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## ELLIPTIC FIBRES ON ENRIQUES SURFACES

G. Angermüller and W. Barth

### 1. Introduction

It is known [5] that an Enriques surface  $X$  admits holomorphic fibrations over  $\mathbb{P}_1$ , where the general fibre is a smooth elliptic curve. The aim of this note is to classify the singular fibres. Since  $X$  is algebraic and  $b_2(X) = 10$ , such a fibre can have at most 9 irreducible components. In Kodaira's notation [6] (see also section 2 below) the following types of (non-multiple) singular elliptic fibres have 9 components or less:

$$I_b, b \leq 9,$$

$$I_b^*, b \leq 4,$$

$$II, II^*, III, III^*, IV, IV^*.$$

For each of these types we give an example of an Enriques surface and an elliptic fibration, in which it appears as singular fibre. So we prove the

**THEOREM:** *For each of the types in the list above, there is an Enriques surface admitting an elliptic fibration with a singular fibre of this type.*

The Enriques surfaces are found as follows [5]: If  $B \subset \mathbb{P}_1 \times \mathbb{P}_1$  is a (reduced) curve of bidegree (4, 4), there is a double cover  $Y \rightarrow \mathbb{P}_1 \times \mathbb{P}_1$  ramified over  $B$ . If  $B$  has at worst simple singularities, the minimal desingularisation  $\bar{Y}$  of  $Y$  is a  $K3$ -surface. If the polynomial defining  $B$  is invariant under an involution  $\tau$  of  $\mathbb{P}_1 \times \mathbb{P}_1$  with 4 fixed points (not on  $B$ ), this  $\tau$  lifts to an involution  $\bar{\sigma}$  of  $\bar{Y}$  without fixed points, and  $X = \bar{Y}/\bar{\sigma}$  is an Enriques surface. Elliptic fibrations of  $X$  are induced by the projections of  $\mathbb{P}_1 \times \mathbb{P}_1$  on its factors and singular fibres on  $X$  are created by singularities of the branch curve  $B$  or special positions of this curve with respect to the projections.

It should be mentioned that an example of a fibre of type  $II^*$  was already given by Horikawa [5]. But a catalog of all possible fibres seems not to exist so far.


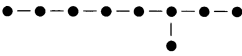
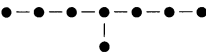
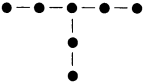
Our method very easily produces examples for all types of fibres, except for the types  $I_5$  and  $I_7$ , where some computations are necessary, and  $I_9$ , where the computations are a little cumbersome. The basefield always is  $\mathbb{C}$ , the field of complex numbers.

The second author wants to thank C. Peters and A. van de Ven, who gave him an introduction to the theory of Enriques surfaces.

### 2. Kodaira's table of singular fibres

First we recall Kodaira's classification [6] of singular fibres in elliptic fibrations, which are not multiple fibres. Since we have to compare these fibres with simple singularities of the branch curve we prefer to use the A–D–E notation instead of Kodaira's one.

Table 1.

Kodaira's notation	Description, resp. dual graph	Notation here
$I_0$	nonsingular elliptic	–
$I_1$	irreducible rational with node	$\tilde{A}_0$
$I_b, b \geq 2$	cycle of $b$ nonsingular rational curves	$\tilde{A}_{b-1}$
$I_b^*, b \geq 0$	 $(b + 5 \text{ curves})$	$\tilde{D}_{b+4}$
II	irreducible rational with cusp	$A'_0$
II*	 $(9 \text{ curves})$	$\tilde{E}_8$
III	two smooth rational curves touching	$A'_1$
III*	 $(8 \text{ curves})$	$\tilde{E}_7$
IV	three smooth rational curves meeting in one point	$A'_2$
IV*	 $(7 \text{ curves})$	$\tilde{E}_6$

An Enriques surface  $X$  is algebraic and  $b_2(X) = 10$ . By Zariski's lemma each curve  $C$  properly contained in a fibre  $X_s$  has selfintersection  $C^2 < 0$ . So the components of  $X_s$  span in  $H^2(X, \mathbb{R})$  a subspace of dimension  $b_2(X_s)$ . Since there also is the class of a hyperplane section, for any fibre  $X_s$  on an Enriques surface,  $b_2(X_s) \leq 9$ . If  $X_s$  is of type  $\tilde{A}_n, \dots, \tilde{E}_8$ , the

subscript is one less than  $b_2(X_s)$ . So only the fibres  $\tilde{A}_n(n \leq 8)$ ,  $A'_0, A'_1, A'_2$ ,  $\tilde{D}_n(n \leq 8)$ ,  $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$  can appear as such fibres.

### 3. Enriques surfaces and double coverings of $\mathbb{P}_1 \times \mathbb{P}_1$

Here we recall the representation of (so-called) non-special Enriques surfaces in terms of double coverings of the quadric  $\mathbb{P}_1 \times \mathbb{P}_1$  [5].

Let  $((u_0 : u_1), (v_0 : v_1))$  be bi-homogeneous coordinates on  $\mathbb{P}_1 \times \mathbb{P}_1$ . A curve  $B \subset \mathbb{P}_1 \times \mathbb{P}_1$  of bidegree  $(4, 4)$  is given by an equation

$$f(u, v) = \sum_{\substack{i_0+i_1=4 \\ j_0+j_1=4}} c_{i_0 i_1 j_0 j_1} u_0^{i_0} u_1^{i_1} v_0^{j_0} v_1^{j_1} = 0.$$

There is a unique normal surface  $Y$  admitting a double covering  $\pi : Y \rightarrow \mathbb{P}_1 \times \mathbb{P}_1$  ramified precisely over  $B$ . If  $B$  is nonsingular, we have

$$K_Y = 0$$

by the adjunction formula, and for  $e(Y) = \sum_0^4 (-1)^i b_i(Y)$  we find

$$\begin{aligned} e(Y) &= 2e(\mathbb{P}_1 \times \mathbb{P}_1) - e(B) = \\ &= 2 \cdot 4 - (2 - 2g(B)) = 24, \end{aligned}$$

because  $g(B) = 9$ . So  $Y$  is a  $K3$ -surface by classification of surfaces. If  $B$  has at most simple singularities, Brieskorn's simultaneous resolution [2] shows that the minimal desingularisation  $\bar{Y}$  of  $Y$  is a deformation of a  $K3$ -surface, hence a  $K3$ -surface itself. Let  $\bar{\pi} : \bar{Y} \rightarrow \mathbb{P}_1 \times \mathbb{P}_1$  be the map induced by  $\pi$ .

Now denote by  $\tau$  the involution

$$((u_0 : u_1), (v_0 : v_1)) \mapsto ((u_1 : u_0), (v_1 : v_0))$$

of  $\mathbb{P}_1 \times \mathbb{P}_1$  and assume that  $f(u, v)$  is invariant under  $\tau$ . Since  $Y$  can be described in terms of  $f$ , the involution  $\tau$  lifts to  $Y$ , such that the points  $y \in Y$  lying over the four fixed points

$$((1 : \pm 1), (1 : \pm 1))$$

of  $\tau$  are invariant. If  $B$  does not pass through any of these fixed points, we may compose the lifted involution on  $Y$  with the involution interchanging the two sheets to obtain an involution  $\sigma : Y \rightarrow Y$  without fixed

points.  $\sigma$  lifts to an involution  $\bar{\sigma}: \bar{Y} \rightarrow \bar{Y}$ , again without fixed points. So  $\bar{Y}/\bar{\sigma} = X$  is an Enriques surface.

Fix one of the projections  $\mathbb{P}_1 \times \mathbb{P}_1$ , say

$$p_1: ((u_0:u_1), (v_0:v_1)) \mapsto (u_0:u_1).$$

Then  $p = p_1 \circ \bar{\pi}: \bar{Y} \rightarrow \mathbb{P}_1$  is a fibration of  $\bar{Y}$  over  $\mathbb{P}_1$ . If  $B$  is reduced, the general fibre  $\bar{Y}_{(u_0:u_1)}$  is a double covering of the line

$$L_{(u_0:u_1)} = p_1^{-1}(u_0:u_1)$$

ramified in the four points of  $L_{(u_0:u_1)} \cap B$ , hence an elliptic curve. The fibration  $p$  commutes with  $\bar{\sigma}$ , hence induces an elliptic fibration  $X \rightarrow \mathbb{P}_1$ . For  $(u_0:u_1) \neq (1:\pm 1)$ , the fibre  $\bar{Y}_{(u_0:u_1)}$  is isomorphic with its image in  $X$ . The two fibres  $\bar{Y}_{(1:\pm 1)}$  however are invariant under  $\bar{\sigma}$ , so they are mapped 2:1 on two 2-fold fibres of  $X$ , which are called the *halfpencils*.

To classify the singular fibres of  $X$ , it suffices therefore to classify the singular fibres of  $\bar{Y}$ .

#### 4. Singular fibres and simple singularities of the branch curve $B$

Here we consider the situation locally above a line on the quadric  $\mathbb{P}_1 \times \mathbb{P}_1$ . So denote by  $L \subset \mathbb{P}_1 \times \mathbb{C}$  the line  $\mathbb{P}_1 \times \{0\}$  and let  $B \subset \mathbb{P}_1 \times \mathbb{C}$  be a curve satisfying:

- a)  $B$  has only  $A$ - $D$ - $E$  singularities along  $L$ ,
- b) the intersection number  $(B.L)$  is 4.

Let  $\bar{Y}$  be the minimal desingularization of  $Y$ , the double cover of  $\mathbb{P}_1 \times \mathbb{C}$  ramified over  $B$ . Denote by  $\bar{L} \subset \bar{Y}$  the curve mapped flatly onto  $L$ . The total inverse image  $F \subset \bar{Y}$  of  $L$  is a fibre in an elliptic fibration containing  $\bar{L}$ . We need the relation between the behaviour of  $B$  near  $L$  and the type of  $F$ . This relation should be well-known, but since we do not have a reference, we give it here explicitly.

The result is contained in Table 2. The first column shows the intersection multiplicities in the (at most four) points of  $B \cap L$  (resp.  $\overline{B \setminus \bar{L}} \cap L$ ). The second column gives the singularities of  $B$  (resp.  $B' = \overline{B \setminus L}$ ) in these points. Except for one case (marked by an \*) the type of  $F$  is determined uniquely by these data. For simplicity, a smooth point of  $B$  is several times considered as singularity of "type  $A_0$ ".

**PROOF:** We check all cases and subcases distinguished by the pattern of intersection numbers.

Table 2.

<i>Case 1: B does not split off L</i>		
intersection numbers	singularities of $B$	type of $F$
1 1 1 1	none	nonsingular
1 1 2	$A_k, k \geq 0$	$\tilde{A}_k$
2 2	$A_k, A_l, k, l \geq 0$	$\tilde{A}_{k+l+1}$
1 3	$A_k, k = 0, 1, 2$ $D_k$ $E_k$	$A'_k$ $\tilde{D}_k$ $\tilde{E}_k$
4	$A_k, k = 0, 1$ $A_k, k \geq 3$ * $D_k$ * $D_5$ $E_6$	$A'_{k+1}$ $\tilde{D}_{k+1}$ $\tilde{D}_{k+1}$ $\tilde{E}_6$ $\tilde{E}_7$
* If $k = 5$ and $L$ touches the nonsingular (resp. singular) branch, we have type $\tilde{D}_6$ (resp. $\tilde{E}_6$ ).		
<i>Case 2: B splits off L, say <math>B = B' \cup L</math></i>		
interaction numbers	singularities of $B'$	type of $F$
1 1 1 1	none	$\tilde{D}_4$
1 1 2	$A_k, k \geq 0$	$\tilde{D}_{k+5}$
2 2	$A_k, A_l, k, l \geq 0$	$\tilde{D}_{k+l+6}$
1 3	$A_k, k = 0, 1, 2$	$\tilde{E}_{k+6}$
4	$A_k, k = 0, 1$	$\tilde{E}_{k+7}$

CASE 1:

1 1 2: The curve  $\bar{L}$  is irreducible rational. If  $B$  is smooth ( $k = 0$ ),  $\bar{L}$  has a node over the point where  $B$  touches  $L$  (type  $\tilde{A}_0$ ). If  $k \geq 1$ , the fibre is reducible, hence contains smooth curves only, and  $\bar{L}$  must meet the  $A_k$ -string in  $\bar{Y}$  over the singularity of  $B$  in two distinct points. This creates a loop and  $F$  must be of type  $\tilde{A}_k$ .

2 2: The curve  $\bar{L}$  decomposes in two copies of  $\mathbb{P}_1$ . Both the trees  $A_k$  and  $A_l$  over the intersections meet these two curves in distinct points. This creates a loop and  $F$  must be of type  $\tilde{A}_{k+l+1}$ .

1 3: The curve  $\bar{L}$  is irreducible and meets the exceptional configuration (over the point where  $B$  touches  $L$ ) in one point. If  $B$  is nonsingular,  $\bar{L}$  has a cusp. If  $B$  has singularities  $A_1$  or  $A_2$ , the fibre must be of type  $A'_1$  or  $A'_2$ , because it contains only 2, resp. 3 components, and no cycle.

Singularities  $A_k$  with  $k \geq 3$  cannot occur by Lemma 1 below. If  $B$  has a singularity of type  $E_k$ , there is only one way to add one vertex for  $\bar{L}$  and to arrive at a diagram from Table 1, namely at  $\tilde{E}_k$ . The situation is analogous for a  $D_k$ -singularity except in the case  $k = 8$ , where the  $D_k$ -diagram could be completed to  $\tilde{D}_8$  or  $\tilde{E}_8$ . But blowing up the singularity once, one finds that  $\tilde{E}_8$  is impossible.

4:  $\bar{L}$  consists of two copies of  $\mathbb{P}_1$ . If  $B$  is smooth, they touch (and  $F$  is of type  $A'_1$ ), if  $B$  has a singularity  $A_1$ , the three curves cannot form a cycle, hence meet in one point ( $F$  is of type  $A'_2$ ). By Lemma 1 below,  $B$  cannot have an  $A_2$  singularity in  $L$ . If  $B$  has an  $A_k$ -singularity,  $k \geq 3$ , then the two components of  $\bar{L}$  intersect the  $A_k$ -tree over this singularity in two distinct points. Except for  $k = 6, 7$ , only a  $\tilde{D}_{k+1}$  can be formed to give a configuration from Table 1. But adding two components to a tree  $A_6$ , resp.  $A_7$ , obtaining an  $\tilde{E}_7$ , resp.  $\tilde{E}_8$ , would not be symmetric under the involution interchanging the sheets of  $\bar{Y}$  over  $\mathbb{P}_1 \times \mathbb{C}$ . If the singularity of  $B$  has type  $D_k$ , the two components of  $\bar{L}$  can complete the  $D_k$ -configuration to give a fibre of type  $\tilde{D}_{k+1}$ . This is the only possibility, symmetric under sheet-exchange, unless  $k = 5$ . But if  $B$  has a  $D_5$ -singularity, there are indeed two possibilities (the asterisk in Table 2):  $L$  touches either the nonsingular or the singular branch. After blowing up once, one finds the former to lead to an  $\tilde{D}_6$  fibre, and the latter to type  $\tilde{E}_6$ . If  $B$  has an  $E_k$ -singularity, the fibre must be of type  $\tilde{E}_{k+1}$ , so  $k = 6$  or  $7$ . But two vertices cannot be added to  $E_7$  in a symmetric way. The only possibility is  $E_6$  leading to  $\tilde{E}_7$ .

CASE 2: Here  $\bar{L}$  is always irreducible, lying bijectively over  $L$ .

1 1 1 1: The branch curve  $B$  has four  $A_1$ -singularities, each introducing one component. The fibre is of type  $\tilde{D}_4$ , because it is the only one with 5 components without a cycle.

1 1 2: The curve  $B$  has two  $A_1$ -singularities and one  $A_3$  (if  $B'$  is smooth), resp.  $D_{k+3}$  (if  $B'$  has an  $A_k$  double point). The only way to combine them to a diagram from Table 1 leads to a  $\tilde{D}_{k+5}$ -fibre.

2 2: Let the singularities of  $B'$  on  $L$  have type  $A_k$  and  $A_l$ ,  $k, l \geq 0$ . Then  $B$  has singularities of type  $A_3$  (if  $k = 0$ ), resp.  $D_{k+3}$ , and  $A_3$  (if  $l = 0$ ), resp.  $D_{l+3}$ . Since the two trees resolving them cannot meet in  $F$ , there is only one type for  $F$ , namely  $\tilde{D}_{k+l+6}$ , except if  $k = 0, l = 2$  which case might lead to  $\tilde{E}_8$ . But this possibility is excluded, because after blowing up once, one finds that  $\bar{L}$  must meet the middle curve in any  $A_3$ -tree.

1 3: By Lemma 1 below,  $B'$  is either smooth near  $L$  or has a double point  $A_1$  or  $A_2$ . So  $B$  has one  $A_1$ -singularity and another one of type  $A_5, D_6$ , or  $E_7$ . The only way to combine their trees, is to form a fibre of type  $\tilde{E}_6, \tilde{E}_7$ , or  $\tilde{E}_8$ .

4: If  $B'$  is smooth,  $B$  has on  $L$  an  $A_7$ -singularity, if  $B'$  has an ordinary double point, then  $B$  has a  $D_8$ . There cannot be other singularities on  $B'$ , because then  $B$  would not have a simple singularity. Blowing up twice in the first case, and once in the second, one checks that  $F$  cannot be of type  $\tilde{D}_7$ , resp.  $\tilde{D}_8$ . Hence the fibre is of type  $\tilde{E}_7$  or  $\tilde{E}_8$ .

LEMMA 1: Let  $P \in B$  be a singularity of type  $A_k$  and  $C$  a smooth curve through  $P$ .

- i) if  $k \geq 3$ , then  $i_P(B, C) \neq 3$
- ii) if  $k = 2$ , then  $i_P(B, C) \neq 4$ .

PROOF: Let  $\sigma$  be the blowing up of  $P$  and  $E = \sigma^{-1}P$  the exceptional curve. The proper transform  $\bar{B}$  of  $B$  has a double point  $A_{k-2}$  in  $\bar{P} \in E$ . Let  $\bar{C}$  be the proper transform of  $C$ . Then

$$i_P(B, C) = 2(E, \bar{C}) + i_{\bar{P}}(\bar{B}, \bar{C}).$$

If  $k \geq 3$ , then  $i_{\bar{P}}(\bar{B}, \bar{C}) = 0$  or  $\geq 2$ . This proves i). If  $k = 2$ , then  $E$  and  $\bar{B}$  touch at  $\bar{P}$ . So, if  $i_{\bar{P}}(\bar{B}, \bar{C}) = 2$ , then  $(E, \bar{C}) \geq 2$  too, a contradiction. This proves ii).

### 5. The SQH-criterion for $A_n$ -singularities, $n \leq 5$

To detect  $A_n$ -singularities we apply a computational technique developed in [4] and [3].

A convergent power series  $f(x, y)$  is called *semi-quasihomogeneous* with respect to weights  $a, b \in \mathbb{Q}$  if all terms in  $f(x^a, y^b)$  have degree  $\geq 1$ . The terms in  $f(x^a, y^b)$  of degree 1 determine the part of  $f(x, y)$  of weight 1. If this part of weight 1 describes an isolated singularity at the origin, the singularity at 0 with equation  $f(x, y) = 0$  is called SQH of weights  $a, b$ .

Recognition principle for  $A_n$ 's [1, 4]: *If a curve singularity is SQH with weights  $a = \frac{1}{2}$  and  $b = \frac{1}{n+1}$ , then it is biholomorphically equivalent with the singularity  $A_n$  of equation*

$$x^2 + y^{n+1} = 0.$$

We shall apply this to a polynomial

$$f(x, y) = Q(x, y) + C(x, y) + D(x, y) + xy\bar{C}(x, y) + x^2y^2\bar{Q}(x, y),$$



where  $Q, C, D$  are homogeneous of degree 2, 3, 4 respectively, and  $\bar{P}(x, y) = P(y, x)$  for any polynomial  $P$ .

**CRITERION:** *If  $Q$  is not a square, then the singularity with equation  $f = 0$  is an  $A_1$ . If there is a linear form  $L(x, y) \neq 0$  with  $Q = L^2$ , then this singularity is of type*

$A_2$  if  $L \nmid C$ ,

$A_3$  if  $C = LQ_1$  and  $L \nmid D - \frac{1}{4}Q_1^2$ ,

$A_4$  if  $D - \frac{1}{4}Q_1^2 = LC_1$  and  $L \nmid xy\bar{L}Q_1 - \frac{1}{2}Q_1C_1$ ,

$A_5$  if  $xy\bar{L}Q_1 - \frac{1}{2}Q_1C_1 = LD_1$  and  $L \nmid (xy\bar{L})^2 - \frac{1}{4}C_1^2 - \frac{1}{2}Q_1D_1$ .

**PROOF:** Assume  $Q = L^2$ . After interchanging  $x$  and  $y$ , if necessary, we may assume  $\partial L / \partial x \neq 0$ . So any coordinate change  $x_1 = L +$  (higher order terms),  $y_1 = y$  will be biholomorphic.

First, let  $x_1 = L$ ,  $y_1 = y$ . Then

$$\begin{aligned} f &= L^2 + C + D + xy\bar{C} + x^2y^2\bar{L}^2 = \\ &= x_1^2 + c_1y_1^3 + r_1(x_1, y_1) \end{aligned}$$

where, for the weights  $\frac{1}{2}, \frac{1}{3}$  all terms in  $r_1$  have degree  $> 1$ . So  $f$  is SQH of weights  $\frac{1}{2}, \frac{1}{3}$  if  $c_1 \neq 0$ , which is the case if  $L = x_1$  does not divide  $C$ .

If next  $C = LQ_1$ , then

$$\begin{aligned} f &= L^2 + LQ_1 + D + xy\bar{L}Q_1 + x^2y^2\bar{L}^2 \\ &= (L + \frac{1}{2}Q_1)^2 + (D - \frac{1}{4}Q_1^2) + xy\bar{L}Q_1 + x^2y^3\bar{L}^2. \end{aligned}$$

Putting  $x_2 = L + \frac{1}{2}Q_1$  and  $y_2 = y$  we obtain

$$f = x_2^2 + c_2y_2^4 + r_2(x_2, y_2),$$

where all monomials in  $r_2$  have degree  $> 1$  for the weights  $\frac{1}{2}, \frac{1}{4}$ . So  $f$  is SQH of weights  $\frac{1}{2}, \frac{1}{4}$  if  $c_2 \neq 0$ . This  $c_2$ , the coefficient of  $y_2^4$  in  $D - \frac{1}{4}Q_1^2$ , as polynomial in  $x_2$  and  $y_2$ , is also the coefficient of  $y^4$  in  $D - \frac{1}{4}Q_1^2$ , as polynomial in  $L$  and  $y$ , because  $L$  and  $x_2$  differ by a quadratic term. So  $f$  is SQH of weights  $\frac{1}{2}, \frac{1}{4}$  if  $L$  does not divide  $D - \frac{1}{4}Q_1^2$ .

If next  $C = LQ_1$  and  $D - \frac{1}{4}Q_1^2 = LC_1$ , then

$$\begin{aligned} f &= (L + \frac{1}{2}Q_1)^2 + LC_1 + xy\overline{LQ}_1 + x^2y^2\overline{L}^2 \\ &= (L + \frac{1}{2}Q_1 + \frac{1}{2}C_1)^2 + (xy\overline{LQ}_1 - \frac{1}{2}Q_1C_1) + (x^2y^2\overline{L}^2 - \frac{1}{4}C_1^2). \end{aligned}$$

Putting  $x_3 = L + \frac{1}{2}Q_1 + \frac{1}{2}C_1$ ,  $y_3 = y$  we obtain

$$f = x_3^2 + c_3y_3^5 + r_3(x_3, y_3).$$

So  $f$  is SQH of weights  $\frac{1}{2}, \frac{1}{5}$  if  $c_3 \neq 0$ . Here  $c_3$  is the coefficient of  $y^5$  in  $xy\overline{LQ}_1 - \frac{1}{2}Q_1C_1$  as polynomial of  $L$  and  $y$ . Then  $c_3 \neq 0$  if and only if  $L$  does not divide this polynomial.

If finally  $C = LQ_1$ ,  $D - \frac{1}{4}Q_1^2 = LC_1$ , and  $xy\overline{LQ}_1 - \frac{1}{2}Q_1C_1 = LD_1$ , then

$$\begin{aligned} f &= (L + \frac{1}{2}Q_1 + \frac{1}{2}C_1)^2 + LD_1 + (x^2y^2\overline{L}^2 - \frac{1}{4}C_1^2) = \\ &= (L + \frac{1}{2}Q_1 + \frac{1}{2}C_1 + \frac{1}{2}D_1)^2 + (x^2y^2\overline{L}^2 - \frac{1}{4}C_1^2 - \frac{1}{2}Q_1D_1) - \\ &\quad - \frac{1}{2}C_1D_1 - \frac{1}{4}D_1^2. \end{aligned}$$

Putting  $x_4 = L + \frac{1}{2}Q_1 + \frac{1}{2}C_1 + \frac{1}{2}D_1$ ,  $y_4 = y$  we obtain

$$f = x_4^2 + c_4y_4^6 + r_4(x_4, y_4).$$

Arguing as above finds that  $f$  is SQH of weights  $\frac{1}{2}, \frac{1}{6}$  if  $L$  does not divide  $(xy\overline{L})^2 - \frac{1}{4}C_1^2 - \frac{1}{2}Q_1D_1$ .

### 6. Examples

On  $\mathbb{P}_1 \times \mathbb{P}_1$  we use bihomogeneous coordinates  $(u_0 : u_1), (v_0 : v_1)$ . Since  $\tau$  interchanges  $u_0$  and  $u_1$  as well as  $v_0$  and  $v_1$ , a polynomial  $f(u_0, u_1, v_0, v_1)$ , bihomogeneous of degree 4, is invariant under  $\tau$  if it is of the form

$$\begin{aligned} f &= a_0(u_0^4v_0^4 + u_1^4v_1^4) + a_1(u_0^3u_1v_0^4 + u_0u_1^3v_1^4) + \\ &\quad + a_2(u_0^2u_1^2v_0^4 + u_0^2u_1^2v_1^4) + a_3(u_0u_1^3v_0^4 + u_0^3u_1v_1^4) + \\ &\quad + a_4(u_1^4v_0^4 + u_0^4v_1^4) + a_5(u_0^4v_0^3v_1 + u_1^4v_0v_1^3) + \\ &\quad + a_6(u_0^3u_1v_0^3v_1 + u_0u_1^3v_0v_1^3) + a_7(u_0^2u_1^2v_0^3v_1 + u_0^2u_1^2v_0v_1^3) + \\ &\quad + a_8(u_0u_1^3v_0^3v_1 + u_0^3u_1v_0v_1^3) + a_9(u_1^4v_0^3v_1 + u_0^4v_0v_1^3) + \\ &\quad + a_{10}(u_0^4v_0^2v_1^2 + u_1^4v_0^2v_1^2) + a_{11}(u_0^3u_1v_0^2v_1^2 + u_0u_1^3v_0^2v_1^2) + \\ &\quad + a_{12}u_0^2u_1^2v_0^2v_1^2. \end{aligned}$$

As line  $L$  we choose the line  $u_1 = 0$ , on which we shall use the points

$$P_1 : (1 : 0)(1 : 0) \quad P_2 : (1 : 0)(0 : 1).$$

The curve  $B$  with equation  $f = 0$  passes through  $P_1$  iff

$$a_0 = 0,$$

which we shall assume from now on.

Since we usually work in  $P_1$ , we also need the inhomogeneous expansion of  $f$  in coordinates  $u = \frac{u_1}{u_0}$  and  $v = \frac{v_1}{v_0}$ :

$$\begin{aligned} f(u, v) = & a_1(u + u^3v^4) + a_2(u^2 + u^2v^4) + a_3(u^3 + uv^4) + \\ & + a_4(u^4 + v^4) + a_5(v + u^4v^3) + a_6(uv + u^3v^3) + \\ & + a_7(u^2v + u^2v^3) + a_8(u^3v + uv^3) + a_9(u^4v + v^3) + \\ & + a_{10}(v^2 + u^4v^2) + a_{11}(uv^2 + u^3v^2) + a_{12}u^2v^2. \end{aligned}$$

*Examples of  $\tilde{A}_k$ -fibres,  $k = 0, 1, 2, 3, 5, 7$ .*

Using the SQH-criterion, one easily writes down conditions on the coefficients of  $f$ , guaranteeing that  $B$ , the curve with equation  $f = 0$ , has at  $P_1$  double points of certain types:

conditions		singularity at $P_1$
$a_5 = 0$	$a_1 \neq 0$	$A_0$
$a_1 = a_5 = 0$	$4a_2a_{10} \neq a_6^2$	$A_1$
$a_1 = a_2 = a_5 = a_6 = 0$	$a_3 \neq 0$	$A_2$
$a_1 = a_2 = a_3 = a_5 = a_6 = 0$	$4a_4a_{10} \neq a_7^2$	$A_3$
$a_i = 0$ for $i \leq 7$	$a_8 \neq 0$	$A_5$
$a_i = 0$ for $i \leq 8$	$a_9 \neq 0$	$A_7$

If  $a_{10} \neq 0$  the curve  $B$  intersects the line  $L$  with equation  $u_1 = 0$  at  $P_1$  with multiplicity 2. If  $a_4$  and  $a_{12}$  vary independently (the first four cases), the linear system  $|B|$  determined by the vanishing conditions on the  $a_i$  has no fixed point other than  $P_1$  and  $\tau P_1$ . By Bertini's theorem, the general curve in  $|B|$  is nonsingular away from  $P_1$  and  $\tau P_1$ . It inter-

sects  $L$  with multiplicities 211, and at  $P_1$  it has the double point  $A_k$ ,  $k = 0, 1, 2, 3$  from the table above. It therefore defines a double covering  $Y \rightarrow \mathbb{P}_1 \times \mathbb{P}_1$  ramified over  $B$ , such that the fibre  $F \subset \bar{Y}$  over  $L$  has type  $\tilde{A}_0, \tilde{A}_1, \tilde{A}_2$ , resp.  $\tilde{A}_3$  (Table 2).

If  $a_i = 0$  for  $i \leq 7$ , the linear system  $|B|$  has the lines  $v_0 \cdot v_1 = 0$  as fixed components. Since  $a_9$  and  $a_{12}$  vary independently, the system  $|B - \{v_0 v_1 = 0\}|$  has only  $P_1$  and  $\tau P_1$  as fixed points. So the general curve in  $|B|$  is nonsingular, except for  $P_1$  and  $\tau P_1$  and two ordinary double points on each of the lines  $v_0 = 0, v_1 = 0$ . It determines a surface carrying over  $L$  a fibre  $F$  of type  $\tilde{A}_5$ , resp.  $\tilde{A}_7$  (Table 2).

*Examples of  $\tilde{A}_k$ -fibres,  $k = 4, 6$ .*

Interchanging  $v_0$  and  $v_1$  leaves in the polynomial  $f$  the coefficients  $a_2, a_7, a_{10}, a_{11}, a_{12}$  invariant, whereas it permutes  $a_0$  with  $a_4, a_1$  with  $a_3, a_5$  with  $a_9$ , and  $a_6$  with  $a_8$ . The linear system of curves  $B$  with equation  $f = 0$  having singularities at  $P_1$  and  $P_2$  therefore is specified by the conditions

$$\tilde{a}_0 = a_1 = a_3 = a_4 = a_5 = a_9 = 0.$$

The inhomogeneous expansion of such an  $f$  at  $P_1$  is

$$\begin{aligned} & a_6(uv + u^3v^3) + a_8(u^3v + uv^3) + \\ & + a_2(u^2 + u^2v^4) + a_7(u^2v + u^2v^3) + \\ & + a_{10}(v^2 + u^4v^2) + a_{11}(uv^2 + u^3v^2) + a_{12}u^2v^2. \end{aligned}$$

Its expansion at  $P_2$  is identical, except for the permutation of  $a_6$  and  $a_8$ . To apply section 5, we give  $Q, C, D$  at  $P_1$  explicitly (to obtain them at  $P_2$ , interchange  $a_6$  and  $a_8$ ):

$$\begin{aligned} Q(u, v) &= a_2u^2 + a_6uv + a_{10}v^2 \\ C(u, v) &= uv(a_7u + a_{11}v) \\ D(u, v) &= uv(a_8u^2 + a_{12}uv + a_8v^2). \end{aligned}$$

Let us put

$$L(u, v) = a_7u + a_{11}v,$$

then  $P_1$  is a singularity of type

- $A_1$  if  $4a_2a_{10} \neq a_6^2$ ,
- $A_2$  if  $4a_2a_{10} = a_6^2, a_6 \neq 0, Q \neq \text{const} \cdot L^2$ ,
- $A_3$  if  $Q = L^2, L \nmid D - \frac{1}{4}u^2v^2$ .

Fix  $a_7 \neq 0$  and  $a_{11} \neq 0$  and put

$$a_2 = a_7^2, a_8 = -2a_7a_{11}, a_{10} = a_{11}^2.$$

Any curve with these coefficients has at  $P_2$  a singularity of type  $A_2$ . Denote by  $f_1$  a polynomial formed with these coefficients and

$$a_6 \neq \pm a_8, a_{12} = 0.$$

Then  $f_1 = 0$  has an  $A_1$  at  $P_1$ . The curve intersects each of the four lines  $u_0u_1v_0v_1 = 0$  in two of the points  $P_1, P_2, \tau P_1, \tau P_2$  only. So for general  $\lambda, a_{12}$  the curve  $B$  with equation  $\lambda f_1 + a_{12}u_0^2u_1^2v_0^2v_1^2 = 0$  is non-singular, except for the points  $P_1, P_2, \tau P_1$ , and  $\tau P_2$ , where it has  $A_1$ , resp.  $A_2$ -singularities. The corresponding Enriques surface carries a fibre of type  $\tilde{A}_4$  over  $L$  (Table 2).

Next denote by  $f_2$  a polynomial with the coefficients  $a_2, a_7, a_8, a_{10}, a_{11}$  as above and

$$a_6 = -a_8, a_{12} = 0.$$

Again, for general  $a_{12}$  the curve  $B$  with equation  $f_2 + a_{12}u_0^2u_1^2v_0^2v_1^2 = 0$  is nonsingular except for  $P_1, P_2, \tau P_1, \tau P_2$ . At  $P_2$  it has an  $A_2$ , and at  $P_1$  an  $A_3$ -singularity, provided that  $L$  does not divide

$$D - \frac{1}{4}u^2v^2 = uv(a_8u^2 + (a_{12} - \frac{1}{4})uv + a_8v^2),$$

which is the case for general  $a_{12}$ . So for general  $a_{12}$  the curve defines an Enriques surface with a fibre of type  $\tilde{A}_6$  over  $L$  (Table 2).

*Example of an  $\tilde{A}_8$ -fibre*

We put  $a_1 = a_3 = a_4 = a_5 = a_9 = 0$  as in the preceding example, and for some  $a \in \mathbb{C}^*$  we put

$$\begin{aligned} a_6 &= 2a^2, a_8 = -2a^2 \\ a_2 &= a_{10} = a^2, a_7 = a_{11} = a \\ a_{12} &= \frac{1}{4} - 4a^2. \end{aligned}$$

Then  $f$  has at  $P_1$  the inhomogeneous expansion

$$\begin{aligned} f(u, v) &= 2a^2(uv + u^3v^3) - 2a^2(u^3v + uv^3) \\ &\quad + a^2(u^2 + u^2v^4) + a(u^2v + u^2v^3) \\ &\quad + a^2(v^2 + u^4v^2) + a(uv^2 + u^3v^2) + \left(\frac{1}{4} - 4a^2\right)u^2v^2. \end{aligned}$$

At  $P_2$ , just as above, the curve  $B$  with equation  $f = 0$  has a singularity  $A_2$ . The SQH-criterion shows that at  $P_1$ , this curve has a singularity  $A_5$ .

The curve  $B$  cannot split off any of the four lines  $u_0u_1v_0v_1 = 0$ . So  $B$  intersects these lines only in the four points  $P_1, P_2, \tau P_1, \tau P_2$ . So any irreducible component of  $B$  passes through one of these points, which is a double point  $A_n$  on  $B$ . This shows that  $B$  is reduced. In any of the four fixed points  $(1: \pm 1), (1: \pm 1)$  of  $\tau$ , the polynomial  $f(\pm 1, \pm 1)$  in  $a$  has constant term  $\frac{1}{4} \neq 0$ . So for general  $a$ , the curve  $B$  does not pass through any of these four points.

Lemma 2 below shows that  $B$  has no other singularities than double points. So it defines a double covering leading to an Enriques surface, which has a fibre of type  $\tilde{A}_8$  (Table 2).

LEMMA 2: For any  $a \neq 0$ , the curve

$$\{(u, v) \in \mathbb{C}^2 : f(u, v) = 0\}$$

has no singularities of multiplicity  $\geq 3$ .

PROOF: Assume that  $(p, q) \neq (0, 0)$  is an  $m$ -fold point. As  $f(u, v) = f(v, u)$ , the point  $(q, p)$  has multiplicity  $m$  too. We distinguish two cases:

1.  $p \neq q$ : The line  $u + v = r$  with  $r = p + q$  passes through both the singular points. Then there must be  $\alpha \neq \beta \in \mathbb{C}$  such that  $(x - \alpha)^m(x - \beta)^m$  divides  $f(x, r - x)$ , which by inspection is a polynomial of degree 5. For  $m \geq 3$  the only possibility is that  $f(x, r - x)$  vanishes identically. Its constant term being  $a^2r^2$ , this is excluded if  $r \neq 0$ . But if  $r = 0$ , we have  $f(x, r - x) = \frac{1}{4}x^2$ , and arrive at a contradiction again.

2.  $p = q \neq 0$ . Now  $x^2(x - p)^m$  divides

$$f(x, x) = 4a^2x^2g(x),$$

where

$$g(x) = x^4 + \frac{1}{2a}x^3 + \left(\frac{1}{16a^2} - 2\right)x^2 + \frac{1}{2a}x + 1.$$

This polynomial  $g$  satisfies the identity

$$g(x) = x^4 g\left(\frac{1}{x}\right).$$

So if  $m \geq 3$ , necessarily  $p = \pm 1$ , and

$$g(x) = (x \pm 1)^4 = x^4 \pm 4x + 6x^2 \pm 4x^3 + 1.$$

Comparing coefficients we find the contradiction

$$\frac{1}{2a} = \pm 4, \frac{1}{16a^2} - 2 = 6.$$

*Examples of  $A'_k$ -fibres,  $k = 0, 1, 2$*

Consider the linear system of curves  $B$  defined by equations  $f = 0$ , where  $a_1 = a_5 = a_6 = a_{10} = 0$ . Since  $a_4$  and  $a_{12}$  vary independently,  $P_1$  and  $\tau P_1$  are its only base points. By Bertini's theorem the general curve in this system will be nonsingular away from these points. The same holds for any linear system containing the one above. We consider three systems contained in the one defined by  $a_5 = a_{10} = 0$ . For the general member  $B$  always  $a_9$  will be nonzero, i.e.,  $B$  will intersect  $L$  in  $P_1$  with multiplicity 3. The following table gives the definition of the linear systems, the singularity at  $P_1$  of its general members (SQH-test from section 5), and the type of the fibre  $F$  lying over  $L$  (Table 2):

linear system	singularity	fibre
$a_5 = a_{10} = 0$	$A_0$	$A'_0$
$a_1 = a_5 = a_{10} = 0$	$A_1$	$A'_1$
$a_1 = a_5 = a_6 = a_{10} = 0$	$A_2$	$A'_2$

*Example of  $\tilde{D}_4$ -fibres*

The linear system of curves  $B$  splitting off  $L$  and  $\tau L$  is given by  $a_4 = a_5 = a_9 = a_{10} = 0$ . As  $a_3$  and  $a_{12}$  vary independently, there are no fixed points away from  $L$  and  $\tau L$ . The restriction of  $B' = B - L$  to  $L$  has equation

$$a_1 v_0^4 + a_6 v_0^3 v_1 + a_{11} v_0^2 v_1^2 + a_8 v_0 v_1^3 + a_3 v_1^4 = 0.$$

So the curves  $B'$  cut out on  $L$  the complete linear system of degree 4. The general curve  $B$  in the linear system will be nonsingular away from

$L$  and will have four distinct double points on  $L$ . By Table 2, over  $L$  will lie a fibre of type  $\tilde{D}_4$ .

*Examples of  $\tilde{D}_k$ -fibres  $k = 5, 6, 7$*

Consider the linear system of curves  $B$  defined by equations  $f = 0$  where

$$a_1 = a_2 = a_4 = a_5 = a_6 = a_7 = a_9 = a_{10} = a_{11} = 0.$$

It splits off  $L$  and  $\tau L$ , and the only fixed points of  $|B'| = |B - L - \tau L|$  are  $P_1$  and  $\tau P_1$ . As long as  $4a_3a_{11} \neq a_8^2$ , i.e.,  $a_8 \neq 0$ , the curve  $B'$  will intersect  $L$  in two distinct points different from  $P_1$ . The following table defines three systems containing the one above, shows the singularity of  $B'$  at  $P_1$  for the general member  $B$ , and gives the type of the fibre over  $L$  (Table 2)

linear system	singularity	fibre
$a_1 = a_4 = a_5 = a_6 = a_9 = a_{10} = 0$	$A_0$	$\tilde{D}_5$
in addition $a_2 = 0$	$A_1$	$\tilde{D}_6$
in addition $a_7 = a_{11} = 0$	$A_2$	$\tilde{D}_7$

*Examples of  $\tilde{D}_8$ -fibres*

Consider the system  $|B| = |B' + L + \tau L|$  defined by  $a_1 = a_3 = a_4 = a_5 = a_6 = a_8 = a_9 = a_{10} = 0$ . It splits off  $L$  and  $\tau L$ , and the system  $|B'|$  has the fixed points  $P_1, P_2, \tau P_1$  and  $\tau P_2$ . As long as  $a_{11}$  and  $a_{12}$  vary independently, there are no other base points. All curves  $B'$  touch  $L$  in both points  $P_1$  and  $P_2$ , as long as  $a_{11} \neq 0$ . The inhomogeneous expansion of  $f/u$  at  $P_1$  is

$$a_2(u + uv^4) + a_7(uv + uv^3) + a_{11}(v^2 + u^2v^2) + a_{12}uv^2.$$

Since  $f$  is invariant under interchanging  $v_0$  and  $v_1$ , the expansion at  $P_2$  is the same. So  $B'$  is smooth at both points, if  $a_2 \neq 0$ , but if  $a_2 = 0$  and  $a_7 = 0$ , it carries an  $A_1$  at both points. From Table 2 we find that the fibre over  $L$  is of type  $\tilde{D}_8$ .

*Examples of fibres  $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$*

Consider the linear system of curves  $B = L + \tau L + B'$  with equations  $f = 0$ , where  $a_1 = a_4 = a_5 = a_6 = a_9 = a_{10} = a_{11} = 0$ . It contains  $L$  and  $\tau L$  as fixed components and the curves  $B'$  intersect  $L$  at  $P_1$  with multiplicity 3 as long as  $a_8 \neq 0$ . The same holds for the subsystem of



curves where additionally  $a_2 = a_7 = 0$ . Since  $a_3$  and  $a_{12}$  vary independently, the system of curves  $B'$  has  $P_1$  and  $\tau P_1$  as its only fixed points. By Bertini's theorem the general curve  $B$  is nonsingular away from  $L$ .

The following table defines three linear systems, shows the singularity at  $P_1$  of  $B' = B - L$  for its general member  $B$  (SQH-Test from section 5), and the type of fibre  $F$  lying over  $L$  (Table 2).

linear system	singularity of $B'$ fibre	
$a_1 = a_4 = a_5 = a_6 = a_9 = a_{10} = a_{11} = 0$	$A_0$	$\tilde{E}_6$
in addition $a_2 = 0$	$A_1$	$\tilde{E}_7$
in addition $a_7 = 0$	$A_2$	$\tilde{E}_8$

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