} \), so that the intersection of all of them with \( \text{Proj } A \) consists of \( d = \deg(\text{Proj } A) \) points \( P_1 = \bar{P}, P_2, \ldots, P_d \), where all the \( P_i \) are smooth. Let \( Y = \pi^{-1}\{\{P_1, \ldots, P_d\} \cup \{0\} \), so that \( Y = \text{Spec}(A/(f_1, \ldots, f_d)) \) where \( f_i \) is homogeneous of degree \( m \) for each \( i \). Since \( A \) is Cohen–Macaulay, and height \( (f_1, \ldots, f_d) = r \), the ring \( B = A/(f_1, \ldots, f_d) \) is a reduced graded ring of dimension 1. The minimal primes \( \mathfrak{P}_1, \ldots, \mathfrak{P}_d \) of \( B \) satisfy \( \text{Spec}(B/\mathfrak{P}_i) = \pi^{-1}(P_i) - \{0\} \). Let \( \bar{B} \) denote the normalisation of \( B \) (i.e. its integral closure in its total quotient ring), and let \( I \subset B \) be the conductor of \( B \to \bar{B} \) (see [12] for the definition). We have the well known exact sequence (where \(* \) denotes the unit group)

\[
0 \to B^* \to \bar{B}^* \to (\bar{B}/I\bar{B})^*/(B/IB)^* \to \text{Pic } B \to \text{Pic } \bar{B} \to 0
\]

From lemma (1.2), \( \bar{B} \cong \bigoplus_{i=1}^d k[t_i] \), so that \( \text{Pic } \bar{B} = 0 \). Thus we have a surjection \( (\bar{B}/I\bar{B})^*/(\text{Image } \bar{B}^*) \to \text{Pic } B \). Since \( \bar{B} \cong \bigoplus_{i=1}^d k[t_i], \) \( \bar{B}^* \cong \bigoplus_{i=1}^d k^*; \)

also \( \bar{B}/I\bar{B} \cong \bigoplus_{i=1}^d k[t_i]/(t_i^{n_i}) \) for some exponents \( n_i > 0 \). Now \( \left( \frac{k[t]}{t^n} \right)^* = k^* R_n \), where \( R_n \) is \( p^* \)-torsion for any \( v \) such that \( p^* \geq n \) (this boils down to the identity

\[
(1 + a_1 t + a_2 t^2 + \ldots + a_d t^d)^{p^*} = 1 + a_1^{p^*} t^{p^*} + \ldots + a_d^{p^*} t^{d \cdot p^*}).
\]

Thus if \( p^* \geq \sup n_i \), then \( p^* \cdot (\text{Pic } B) = 0 \). If we can find a value of \( v > 0 \)
such that \( v \) is independent of the choice of the initial point \( P \in U \), and the hyperplane sections \( f_1, \ldots, f_r \), then as in the case of plane curves we can conclude that \( p^r \cdot A_0(\text{Spec} \, A) = 0 \); hence \( A_0(\text{Spec} \, A) = 0 \) by divisibility. The rest of the proof will consist in showing that by shrinking \( U \) to some non-empty open subset \( V \), and for any choices of \( f_1, \ldots, f_r \) corresponding to any given point \( P \in V \), the resulting exponent \( v \) is bounded by some preassigned number depending only on \( A \).

Factor the inclusion \( B \subset \tilde{B} \) as \( B \subset \bigoplus_{i=1}^d B/\mathcal{P}_i \) and

\[
\bigoplus_{i=1}^d B/\mathcal{P}_i \subset \bigoplus_{i=1}^d (B/\mathcal{P}_i)^\sim \cong \bigoplus_{i=1}^d k[t].
\]

If \( J_1 \) and \( J_2 \) are the respective conductors, then \( J_1J_2 \tilde{B} \subset I \tilde{B} = I \). Thus if \( J_1 \tilde{B} = \bigoplus t^{m_i}k[t] \), and \( J_2 \tilde{B} = \bigoplus t^{m_2}k[t] \), it is enough to separately bound the exponents \( n_{i1} \) and \( n_{i2} \) for all \( i \).

Let \( A = k[\phi_1, \ldots, \phi_s] \) where \( \phi_i \in A \) is homogeneous of degree \( m_i \) for each \( i \). For any homogeneous prime \( \mathcal{P}_i \) of \( A \) of height \( r \) (corresponding to a point \( \mathcal{P}_i \in \text{Proj} \, A \), \( A/\mathcal{P}_i \cong k[t] \), where \( t \) has degree \( e \) (say) in the grading induced from \( A \). Thus \( A/\mathcal{P}_i \subset k[u] \), where \( u^e = t \), and \( u \) has degree 1. Clearly \( \phi_i \) is mapped to an element of degree \( m_i \) in \( k[u] \), which is homogeneous i.e. \( \phi_i \mapsto \alpha_i \cdot u^{m_i} \) for some \( \alpha_i \in k \). So \( A/\mathcal{P}_i \cong k[\alpha_1u^{m_1}, \ldots, \alpha_su^{m_s}] \subset k[u] \). Now \( \alpha_i = 0 \Leftrightarrow \phi_i \) (considered as a section of \( \mathcal{O}(m_i) \)) vanishes at \( \mathcal{P}_i \in \text{Proj} \, A \). Hence deleting the finite set of zeroes of the sections \( \phi_1, \ldots, \phi_s \in \Gamma(\text{Proj} \, A, \bigoplus \mathcal{O}(m)) \), and correspondingly shrinking our open set \( U \subset \text{Spec} \, A - \{0\} \) to a smaller open set \( V \), we may assume that none of the \( \alpha_i \) vanish when \( \mathcal{P}_i \) is one of the primes we construct by taking hyperplane sections of \( \text{Proj} \, A \). But if \( \alpha_1, \ldots, \alpha_s \) are non-zero, then \( k[\alpha_1u^{m_1}, \ldots, \alpha_su^{m_s}] = k[u^{m_1}, u^{m_2}, \ldots, u^{m_s}] \) i.e. all the \( B/\mathcal{P}_i \) are isomorphic. Since \( m_1, \ldots, m_s \) depends only on \( A \), this bounds the exponents \( n_{i2} \) for \( J_2 \).

We claim that if \( J = \bigcap_i (\mathcal{P}_i + \bigcap_{j \neq i} \mathcal{P}_j) \), then \( J \subset J_1 \). By definition of the conductor, \( J_1 \) is the largest ideal in \( B \) which is a \( \bigoplus_{i=1}^d B/\mathcal{P}_i \)-module.

Hence, to verify the claim, it is enough to prove the following –

given \( a_1, \ldots, a_d \in J \), there exists \( a \in B \) such that \( a - a_i \in \mathcal{P}_i \) (i.e. \( a \mapsto (\bar{a}_1, \ldots, \bar{a}_d) \) under \( B \subset \bigoplus B/\mathcal{P}_i \)). But if \( a_i = b_i + c_i \), with \( b_i \in \mathcal{P}_i \), \( c_i \in \bigcap_{j \neq i} \mathcal{P}_j \), then \( a = \sum_{i=1}^d c_i \) works, since \( a - a_i = \sum_{j \neq i} c_j - b_i \in \mathcal{P}_i \) as desired. Now suppose \( f \in J \) satisfies \( f = (\beta_1t^{r_1}, \ldots, \beta_dt^{r_d}) \) where \( \beta_1 \ldots \beta_d \neq 0 \). Then clearly \( v_i \geq n_{i2} \) for all \( i \). So if we can suitably bound \( v_i \), we will be done. In fact, by symmetry it suffices to find \( f \) with suitably bounded \( v_1 \), and \( \beta_1 \neq 0 \).
Consider the homomorphisms $A \to B \to B/\mathcal{P}_i$, where we identify $B/\mathcal{P}_i$ with $k[u^m, \ldots, u^m] \subset k[u]$. Then for each $i \neq 1$, we can find $\mu, \nu$ such that $1 \leq \mu, \nu \leq s$, and $\alpha_{1i}^m \cdot \alpha_{1v}^m - \alpha_{1i}^m \cdot \alpha_{1v}^m \neq 0$ (i.e. if two points of $\text{Proj} A_i$ namely $P_i$ and $P_v$ are distinct, then they have distinct "weighted homogeneous" coordinates – recall that $\phi_\mu \mapsto \alpha_{1i}^m \cdot u^m$ under $A \to B \to B/\mathcal{P}_j$).

Let $\gamma \in A$ be the element defined by $\gamma = \gamma_{\mu, \nu} = \alpha_{1i}^m \cdot \phi_{\mu}^m - \alpha_{1i}^m \cdot \phi_{\nu}^m$. Then the image of $\gamma$ in $B/\mathcal{P}_i$ is

$$\alpha_{1v}^m (x_{1i} u^m) u^m = \alpha_{1i}^m (x_{1i} u^m) u^m = (\alpha_{1i}^m \cdot \alpha_{1i}^m - \alpha_{1i}^m \cdot \alpha_{1i}^m) \cdot u^m = 0$$

i.e. the image $\tilde{\gamma}$ of $\gamma$ in $B$ actually lies in $\mathcal{P}_i$. The image of $\gamma$ in $B/\mathcal{P}_1$ is $[\alpha_{1i}^m \cdot \alpha_{1i}^m - \alpha_{1i}^m \cdot \alpha_{1i}^m] \cdot u^m = \beta_i \cdot u^m \cdot \nu$, where $\beta_i \neq 0$, and $t_i$ is bounded by a number depending only on $A$. Since $\mathcal{P}_1 + J = \mathcal{P}_1 + \bigcap_{j \neq 1} \mathcal{P}_j$, the element $\gamma_0 = \prod_{i=2}^d \gamma_{\mu, \nu}$ is such that the image of $\gamma_0$ in $B/\mathcal{P}_1$ is actually in $J \cdot B/\mathcal{P}_1$, and hence in $J_1 \cdot B/\mathcal{P}_1$. But $\gamma_0 \cdot k[u] = t^2 + \ldots + t_k k[u]$, and $t_2 + \ldots + t_d$ is bounded by a number depending only on $A$.

This completes the proof of Theorem 1.

**Corollary (1.3):** Let $A$ be as in Theorem 1. Then if $\dim A = 2$, we have $K_0(A) = \mathbb{Z}$. Hence all vector bundles on $\text{Spec} A$ are trivial.

**Proof:** By a remark of Murthy (see [1]) we know that $\text{Pic} A = (0)$. By the standard argument using the cancellation theorem of Bass, it suffices to prove that vector bundles of rank 2 represent trivial elements of $K_0(A)$ to prove that $K_0(A) = \mathbb{Z}$. Now we can find a section of a given vector bundle of rank 2 which has isolated zeroes at smooth points of $\text{Spec} A$. If $P$ is the projective $A$-module of global sections of the bundle, we have an exact sequence $0 \to L \to P^* \to I \to 0$, where $I \subset A$ is the ideal of zeroes of the chosen section, $P^*$ is the dual projective module. An argument using the determinant shows that $L \cong A^2 P^* \in \text{Pic} A$ i.e. $L \cong A$. Next, $[A/I] \in K_0(A)$ gives an element of $A_0(\text{Spec} A)$ which is trivial by Theorem 1. Hence $[A] = [I]$ in $K_0(A)$. Putting these facts together gives $[P^*] \cong [A^{\otimes 2}]$ i.e. all vector bundles of rank 2 are stably trivial. This proves $K_0(A) = \mathbb{Z}$. Now the cancellation theorem of Murthy and Swan [4] proves that all vector bundles are trivial.

The argument needed to deduce the triviality of vector bundles from the vanishing of the Chow group works in all characteristics (see section 2 of this paper).

**§2) Some positive results in characteristic 0**

In this section we obtain partial positive results for cones over
smooth projective curves in characteristic zero. Our result is a slight
improvement on results of Varley (see [3]). The proof is based on an
idea of Ojanguren [8], who used it to prove the result for plane cubics.

**Theorem 2:** Let $X \subset \mathbb{P}^n_k$ be a projectively normal curve over the
algebraically closed field $k$ of characteristic 0. Assume that $X$ is not con-
tained in a hyperplane, and has degree atmost $2n - 1$. Then $A_0(C(X)) = 0$,
where $C(X) \subset \mathbb{A}^{n+1}$ is the affine cone over $X$. Hence vector bundles on
$C(X)$ are trivial. (See the proof of Corollary (1.3).)

**Proof:** The triviality of vector bundles follows from the vanishing of
the Chow group, using the triviality of line bundles (a remark of Murthy
— see [1]) and the cancellation theorem of Murthy and Swan [4]. The
proof of the vanishing of $A_0(C(X))$ is based on two lemmas.

Let $\deg X = d \leq 2n - 1$, and set $r = d - n$.

**Lemma (2.1):** Assume that $P \in X$ is not a Weierstrass point. Let $H \subset \mathbb{P}^n$ be
the osculating hyperplane to $X$ at $P$, so that the zero cycle $(H \cdot X)
= n(P) + \sum_{i=1}^r (P_i)$ (where the $P_i \in X$ may not be distinct from each other
or from $P$, in general). Then $\{P, P_1, \ldots, P_r\}$ span a $\mathbb{P}^r \subset \mathbb{P}^n$.

**Proof:** Suppose that $\{P, P_1, \ldots, P_r\} \subset L \subset \mathbb{P}^n$, where $L$ is a linear
space of dimension $r - 1$. Since the space $\mathcal{L}$ of hyperplanes (in the dual
projective space) which contain $L$ is a $\mathbb{P}^{n-r}$, we have

$$h^0(\mathcal{O}_X(D - P - \sum_{i=1}^r P_i)) \geq n - r + 1$$

(where $\mathcal{O}_X(D) = \mathcal{O}_X(1)$)

Choosing the representative $n(P) + \sum_{i=1}^r (P_i) \in |D|$, we have

$$h^0(\mathcal{O}_X(n - 1)P)) \geq n - r + 1.$$

Now $\deg X = d$, and $\dim |D| = n$. Since $n > d/2$, the divisor $D$ is non-
special by Clifford’s Theorem (see [7], ch. IV). Hence by the Riemann–
Roch theorem, the genus $g$ of $X$ satisfies

$$g = \deg D - \dim |D| = d - n = r.$$

Since $n - 1 = g + (n - r - 1),

$$h^0(\mathcal{O}_X(n - 1)P)) \geq n - r + 1 \Rightarrow h^0(\mathcal{O}_X(gP)) \geq (n - r + 1) - (n - r - 1) = 2$$

i.e. $P \in X$ is a Weierstrass point.
**Lemma (2.2):** There is a non-empty open set $U \subset X$ with the following property – if $P \in U$, then there exist points $P_0, P_1, \ldots, P_{r-1}$ of $X$ such that (i) $P, P_0, \ldots, P_{r-1}$ span a $\mathbb{P}^r \subset \mathbb{P}^n$, and (ii) if $H$ is the osculating hyperplane to $P_0$, then $(H \cdot X) = n(P_0) + (P) + \sum_{i=1}^{r-1} P_i$.

**Proof:** The set $S$ in the dual projective space of hyperplanes which parametrizes osculating hyperplanes is birational to $X$. As $s$ ranges over $S$, the hyperplane sections $(H(s) \cdot X)$ have the form $(H(s) \cdot X) = n(P_0) + (P) + \sum_{i=1}^{r-1} P_i(s)$ (through the individual $P_i(s)$ don’t make sense, the zero cycle $\sum_{i=1}^{r-1} P_i(s)$ does). Then the lemma amounts to the claim that $\sum_{i=1}^{r-1} P_i(s)$ is not independent of $s$. Suppose that the lemma is false. Let $L \subset \mathbb{P}^n$ be the $\mathbb{P}^{r-1}$ spanned by $P_1, \ldots, P_r$ (for general $s \in S$, $P_1(s), \ldots, P_r(s)$ span a $\mathbb{P}^{r-1}$, by Lemma (2.1)). Projection from $L$ to a suitable $\mathbb{P}^{n-r}$ yields a curve $X \subset \mathbb{P}^{n-r}$ with the following property – if $P \in X$ is general, then there exists a hyperplane $H_P \subset \mathbb{P}^{n-r}$ such that the local intersection multiplicity $(H_P \cdot X)_P \geq n$ (choose $H_P$ to be the image of a suitable osculating hyperplane to $X$). But this is impossible – at a general point of a curve in $\mathbb{P}^{n-r}$, the maximum local intersection multiplicity with a hyperplane is $n - r$. This contradiction finishes the proof of the lemma.

We now prove theorem 2. Let $0 \in C(X)$ be the vertex, and $\phi : C(X) - (0) \to X$ the projection. Let $P \in \phi^{-1}(U)$, where $U \subset X$ is the open set of lemma (2.2). Then if $\bar{P} = \phi(P)$, we can find $P_0, \ldots, P_{r-1} \in X$ and a hyperplane $H$ such that $(H \cdot X) = n(P_0) + (P) + \sum_{i=1}^{r-1} (P_i)$. Then $\phi^{-1}(H) \cup \{0\} \cong \mathbb{A}^n \subset \mathbb{A}^{n+1}$ (by abuse of notation, let $\phi$ also denote the projection $\mathbb{A}^{n+1} - \{0\} \to \mathbb{P}^n$). Also, $(\phi^{-1}(H) \cup \{0\}) \cap C(X) = l_0 \cup l_1 \cup \ldots \cup l_{r-1} \cup \bar{T}$, where $l_i$ and $\bar{T}$ are lines through 0 in $\phi^{-1}(H) \cup \{0\}$. Since $P, P_0, \ldots, P_{r-1}$ can be chosen to span a $\mathbb{P}^r \subset \mathbb{P}^n$ by lemma (2.2), the lines $l_0, l_1, \ldots, l_{r-1}$ span an $\mathbb{A}^{r+1} \subset \phi^{-1}(H) - \{0\}$, and satisfy $\phi(l_i - \{0\}) = P_i$, $\phi(\bar{T} - \{0\}) = \bar{P}$, and $P \in \bar{T} - \{0\}$. The lines $l_i, l_1, \ldots, l_{r-1}$ occur with multiplicity 1 in the intersection $(\phi^{-1}(H) \cup \{0\}) \cap C(X)$, while $l_0$ occurs with multiplicity $n$.

There exists a unique linear subspace $L \cong \mathbb{A}^r$, with $L \subset \text{span} \{l_0, \ldots, l_{r-1}\}$, such that $P \in L$, and $L \cap \text{span} \{l_0, \ldots, l_{r-1}\} = \phi$. (This is just the unique $\mathbb{A}^r$ through $P$ which is parallel to span $\{l_0, \ldots, l_{r-1}\} \cong \mathbb{A}^r$). If $f = 0$ is the equation of $L$ in the affine space $\mathbb{A}^{r+1} \cong \text{span} \{l_0, \ldots, l_{r-1}\}$, then the restriction of $f$ to the curve $Y = (\phi^{-1}(H) \cup \{0\}) \cap C(X)$ is a regular function on $Y$ whose divisor of zeroes is $(P)$. Thus $(P) = 0$ in $\text{Pic}^0(Y)$, and hence in $A_0(C(X))$. By lemma (1.1), this proves the result.

We easily deduce theorem 2' from theorem 2.
Theorem 2': Let $X$ be a smooth curve of genus $g$ over an algebraically closed field of characteristic 0. Let $D$ be a divisor on $X$ such that $\deg D \geq 2g + 1$. (Thus $D$ is very ample – see [7], ch. IV). Assume that $X$ is projectively normal in this embedding. Then $A_0(A) = 0$, where $A = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))$.

Proof: Since $\deg D \geq 2g + 1$, $D$ is non-special. Hence by the Riemann-Roch theorem, $n = \dim |D| = \deg D - g$. We claim that $\deg D \geq 2n - 1$ (so that theorem 2 applies). For

$$2n - 1 - \deg D = 2(\deg D - g) - 1 - \deg D = \deg D - (2g + 1) \geq 0.$$  

Remark: In fact, a result of Castelnuovo implies that for the range of degrees in theorem 2', $X$ will always be projectively normal. See [12], p. 52.

§3) A counterexample in characteristic 0

In this section we construct examples of cones over projectively normal complex curves which admit non-trivial vector bundles. Let $L$ denote the field of algebraic numbers.

Theorem 3: Let $X \subset \mathbb{P}_L^n$ be a projectively normal curve such that $H^1(X, \mathcal{O}_X(1)) \neq 0$. Then $K_0(C(X_C)) \neq \mathbb{Z}$.

Corollary (3.1): Let $X$ be a non-hyperelliptic curve, defined over $L$. Then $K_0(A) \neq \mathbb{Z}$, where $A = \bigoplus_{n \geq 0} H^0(X_C, \mathcal{O}_{X_C}^*)$. (The cone over the canonical embedding.)

Corollary (3.2): Let $X \subset \mathbb{P}_L^2$ be a smooth curve of degree at least 4. Then $C(X_C)$ admits non-trivial vector bundles.

This is in contrast to the situation in characteristic $p > 0$, and to the situation for analytic vector bundles (since any analytic vector bundle on a contractible Stein space is trivial). The method of proof is based on an idea of Spencer Bloch. He showed that $\mathbb{C}[x, y, z]/(z^7 - x^2 - y^3)$ provides a counterexample to the statement of theorem 1 in characteristic 0. Let me sketch his idea.

Let $X = \text{spec } \mathbb{C}[x, y, z]/(z^7 - x^2 - y^3)$. Then the origin is the only sin-
gular point of $X$. Let $\tilde{X}$ be a projective surface containing $X$ as an open
subset, such that $\tilde{X}_{\text{sing}} = X_{\text{sing}} = \{0\}$, the origin. Let $\pi: \tilde{X} \to X$ be a re-
solution of the singularity. Then $\tilde{X}$ can be chosen so that $\pi^{-1}(\{0\})$ is a
cuspidal rational curve $E$. Now $K_0(\tilde{X}) = \mathbb{Z} \oplus \text{Pic}(\tilde{X})$, and $SK_1(\tilde{X}) \cong 
\cong \text{Pic}(\tilde{X}) \otimes \mathbb{C}^* \cong (\mathbb{C}^*)^\otimes n$ for some $n$, since $\tilde{X}$ is a rational surface. Also
$SK_1(E) \cong \Omega^1_{\mathcal{O}/\mathbb{C}}$, the module of Kahler differentials of $\mathbb{C}$ (see [7] for the
definition, some properties and references). Since $\mathbb{C}$ has uncountable
transcendence degree over $\mathbb{Q}$, and $\text{Hom}_{\mathbb{C}}(\Omega^1_{\mathcal{O}/\mathbb{C}}, \mathbb{C}) = (\text{vector space of all derivations } \mathbb{C} \to \mathbb{C})$, $\Omega^1_{\mathcal{O}/\mathbb{C}}$ is a $\mathbb{C}$-vector space of uncountable dimension.

Now one considers the diagram

$$
\begin{array}{c}
K_1(\tilde{X}) \xrightarrow{\alpha'} K_1(E) \xrightarrow{\beta} K_0(\tilde{X}, E) \to K_0(\tilde{X}) \to K_0(E) \\
\uparrow \quad \uparrow \quad \pi^* \quad \uparrow \quad \uparrow
\end{array}
$$

Here $K_0(\tilde{X}, E)$ and $K_0(\tilde{X}, P)$ are relative $K$-groups (we give the defini-
tions below). Clearly $\alpha'$ is onto, as $K_1(\{0\}) = \mathbb{C}^*$; hence $\beta' = 0$. It turns
out that points of $\tilde{X} - \{0\}$ admit cycle classes in $K_0(\tilde{X}, \{0\})$, and similarly
for $K_0(\tilde{X}, E)$. Define $F_0K_0(\tilde{X}, E)$ to be the subgroup of $K_0(\tilde{X}, E)$
gen-
erated by classes of points of $\tilde{X} - E$, and similarly define $F_0K_0(\tilde{X}, \{0\})$.
Evidently $\pi^*: F_0K_0(\tilde{X}, \{0\}) \to F_0K_0(\tilde{X}, E)$ as $\pi: \tilde{X} - E \cong \tilde{X} - \{0\}$. One
main ingredient of the proof is a geometric description of $\beta|_{SK_1(E)}$. A
class in $SK_1(E)$ is represented by finite sets of points of $E - E_{\text{sing}}$, to-
gether with non-zero elements of the residue fields of each of the points.
If $P_1, \ldots, P_r \in E - E_{\text{sing}}$, and $\alpha_1, \ldots, \alpha_r \in \mathbb{C}^*$ (where we think of $\alpha_i \in \mathbb{C}(P_i)^*$),
we choose a curve $C \subset \tilde{X}$ which meets $E$ transversally at $P_1, \ldots, P_r$. If $C$
meets $E$ at additional points $P_{r+1}, \ldots, P_s$, assume that these intersections
are also transverse and the points $P_i \in E$ are all smooth. Let $\alpha_i \in \mathbb{C}(P_i)^*$
be set equal to 1 for $r + 1 \leq i \leq s$. Choose a rational function $f \in \mathbb{C}(C)^*$,
such that $f(P_i) = \alpha_i (1 \leq i \leq s)$. Then the element $\beta(\{P_1, \alpha_1\}, \ldots, \{P_s, \alpha_s\})$
= (cycle class of the divisor of $f) \in F_0K_0(\tilde{X}, E)$. Once one has this, one
can show that $F_0K_0(\tilde{X}, E) \neq 0$, and hence $F_0K_0(\tilde{X}, \{0\}) \neq 0$, as desired.

In our case, we have to work harder, because $SK_1$ of the ambient
surface maps onto $SK_1$ of the exceptional set when we resolve the sin-
gularity of the cone. However, if we work with a multiple of the excep-
tional set, then obstructions to the triviality of vector bundles appear.

Let $X_L \subset \mathbb{P}^n_L$ be our given curve, and let $Y_L$ be the affine cone over $X_L$.
We will make use of the following convention – unless “$L$” appears as a
subscript on the symbol for a variety, we will be working over $\mathbb{C}$. Let $\tilde{Y}$
be the blow up of $Y$ at $P$. Then $\tilde{Y} \cong \mathbb{V}(\mathcal{O}_X(-1))$, a ruled surface over $X$, and
the exceptional set $\pi^{-1}(P) = E_0$ (where $P \in Y$ is the vertex) is a sec-
tion of $\tilde{Y} \to X$ with normal bundle $\cong \mathcal{O}_X(-1)$. Let $E$ be the subscheme
2E_0; thus if I is the sheaf of ideals of E_0 on \( \bar{Y} \), then E is defined by the sheaf \( I^2 \). We write “2P” for the scheme \( \text{spec } A/M^2 \), where \( Y = \text{spec } A \), and M is the maximal ideal of P.

For any scheme T, let \( \mathcal{P}(T) \) denote the category of locally free sheaves of finite rank; \( \mathcal{H}(T) \) will denote the category of coherent sheaves of finite homological dimension on T. If \( S \subset T \) is a subscheme, let \( \mathcal{H}(S, T) \) denote the category of \( \mathcal{O}_T \)-modules which are coherent, of finite homological dimension, and vanish on \( T - S \). If S is a single point \( x \in T \), we may write \( \mathcal{H}_x \) for \( \mathcal{H}(x, T) \).

Now we define the relative K-groups and cycle classes in them. Let \( i: Y \hookrightarrow X \) be a closed immersion. We have a natural map \( i^*: \mathcal{P}(X) \to \mathcal{P}(Y) \). For any exact category \( \mathcal{C} \), let \( BQ\mathcal{C} \) be the topological space (together with its natural base point) as defined by Quillen [9]. Then we have a natural map (of based spaces) \( BQ\mathcal{P}(X) \to BQ\mathcal{P}(Y) \).

Let \( F(i^*) \) denote the homotopy fiber of \( i^* \) (for a map \( (X, P) \to (Y, P') \) of based spaces, the homotopy fiber is the set of pairs \( (\omega, x) \) where \( x \in X \), \( \omega: [0, 1] \to Y \) is a path, with \( \omega(0) = P' \), \( \omega(1) = f(x) \). The base point is \( (\omega_0, P) \) where \( \omega_0: [0, 1] \to P' \). One of the basic properties of the homotopy fiber is that its homotopy groups fit into a long exact sequence with those of the domain and range. So if we set \( K_n(X, Y) = \pi_{n+1}(F(i^*), *) \), where \( * \in F(i^*) \) is the base point, then we have a long exact sequence

\[
\ldots \to K_n(X, Y) \to K_n(X) \to K_n(Y) \to K_{n-1}(X, Y) \to \ldots
\]

A general reference for the definitions and basic properties of higher K-groups is the fundamental paper [9] of Quillen. A summary of Quillen’s results, and some applications to questions in the theory of algebraic cycles, can be found in Bloch’s lecture notes [13].

Let \( Z \subset X - Y \) be a subscheme, closed in \( X \), and of finite homological dimension. Then we claim that there is a natural cycle class \( [Z] \in K_0(X, Y) \). To construct it, we use the category \( \mathcal{H}_0(X) \subset \mathcal{H}(X) \), defined to be the full subcategory consisting of all coherent \( \mathcal{O}_X \)-modules \( F \) satisfying \( \text{Tor}^i_F(\mathcal{F}, \mathcal{O}_Y) = 0 \) for \( i > 0 \). Then the map \( i: Y \hookrightarrow X \) induces a functor \( i^*: \mathcal{H}_0(X) \to \mathcal{H}(Y) \). By Quillen’s resolution theorem [9], the maps \( BQ\mathcal{P}(X) \to BQ\mathcal{H}_0(X) \) and \( BQ\mathcal{P}(Y) \to BQ\mathcal{H}(Y) \) are homotopy equivalences. Hence the natural induced map between the homotopy fibers \( F(i^*) \) and \( F(i^*) \) is also a homotopy equivalence. The inclusion \( j: Z \hookrightarrow X \) induces a functor \( j_*: \mathcal{P}(Z) \to \mathcal{H}_0(X) \), since \( Z \cap Y = \emptyset \); and the composite functor \( i^* \circ j_*: \mathcal{P}(Z) \to \mathcal{H}(Y) \) is the 0-functor. Hence the induced map \( BQ\mathcal{P}(Z) \to BQ\mathcal{H}(Y) \) maps everything to the base point.
Hence we have an induced natural map $BQ\mathcal{P}(Z) \rightarrow F(\tilde{f}^*)$, and thus a map $K_0(Z) \rightarrow K_0(X, Y)$. The image of $[\mathcal{O}_Z] \in K_0(Z)$ under this map is the required cycle class; by construction, it maps to the usual cycle class in $K_0(X)$ under the natural map $K_0(X, Y) \rightarrow K_0(X)$.

The relative $K$-groups, and the cycle classes, enjoy the following naturality properties. If $i: Y \hookrightarrow X$ and $i': Y' \hookrightarrow X'$, and $\pi: X \rightarrow X'$ is a morphism such that $\pi^{-1}(Y') = Y$, then we have a diagram

$$
\begin{array}{cccccc}
\vdots & \rightarrow & K_\pi(X, Y) & \rightarrow & K_\pi(X) & \rightarrow & K_{\pi-1}(X, Y) & \rightarrow & \vdots \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\vdots & \rightarrow & K_\pi(X', Y') & \rightarrow & K_\pi(X') & \rightarrow & K_{\pi-1}(X', Y') & \rightarrow & \vdots
\end{array}
$$

Further, let $Z' \subset X'$ be a subscheme of finite homological dimension, satisfying $Z' \cap Y' = \emptyset$, and $\text{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_{Z'}, \mathcal{O}_X) = 0$ for $i > 0$. Let $\mathcal{H}_1(X')$ denote the category of coherent $\mathcal{O}_{X'}$-modules $\mathcal{F}$ of finite homological dimension satisfying $\text{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_Y) = \text{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X) = \text{Tor}_i^{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_Y) = 0$ for $i > 0$. Then consider the commutative square of categories (where $Z = \pi^{-1}(Z')$)

$$
\begin{array}{ccc}
\mathcal{P}(Z) & \rightarrow & \mathcal{H}_0(X) \\
\pi^* \uparrow & & \uparrow \pi^* \\
\mathcal{P}(Z') & \rightarrow & \mathcal{H}_1(X')
\end{array}
$$

This gives the equation $\pi^*([Z']) = [Z]$ in $K_0(X, Y)$. Two cases where the hypothesis are satisfied are when $X - Y \hookrightarrow X' - Y'$ is an open immersion, and when the map $\pi$ is flat. In our applications, we only work with $K_0(X, Y)$ where $X - Y$ is smooth.

There is one technical point that we systematically ignore. When we say that a diagram of categories

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
H \downarrow & & \downarrow G \\
C & \xrightarrow{K} & D
\end{array}
\]

commutes, what we often mean is that the functors $G \circ F$ and $K \circ H$ are naturally equivalent. Thus, the corresponding diagram of classifying spaces

\[
\begin{array}{ccc}
B \mathcal{A} & \xrightarrow{B(F)} & B \mathcal{B} \\
B(H) \downarrow & & \downarrow B(G) \\
B \mathcal{C} & \xrightarrow{B(K)} & B \mathcal{D}
\end{array}
\]
only commutes up to homotopy. The induced map on homotopy fibers $F(B(F)) \to F(B(K))$ depends on the choice of this homotopy i.e. the choice of the equivalence of functors $G \circ F \cong K \circ H$. However, in all our situations, there is always one “natural” choice of the equivalence – for example, there is an obvious choice of a natural isomorphism $(M \otimes_A N) \otimes_B P \cong M \otimes_A (N \otimes_B P)$; this is the kind of choice which has to be made consistently. More details will appear in my thesis. I wish to thank Professor Swan for pointing out that some care is needed here.

We need to make use of certain results from $K$-theory. We give them in a sequence of lemmas. Recall that $L$ denotes the field of algebraic numbers.

**Lemma (3.2) (Van der Kallen [14]):** Let $\mathcal{O}$ be a regular local ring containing $L$, and let $\mathcal{O}[[t]]/(t^2)$ be the ring of dual numbers over $\mathcal{O}$. Then $K_2(\mathcal{O}[[t]]/(t^2))$ fits into the exact sequence

$$0 \to \Omega^1_{\mathcal{O}/L} \to K_2(\mathcal{O}[[t]]/(t^2)) \to K_2(\mathcal{O}) \to 0.$$ 

The isomorphism $\ker(K_2(\mathcal{O}[[t]]/(t^2)) \to K_2(\mathcal{O})) \cong \Omega_{\mathcal{O}/L}^1$ is given as follows: the kernel is generated by symbols of the form \{u, 1 + vt\} where $u \in \mathcal{O}^*$, $v \in \mathcal{O}$, and

$$\{u, 1 + vt\} \to v \cdot \frac{du}{u} \in \Omega^1_{\mathcal{O}/L}.$$

(Note that $\Omega^1_{\mathcal{O}/L} = \Omega^1_{\mathcal{O}/\mathbb{Z}}$, since $L/\mathbb{Q}$ is separable algebraic.)

From now on, all differentials will be relative to $L$ unless indicated otherwise.

**Lemma (3.3) (Localisation sequence [11]):** Let $U \to X$ be an open immersion, where $U$ is affine, and $X - U$ is defined by an ideal sheaf which is locally principal and generated by a non zero-divisor. Let $H$ be the category of coherent $\mathcal{O}_X$-modules which are 0 on $U$ and have homological dimension 1 on $X$. Then we have a localisation sequence

$$\ldots \to K_{q+1}(U) \to K_q(H) \to K_q(X) \to K_q(U) \to \ldots.$$

Now we come back to cones. Recall that $E \subset \bar{Y}$ is the non-reduced scheme “$2E_0$” where $E_0$ is the exceptional set. For any finite subscheme $S \subset E$, the localisation sequence gives

$$\ldots \to K_2(E) \to K_2(E - S) \to K_1(\mathcal{M}(S, E)) \to K_1(E) \to K_1(E - S) \to \ldots$$
Taking limits over all such $S$ (see Quillen [9], p. 96) we get

$$\ldots \to K_2(E) \to K_2(F) \to \bigoplus_{x \in E} K_1(\mathcal{H}_x) \to K_1(E) \to K_1(F) \to \ldots$$

where $F$ is the local ring at the generic point of $E$. Define $SK_1(E) = \ker(K_1(E) \to K_1(F))$. Then we have a presentation

$$K_2(F) \to \bigoplus_{x \in E} K_1(\mathcal{H}_x) \to SK_1(E) \to 0.$$ 

Since it is difficult to work with $\mathcal{H}_x$, we wish to obtain another viewpoint on $SK_1(E)$. To do this, replace $E$ by $\mathcal{O}_{x,E}$, for any closed point $x \in E$, in the above argument. We obtain an exact sequence

$$K_2(\mathcal{O}_{x,E}) \to K_2(F) \to K_1(\mathcal{H}_x) \to 0,$$

because $K_1(\mathcal{O}_{x,E}) \subset K_1(F)$ (since $K_1(\text{local ring}) = \text{units}$). Now let $x \in E_0$ be a smooth closed point; since any infinitesimal deformation of the regular local ring $\mathcal{O}_{x,E_0}$ is trivial (as $\mathcal{O}_{x,E_0}$ is essentially of finite type over $\mathbb{C}$), we see that $\mathcal{O}_{x,E} \cong \mathcal{O}_{x,E_0}[t]/(t^2)$. Hence lemma (3.2) applies to give a diagram

$$0 \to \Omega^1_{\mathcal{O}_{x,E_0}} \to K_2(\mathcal{O}_{x,E}) \to K_2(\mathcal{O}_{x,E_0}) \to 0$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$0 \to \Omega^1_{\mathcal{C}(E_0)} \to K_2(F) \to K_2(\mathcal{C}(E_0)) \to 0$$

By a result of Dennis and Stein [10], $K_2(\mathcal{O}_{x,E_0}) \subset K_2(\mathcal{C}(E_0))$. Also, $\Omega^1_{\mathcal{O}_{x,E_0}} \subset \Omega^1_{\mathcal{C}(E_0)}$, since $\Omega^1_{\mathcal{O}_{x,E_0}}$ is a free $\mathcal{O}_{x,E_0}$-module, and the inclusion is just the localisation at the generic point. Hence $K_2(\mathcal{O}_{x,E}) \subset K_2(F)$.

Let $\eta \in E$ be the generic point; for any point $x \in E$ let $i_x: \{x\} \subset E$. Then we have constructed an exact sequence of sheaves (for the Zariski topology)

$$0 \to \mathcal{K}_{2,E} \to (i_\eta)_* K_2(F) \to \bigoplus_{x \in E} (i_x)_* K_1(\mathcal{H}_x) \to 0,$$

where $\mathcal{K}_{2,E}$ is the sheaf associated to the presheaf $U \mapsto K_2(\Gamma(U, \mathcal{O}_E))$. Here $K_2(F), K_1(\mathcal{H}_x)$ are regarded as constant sheaves supported on a subvariety.

Since $(i_\eta)_* K_2(F), (i_x)_* K_1(\mathcal{H}_x)$ are flasque, they have no higher cohomology, and so we can use the above resolution of $\mathcal{K}_{2,E}$ to compute its
cohomology. Hence, we obtain an isomorphism $SK_1(E) \cong H^1(E, \mathcal{K}_2, E)$, since both are presented as $\text{coker}(K_2(F) \to \bigoplus_{x \in E} K_1(\mathcal{K}_x))$.

Next, we go back to the identification of $\mathcal{O}_{x,E}$ with $\mathcal{O}_{x,E_0}[t]/(t^2)$. The map $\text{Ker}(K_2(\mathcal{O}_{x,E}) \to K_2(\mathcal{O}_{x,E_0})) \to \Omega^1_{\mathcal{O}_{x,E_0}}$ was given by

$$\{u, 1 + vt\} \mapsto v \frac{du}{u}.$$

This is not quite canonical, as it involves the choice of $t$ generating $\text{Ker}(\mathcal{O}_{x,E} \to \mathcal{O}_{x,E_0})$. However, $v \frac{du}{u} \otimes t \in \Omega^1_{\mathcal{O}_{x,E_0}} \otimes \mathcal{O}_{E_0}/I/I^2$ is clearly canonical. Thus we obtain an exact sequence of sheaves

$$0 \to I/I^2 \otimes \mathcal{O}_{E_0} \Omega^1_{\mathcal{O}_{E_0}} \to \mathcal{K}_2, E \to \mathcal{K}_2, E_0 \to 0.$$

In fact, this exact sequence splits naturally, using the fibration $\tilde{Y} \to X$ together with the isomorphism $X \cong E_0$ to split the inclusion $E_0 \subset E$. Hence, we have a naturally split exact sequence

$$0 \to H^1(E_0, \Omega^1_{\mathcal{O}_{E_0}} \otimes \mathcal{O}_{E_0}/I/I^2) \to SK_1(E) \to SK_1(E_0) \to 0.$$

Now $E_0 \cong X_L \times_L \mathbb{C}$. Hence $\Omega^1_{E_0} \cong (\Omega^1_{X_L} \otimes_L \mathbb{C}) \oplus (\mathcal{O}_{X_L} \otimes_L \Omega^1_L)$. This gives a corresponding splitting of $\Omega^1_{E_0} \otimes \mathcal{O}_{E_0}/I/I^2$. Since $\Omega^1_{X_L} \otimes_L \mathbb{C} = \Omega^1_{X/L}$, and $I/I^2 \cong \mathcal{O}_X(1)$, we have

$$H^1(E_0, (\Omega^1_{X_L} \otimes_L \mathbb{C}) \otimes \mathcal{O}_{E_0}/I/I^2) = H^1(X, \mathcal{O}_X(1) \otimes \Omega^1_{X/L}) = 0$$

by Serre duality.

Again using $E_0 \cong X_L \times \mathbb{C}$, and the Künneth formula, the other direct summand reduces to $H^1(X_L, \mathcal{O}_{X_L}(1)) \otimes \Omega^1_L$.

**Lemma (3.4):** $K_a(\tilde{Y}) \cong K_a(X)$, and the natural maps $K_a(\tilde{Y}) \to K_a(E)$ are injective. Further, $\text{coker}(K_a(\tilde{Y}) \to K_a(E)) \cong \text{Ker}(K_a(E) \to K_a(E_0))$.

**Proof:** Since $\tilde{Y} \to X$ is an $\mathbb{A}^1$-bundle, the first claim follows from [9], sec. 7, prop. (4.1). The remaining claims just exploit the fact that in $E_0 \subset E \subset \tilde{Y} \to X$, the composite $E_0 \to X$ is an isomorphism.

In particular, $H^1(X_L, \mathcal{O}_{X_L}(1)) \otimes \Omega^1_L \subset K_0(\tilde{Y}, E)$. The next task is to imitate the geometric construction of the boundary map used by Spencer Bloch to show that at least some of these elements land in $F_0K_0(\tilde{Y}, E)$.
From the diagram

\[
\begin{align*}
K_1(\overline{Y}) & \longrightarrow K_1(E) \overset{\delta}{\longrightarrow} K_0(\overline{Y}, E) \longrightarrow K_0(\overline{Y}) \\
\uparrow \quad \pi^\ast \uparrow & \quad \pi^\ast \uparrow \quad \uparrow \\
K_1(Y) & \longrightarrow K_1(2P) \overset{\psi}{\longrightarrow} K_0(Y, 2P) \longrightarrow K_0(Y)
\end{align*}
\]

and the fact that \( SK_1(2P) = 0 \), we claim that if \( \alpha \in H^1(X_L, \mathcal{O}_{X_L}(1)) \otimes \Omega^1_L \) is non-zero, and \( \partial \alpha = \pi^\ast \delta \), then \( \phi(\delta) \in K_0(Y) \) is also non-zero. For suppose \( \delta = \psi(\gamma) \). Then \( \pi^\ast(\gamma) = \alpha \in \text{Image}(K_1(\overline{Y}) \rightarrow K_1(E)) \), which maps isomorphically to \( K_1(E_0) \). But, by changing \( \gamma \) by an element of \( K_1(Y) \) (in fact, an element of \( \mathbb{C}^\ast \)) we may assume \( \gamma \mapsto 0 \) in \( K_1(P) \). Clearly \( \pi^\ast(\gamma) \mapsto 0 \) in \( K_1(E_0) \), from

\[
\begin{align*}
K_1(E) & \rightarrow K_1(E_0) \\
\uparrow & \uparrow \\
K_1(2P) & \rightarrow K_1(P)
\end{align*}
\]

Since \( \alpha \mapsto 0 \) in \( K_1(E_0) \), \( \pi^\ast(\gamma) \mapsto 0 \) in \( K_1(E_0) \). But this forces \( \pi^\ast(\gamma) = \alpha = 0 \) i.e. \( \pi^\ast(\gamma) = \alpha \). Since \( K_1(2P) \overset{\pi^\ast}{\longrightarrow} K_1(E) \rightarrow K_1(F) \) is injective (use the grading) while \( \alpha \in SK_1(E) \), this forces \( \alpha = 0 \).

Now let \( C \subset \overline{Y} \) be a smooth (possibly disconnected) affine closed curve. Then we claim there is a map between the sequence of \((C, C \cap E)\) and \((\overline{Y}, E)\). Let \( j : C \hookrightarrow \overline{Y}, j' : C \cap E \hookrightarrow E \). Then we have a diagram

\[
\begin{align*}
\mathcal{H}_0(\overline{Y}) & \rightarrow \mathcal{H}(E) \\
\downarrow j^\ast & \quad \uparrow j^\ast \\
\mathcal{H}(C) & \rightarrow \mathcal{H}(C \cap E)
\end{align*}
\]

(\( \mathcal{H}_0 \) was defined when we introduced cycle classes).

This induces the maps between the sequences.

**Lemma (3.5):** Let \( C \) be a smooth affine curve, \( S \subset C \) a finite subscheme, \( \mathcal{O}_{S, C} \) the semilocal ring of \( S \) on \( C \). Then there is a commutative diagram (upto sign)

\[
\begin{align*}
\mathcal{O}_{S, C} & \overset{\partial}{\longrightarrow} K_1(S) \\
\mathcal{O}_{S, C} & \overset{\varepsilon}{\longrightarrow} K_0(C, S)
\end{align*}
\]
where $\partial : K_1(S) \to K_0(C, S)$ is the boundary map of the pair $(C, S)$, $\varepsilon : O^*_S.C \to K_1(S)$ is the natural map on units, and $\eta$ sends $f \in O^*_S.C$ to the cycle class of the divisor $(f)$ of $f$ on $C$.

**Proof:** It clearly suffices to check that $\partial \circ \varepsilon(f) = \eta(f)$ for all $f \in \text{Image}(O_C \to O_S^*, C) \cap O^*_S.C$. Such an $f$ can be regarded as a morphism $C \to \mathbb{A}^1$, and $[(f)] \in K_0(C, S)$ is just $f^*([0])$, where $[0] \in K_0(\mathbb{A}^1, f(S))$, and $f(S) \subset \mathbb{A}^1 - \{0\}$. So we are reduced to checking the claim in the case when $C \cong \mathbb{A}^1$, $S \subset \mathbb{A}^1 - \{0\}$, and $f = t$, the standard function on $\mathbb{A}^1$. The image of $t$ in $K_1(S)$ is a unit. If $\mathbb{A}^1 = \text{Spec } k[t]$, $S = \text{Spec } k[t]/I$, then we have a diagram of rings

$$
\begin{align*}
k[t] &\to k[t, t^{-1}] \\
\downarrow &\downarrow \\
k[t] &\to k[t]/I
\end{align*}
$$

This induces a map between the localisation sequence for $(k[t], k[t, t^{-1}])$ and the exact sequence of the pair $(\mathbb{A}^1, S)$. In terms of categories, we have a diagram

$$
\begin{align*}
\mathcal{H}_0(\mathbb{A}^1) &\to \mathcal{H}_0(\mathbb{G}_m) \\
\downarrow &\downarrow \\
\mathcal{H}_0(\mathbb{A}^1) &\to \mathcal{H}(S)
\end{align*}
$$

Hence we have a diagram of spaces

$$
\begin{align*}
BQ\mathcal{P}({0}) &\to BQ\mathcal{H}_0(\mathbb{A}^1) \to BQ\mathcal{H}_0(\mathbb{G}_m) \\
\downarrow &\downarrow \downarrow \\
F(i^*) &\to BQ\mathcal{H}_0(\mathbb{A}^1) \to BQ\mathcal{H}(S)
\end{align*}
$$

since the homotopy fiber in the localisation sequence is known to be homotopy equivalent to $BQ\mathcal{P}({0})$. The induced map $BQ\mathcal{P}({0}) \to BQ\mathcal{H}_0(\mathbb{A}^1)$ is the one which was used to define the cycle class of $[0]$ in $K_0(\mathbb{A}^1, S)$. So the lemma will follow if we can show that for $t \in K_1(k[t, t^{-1}])$, $\partial(t) = \pm [0] \in K_0({0})$ in the localisation sequence. This is proved in Quillen [9].

We need one more lemma. Let $C \subset \mathbb{P}$ be as before, and let $\Pi \in O_{C \cap E, Y}$ be a local generator for the ideal sheaf of $C$ on $\mathbb{P}$. Then we have a diagram of localisation sequences

$$
\begin{align*}
K_2(O_{C \cap E, E}) &\to K_2(F) \to K_1(\mathcal{H}(C \cap E, E)) \to 0 \\
\uparrow &\uparrow \uparrow \\
K_2(O_{C \cap E, Y}) &\to K_2(O_{C \cap E, Y}[\Pi^{-1}]) \to K_1(O_{C \cap E, C}) \to 0
\end{align*}
$$
Here $\mathcal{H}_0(\mathcal{O}_{C \cap E,Y})$ is the category of coherent $\mathcal{O}_{C \cap E,Y}$ modules $M$ satisfying $\text{Tor}_i(M, \mathcal{O}_{C \cap E,E}) = 0$ for $i > 0$, and similarly for $\mathcal{O}_{C \cap E,Y}[\Pi^{-1}]$. Note that both rings are regular. There is another map

$$\beta : K_1(\mathcal{O}_{C \cap E,C}) = \mathcal{O}_{C \cap E}^* \xrightarrow{\epsilon} K_1(C \cap E) \xrightarrow{j_*} K_1(\mathcal{H})$$

(where $\mathcal{H} = \mathcal{H}(C \cap E, E)$).

**Lemma (3.6):** $\alpha = \beta$.

**Proof:** $\mathcal{P}(\mathcal{O}_{C \cap E,C}) \to \mathcal{H}(C \cap E, E)$ factors through the full subcategory $\mathcal{P}(C \cap E) \subset \mathcal{H}(C \cap E, E)$.

The point of Lemma (3.6) is to use symbols for calculations, to avoid dealing with $\mathcal{H}(C \cap E, E)$. Lemmas (3.5) and (3.6) give the geometric description of the boundary map $K_1(E) \to K_0(\mathcal{Y}, E)$, since we know that $K_2(\mathcal{O}_{C \cap E,Y}[\Pi^{-1}]) \to K_1(\mathcal{O}_{C \cap E,C})$ is the tame symbol (see Quillen [9]).

Finally we are ready to prove the theorem. Let $x_0, \ldots, x_n$ be homogeneous coordinates on $\mathbb{P}^n_C$. Let $a_0, \ldots, a_n \in C$ be algebraically independent over $L$, and let $H = a_0 + a_1 x_1/x_0 + \ldots + a_n x_n/x_0$ be a rational function on $\mathcal{Y}$. The divisor of zeros of $H$ consists of a union of fibres of the map $p : \mathcal{Y} \to X$; indeed, if $H \subset \mathbb{P}^n_C$ is the hyperplane $a_0 x_0 + \ldots + a_n x_n = 0$, the divisor is just $p^{-1}(H \cap X)$. Let $D$ be a derivation of $C$ extending $\partial/\partial a_0$ of $L[a_0, \ldots, a_n]$. Then $(dH, D) = 1$, where we regard $D$ as a derivation on functions on $\mathcal{Y}$ which is 0 on functions defined over $L$. Now the homogeneous coord-$x_0$ may be regarded as a regular function on $\mathcal{Y}$ in the ideal of $E_0$, which generates that ideal at points of $X - \{x_0 = 0\}$ (under the identification of $X$ with $E_0$). Since $(\Pi = 0) \cap E_0$ consists of points not defined over $L$, $x_0$ generates $I$ (and hence $I/I^2$) at these points.

Let $\phi$ be a rational function on $\mathcal{Y}$ which is regular at the points $\{t_1, \ldots, t_p\} = (x_0 = 0) \cap E_C$. Then $S = \phi(dH/\Pi) \otimes x_0$ (where $\phi, \Pi$ are restricted to $E_0$) represents an element of $H^1(E_0, \mathcal{O}_{E_0}^1 \otimes I/I^2) \subset SK_1(E)$, whose boundary is a relative 0-cycle. Hence Theorem 3 is proved if this
class is nonzero. Clearly it is enough to show that \((S, D) \in H^1(E_0, \mathcal{I}/\mathcal{I}^2)\) is nonzero (where \((S, D)\) denotes contraction with the derivation \(D\)). We will do this using Serre Duality, in its formulation in terms of residues (see [7]). Let \(\omega \in H^0(E_0, \mathcal{O}_{E_0}(-1) \otimes \mathcal{O}_{E_0})\), the dual vector space to \(H^1(E_0, \mathcal{I}/\mathcal{I}^2)\); assume \(\omega\) is defined over \(L\). Then \(\omega\) is nonzero at \(t_1, \ldots, t_p\), and \(x_0\omega/\Pi\) has a simple pole with nonzero residue at each \(t_i\). On the affine curve \(\mathcal{I} = 0\), we can find a regular function \(\phi\) with prescribed values at each \(t_i\). Thus, by properly choosing \(\phi\), we can arrange that \(\sum_{i=1}^p \text{res}_{t_i}[(\phi/\Pi) \otimes x_0\omega]\) is nonzero. This finishes the proof.

**REMARKS:**

1) The proof in fact shows that \(K_0(Y)\) is uncountably generated, since there are uncountably many mutually algebraically independent choices of numbers \(a_0, \ldots, a_n\).

2) Since derivations of the form \(\partial/\partial a\) (with a running through a transcendence base for \(\mathbb{C}\)) span the dual of \(\Omega^1_{\mathcal{O}_X}\), at least if we allow infinite linear combinations, one can show that \(\text{image}(SK_1(E) - K_0(\mathcal{I}, E)) = F_0K_0(\mathcal{I}, E)\). Hence \(K_0(\mathcal{I}, E)\) is generated by algebraic cycles. This is no longer clear if the curve \(X\) is not defined over a number field, since the vector space \(\Omega^1_X\) may play some role, where \(k\) is a field of definition of \(X\). However, theorem 3 is still valid; in the final step of the proof, choose \(a_0, \ldots, a_n\) to be algebraically independent over \(k\).

**REFERENCES**


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