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## AN ASYMPTOTIC SERIES EXPANSION OF THE MULTIDIMENSIONAL RENEWAL MEASURE

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### 1. Introduction and main theorem

Let  $p(x)dx$  be an absolutely continuous probability distribution on Euclidean  $d$ -dimensional space with non-vanishing mean vector  $\mu$ . As usual we define the renewal measure  $\nu$  by the formula

$$\nu(E) = \sum_{n=0}^{\infty} \int_E p^{n*}(x) dx$$

for any Borel set  $E$ . Then it is well known that, if say  $Q$  is the unit cube,

$$\nu(Q + \lambda\mu) \sim \frac{C}{\lambda^\rho}, \quad \lambda \rightarrow +\infty,$$

where  $\rho = \frac{1}{2}(d - 1)$ . See [1], [2], [4], and [6]. We are concerned here with the error

$$E(\lambda) = \nu(Q + \lambda\mu) - \frac{C}{\lambda^\rho}.$$

In one dimension the decay of  $E(\lambda)$  is to a large extent independent of the distribution  $p(x)$ , provided  $p(x)$  has sufficiently many moments. For example if  $\int |x|^k p(x) dx < +\infty$ ,  $k = 2, 3, 4, \dots$ , then

$$E(\lambda) = o(\lambda^{-k+1}), \quad \lambda \rightarrow +\infty.$$

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(See however Stone [7] where it is shown that

$$E(\lambda) = R(\lambda) + o(\lambda^{-k}), \quad \lambda \rightarrow +\infty,$$

where  $R(\lambda)$  does depend on  $p(x)$ .) In contrast to this, in more than one dimension it is not true in general that

$$E(\lambda) = o(\lambda^{-(\rho+1)}), \quad \lambda \rightarrow +\infty,$$

no matter how many moments are assumed finite. In fact  $E(\lambda)$  has an asymptotic series expansion which is very much dependent on the complete structure of  $p(x)$ .

One may get a feeling for this phenomena by considering the following special example. Suppose  $dP$  is a singular measure in the plane supported on the line  $x = 1$  and smooth on that line. So for a  $C_0^\infty$  function  $f$ ,

$$dP(f) = \int f(1, y)g(y)dy, \quad \int g(y)dy = 1$$

with  $g \in C_0^\infty$ . Suppose  $\int yg(y)dy = 0$ . Then  $dP^{*n}$  will be supported on the line  $x = n$  and its distribution on this line is governed by the local central limit theorem which is very distributional dependent. It follows obviously that  $v(Q + n)$  is also very dependent on the distribution of  $g$ .

To state our theorem we need some notation. Let  $\pi$  be the hyperplane through the origin perpendicular to  $\mu$ . For  $x$  in  $\mathbb{R}^d$  let  $x_1$  be the projection of  $x$  on  $\mu$  and let  $x'$  be the projection on  $\pi$ . We let  $x_2, \dots, x_d$  denote the standard coordinates in  $\pi$  of  $x'$ . Then we consider the covariance matrix

$$B = (E[X_i X_j])_{i,j=2,\dots,d}$$

where  $X = (X_1, \dots, X_d)$  is a random vector with distribution  $p(x)dx$ . By  $\omega_j$  we mean an expression of the form

$$Q_0(x') + \frac{Q_1(x')}{x_1^{1/2}} + \dots + \frac{Q_{n_j}(x')}{x_1^{n_j/2}} \tag{1}$$

where each  $Q_k$  is a homogeneous polynomial of degree  $k$  in  $x'$  whose coefficients are determined by the moments of  $p(x)$ .

We shall prove the following result.

**THEOREM:** *If  $p(x)$  has a sufficient number of finite moments then*

$$E_k(x) = v(Q + x) - \int_{Q+x} h_k(y) dy = o\left(\frac{1}{x_1^{\rho+k/2}}\right), \quad x_1 \rightarrow +\infty, \quad (2)$$

where

$$h_k(x) = \frac{|\mu|^\rho}{(\det B)^{1/2}(2\pi x_1)^\rho} \exp\left(-\frac{|\mu|B^{-1}(x', x')}{2x_1}\right) \cdot \left(1 + \frac{\omega_1(x)}{x_1^{1/2}} + \frac{\omega_2(x)}{x_1} + \dots + \frac{\omega_k(x)}{x_1^{k/2}}\right).$$

The estimate is uniform in  $x'$ .

**REMARK 1:**  $\omega_j$  is bounded in any paraboloid  $|x'|^2 \leq Cx_1$  and thus the behavior of  $v(Q + x)$  in such a paraboloid is determined by (2) up to “order”  $\rho + \frac{k}{2}$ .

**REMARK 2:** The explicit form of (1) in terms of the moments of  $p(x)$  is rather complicated. For instance in  $\mathbb{R}^2$  we have

$$\begin{aligned} \omega_1(x) = & \frac{1}{2} \frac{x_2}{x_1^{1/2}} \left( \mu_{0,3} \frac{\mu_1^2}{\mu_2^2} - \mu_{1,1} \frac{\mu_1}{\mu_2} \right) + \\ & + \frac{x_2^3}{x_1^{3/2}} \left( \frac{1}{2} \mu_{1,1} \frac{\mu_1^2}{\mu_2^2} - \frac{1}{6} \mu_{0,3} \frac{\mu_1^3}{\mu_2^3} \right) \end{aligned}$$

where  $\mu_i = E[X_i^i]$  and  $\mu_{i,j} = E[X_1^i X_2^j]$ .

**REMARK 3:** We will prove (2) if

$$E[|X_1|^{\max(1, \rho) + \frac{1}{2}(k+3)}] < +\infty$$

and

$$E[|X'|^{k+5}] < +\infty.$$

In fact one can prove the sharper result

$$E_k(x) = o\left(\frac{1}{x_1^{\rho + \frac{1}{2}(k+a)}}\right), \quad x_1 \rightarrow +\infty,$$

for any  $\alpha < 1$  if one assumes that

$$E[|X_1|^{\max(1, \rho) + \frac{1}{2}(k + \alpha) + \varepsilon}] < +\infty$$

and

$$E[|X'|^{2+k+\alpha+\varepsilon}] < +\infty$$

for some  $\varepsilon > 0$ . For the stronger result one needs to use more delicate techniques such as those developed in [1].

In proving the theorem we need the “correct” method of expanding  $f(t) = \hat{p}(t)$ . We write

$$f(t) = \sum_{j=0}^m p_j(t) + E_m(t),$$

where  $p_j(\lambda^2 t_1, \lambda t') = \lambda^j p_j(t_1, t')$ , for  $\lambda > 0$ . This method of counting degrees has been employed in several complex variables and operators on nilpotent groups. See [3] and [5].

### 2. Proof of the theorem

Let  $\phi(x)$  be  $C^\infty$  and have support in the unit cube  $Q = \{x; |x_i| \leq 1\}$  and put  $\phi_\varepsilon(x) = \varepsilon^{-d} \phi(x/\varepsilon)$ . Let  $\chi_E$  denote the indicator function of the set  $E$ , and put  $Q_r = \{x; |x_i| \leq r\}$ . Let  $\Omega_k$  be the measure with density  $h_k$ , let  $N$  denote the integer  $[\rho + \frac{1}{2}k] + 1$  and put  $M = 2(N - \rho)$ . We shall prove

$$|\phi_\varepsilon * \chi_{Q_r} * (v - \Omega_M)(x)| \leq C \frac{\log^d \frac{1}{\varepsilon}}{x_1^N}, \tag{3}$$

where  $C$  can be chosen uniformly for  $r$  bounded, and

$$\begin{aligned} \phi_\varepsilon * \chi_{Q_{1-\varepsilon}} * (v - \Omega_M)(x) - C\varepsilon &\leq (v - \Omega_M)(Q + x) \leq \\ &\leq \phi_\varepsilon * \chi_{Q_{1+\varepsilon}} * (v - \Omega_M)(x) + C\varepsilon. \end{aligned} \tag{4}$$

(2) follows easily from (3) and (4) by putting  $\varepsilon = x_1^{-m}$  with  $m$  large enough. (As  $N \geq \rho + \frac{1}{2}(k + 1)$  and  $M \geq k + 1$ , we obtain a sharper result than (2). This is possible as we assume a stronger moment condition than necessary; compare Remark 3.)

We assume without loss of generality that  $|\mu| = 1$  and that  $B$  is the identity matrix.

Let us turn to the estimate (3). Set  $f(t) = \int e^{-it \cdot x} p(x) dx$ . As observed in [1],  $\hat{\nu} = (1 - f)^{-1} \in L^1_{loc}(\mathbb{R}^d)$ . So

$$\phi_\varepsilon * \chi_{Q_r} * \nu(x) = \frac{1}{(2\pi)^d} \int e^{it \cdot x} \frac{\hat{\phi}(\varepsilon t) \hat{\chi}_{Q_r}(t)}{1 - f(t)} dt. \tag{5}$$

Let  $\psi(t)$  be  $C^\infty_0$  and 1 near the origin. Then, as  $(1 - f(t))^{-1}$  is bounded if  $t$  is bounded away from the origin and  $\chi_{Q_r}$  and its derivatives are bounded by a constant times  $\prod_{i=1}^d |t_i|^{-1}$ ,  $|t| \rightarrow +\infty$ ,  $N$  integrations by parts gives

$$\left| \int e^{it \cdot x} \frac{\hat{\phi}(\varepsilon t) \hat{\chi}_{Q_r}(t)}{1 - f(t)} (1 - \psi(t)) dt \right| \leq \frac{C \log^d \frac{1}{\varepsilon}}{x_1^N}. \tag{6}$$

It remains to consider

$$I(x) = \frac{1}{(2\pi)^d} \int e^{it \cdot x} \frac{\hat{\phi}(\varepsilon t) \hat{\chi}_{Q_r}(t) \psi(t)}{1 - f(t)} dt. \tag{7}$$

The general idea is to use the Taylor expansion of  $f$  at the origin to prove that

$$I(x) = \phi_\varepsilon * \chi_{Q_r} * \Omega_M(x) + O(x_1^{-N}), \quad x_1 \rightarrow +\infty. \tag{8}$$

By expanding  $e^{-it \cdot x}$  in a Taylor series and integrating term by term, we get

$$f(t) = 1 - \sum_{j=2}^n P_j(t) + R_n(t), \tag{9}$$

where  $P_j(t) = P_j(t_1, t')$  is a polynomial homogeneous in the sense that

$$P_j(\lambda^2 t_1, \lambda t') = \lambda^j P_j(t).$$

The coefficients of  $P_j$  are determined by the moments of  $p(x)$ . From the Taylor expansion of  $e^{-it \cdot x}$  and the moment condition imposed on  $p(x)$ , we get if  $n \leq M + 2$  that

$$\begin{aligned} R_n(t) &= \int \left( e^{-it \cdot x} - \sum_{\|\alpha\| \leq n} \frac{(-it \cdot x)^\alpha}{\alpha!} \right) p(x) dx = \\ &= O(|t_1| + |t'|^2)^{\frac{1}{2}(n+1)}, \quad t \rightarrow 0, \end{aligned} \tag{10}$$

where  $\|\alpha\| = 2\alpha_1 + \alpha_2 + \dots + \alpha_d$ . Furthermore, if  $l \leq n/2$

$$\begin{aligned} \frac{\partial^l R_n}{\partial t_1^l}(t) &= \int (-ix_1)^l \left( e^{-it \cdot x} - \sum_{\|\alpha\| \leq n-2l} \frac{(-it \cdot x)^\alpha}{\alpha!} \right) p(x) dx = \\ &= O((|t_1| + |t|^2)^{\frac{1}{2}(n+1)-l}), \quad t \rightarrow 0, \end{aligned} \tag{11}$$

and  $\partial^l R_n / \partial t_1^l$  is bounded if  $n/2 < l \leq N$ .

Write

$$\begin{aligned} \frac{1}{1-f(t)} &= \frac{1}{P_2(t)} + \frac{1}{1-f(t)} - \frac{1}{P_2(t)} = \\ &= \frac{1}{P_2(t)} + \frac{R_2(t)}{P_2(t)(1-f(t))}. \end{aligned}$$

By iteration, we find

$$\frac{1}{1-f(t)} = \sum_{j=0}^M \frac{R_2^j(t)}{P_2^{j+1}(t)} + \frac{R_2^{M+1}(t)}{P_2^{M+1}(t)(1-f(t))}. \tag{12}$$

We write

$$R_2(t) = - \sum_{k=3}^{M+2} P_k(t) + R_{M+2}(t)$$

and expand  $R_2(t)$  by the multinomial theorem to obtain

$$\frac{1}{1-f(t)} = \sum_{j=0}^M \frac{\left( - \sum_{k=3}^{M+2} P_k(t) \right)^j}{P_2^{j+1}(t)} + S_M(t). \tag{13}$$

Here

$$\begin{aligned} S_M(t) &= \frac{R_2^{M+1}(t)}{P_2^{M+1}(t)(1-f(t))} + \\ &+ \sum_{j=1}^M \Sigma' C_{i_1, \dots, i_m} \frac{R_{M+2}^{i_1}(t) P_3^{i_3}(t) \dots P_{M+2}^{i_m}(t)}{P_2^{j+1}(t)} \end{aligned}$$

where  $\Sigma'$  is a finite sum with  $i_1 + i_3 + \dots + i_{M+2} = j$  and  $i_1 \geq 1$ . We claim that  $S_M$  has  $N$  derivatives with respect to  $t_1$  that are locally integrable. Granted this we can integrate by parts  $N$  times to obtain

$$\left| \int e^{it \cdot x} \hat{\phi}(\varepsilon t) \hat{\chi}_{Q_r}(t) \psi(t) S_M(t) dt \right| \leq \frac{C}{x_1^N}. \tag{14}$$

To see that  $S_M(t)$  has  $N$  locally integrable derivatives, note that the moment assumption on  $p(x)$  implies that  $f(t)$  is differentiable  $N$  times with respect to  $t_1$ . Hence  $S_M(t)$  is differentiable  $N$  times if  $t \neq 0$ . Furthermore, in a neighborhood of the origin every quotient occurring in  $S_M$  is bounded by a constant times  $(|t_1| + |t'|^2)^{\frac{1}{2}(M-1)}$  (see (10)). As each differentiation with respect to  $t_1$  introduces at worst a multiplicative factor of size  $(|t_1| + |t'|^2)^{-1}$  (see (11)), we get

$$\left| \frac{\partial^N S_M(t)}{\partial t_1^N} (t) \right| \leq \frac{C}{(|t_1| + |t'|^2)^{\rho+1/2}} \in L^1_{\text{loc}}(\mathbb{R}^d).$$

In view of (7), (13) and (14) we have

$$I(x) = \sum_{j=0}^M \sum_{l=3j}^{(M+2)j} \frac{1}{(2\pi)^d} \int e^{it \cdot x} \hat{\phi}(\varepsilon t) \hat{\chi}_{Q_r}(t) \psi(t) \frac{q_{l,j}(t)}{P_2^{j+1}(t)} dt + O(x_1^{-N}),$$

(15)

where  $q_{l,j}(t)$  is a polynomial satisfying

$$q_{l,j}(\lambda^2 t_1, \lambda t') = \lambda^l q_{l,j}(t_1, t').$$

(16)

Write

$$\psi(t) \frac{q_{l,j}(t)}{P_2^{j+1}(t)} = \frac{q_{l,j}(t)}{P_2^{j+1}(t)} + (\psi(t) - 1) \frac{q_{l,j}(t)}{P_2^{j+1}(t)}.$$

Now sufficiently high order derivatives with respect to  $t_1$  to

$$(\psi(t) - 1) q_{l,j}(t) P_2^{-(j+1)}(t)$$

are integrable functions, so  $\{(\psi(t) - 1) q_{l,j}(t) P_2^{-(j+1)}(t)\}^\vee$  is a function in  $x_1 > 0$  and is  $O(x_1^{-k})$ ,  $x_1 \rightarrow +\infty$ , for any  $k$ . Thus we get from (15) that

$$I(x) = \sum_{j=0}^M \sum_{l=3j}^{(M+2)j} \phi_\varepsilon * \chi_{Q_r} * Q_{l,j}(x) + O(x_1^{-N}), \quad x_1 \rightarrow +\infty,$$

(17)

where  $\hat{Q}_{l,j} = q_{l,j} P_2^{-(j+1)}$ .

From (16) we see that

$$\frac{q_{l,j}(\lambda^2 t_1, \lambda t')}{P_2^{j+1}(\lambda^2 t_1, \lambda t')} = \lambda^{l-2(j+1)} \frac{q_{l,j}(t)}{P_2^{j+1}(t)}$$

which implies

$$Q_{l,j}(\lambda^2 x_1, \lambda x') = \frac{1}{\lambda^{d-1+l-2j}} Q_{l,j}(x).$$

By letting  $x_1 = 1$ ,  $x' = 0$  and  $y_1 = \lambda^2$  we get

$$Q_{l,j}(y_1, 0) = \frac{C_{l,j}}{y_1^{\rho+l/2-j}}, \quad y_1 > 0.$$

Now if we define  $\omega_k$  by

$$\frac{1}{(2\pi x_1)^\rho} \exp\left(-\frac{|x'|^2}{2x_1}\right) \frac{\omega_k(x)}{x_1^{k/2}} = \sum_{l-2j=k} Q_{l,j}(x), \quad (18)$$

we have  $\omega_k(x_1, 0) = c_k$  which agrees with (1). To see that  $\omega_k(x)$  has the representation (1) also when  $x' \neq 0$ , we first observe that the Fourier transform of

$$w(x) = \begin{cases} \frac{1}{(2\pi x_1)^\rho} \exp\left(-\frac{|x'|^2}{2x_1}\right), & x_1 > 0 \\ 0, & x_1 \leq 0 \end{cases}$$

is

$$\hat{w}(t) = P_2^{-1}(t),$$

see [1]. From this we obtain

$$\left(\frac{\partial^\alpha}{\partial x^\alpha} x_1^\rho w(x)\right)^\wedge(t) = C_{\alpha,p} \frac{t^\alpha}{P_2^\rho(t)},$$

where the derivatives and the Fourier transform are interpreted in the sense of distributions. As  $x_1^\rho w(x) \in C_0^\infty$  if  $x \neq 0$ , we see that  $\frac{\partial^\alpha}{\partial x^\alpha} x_1^\rho w(x)$  for  $x \neq 0$  is a function obtained from  $x_1^\rho w(x)$  by (ordinary) differentiation. From this the representation (1) follows easily by induction. We also see that the terms  $\phi_\varepsilon * \chi_{Q_r} * Q_{l,j}$  in (17) with  $l - 2j > M$  is  $O(x_1^{-N})$ ,  $x_1 \rightarrow +\infty$ . Thus (9) follows from (17). This completes the proof of (3), as it is an immediate consequence of (5), (6), (7) and (9).

To prove (4) we observe that if  $y \in \text{supp } \phi_\varepsilon$  then

$$Q_{1-\varepsilon} + x - y \subseteq Q + x \subseteq Q_{1+\varepsilon} + x - y,$$

which implies

$$\phi_\varepsilon * \chi_{Q_{1-\varepsilon}} * v(x) \leq v(Q + x) = \int v(Q + x) \phi_\varepsilon(y) dy \leq \phi_\varepsilon * \chi_{Q_{1+\varepsilon}} * v(x) \tag{19}$$

as  $v$  is a nonnegative measure. Furthermore,

$$\begin{aligned} &|\phi_\varepsilon * \chi_{Q_{1\pm\varepsilon}} * \Omega_M(x) - \Omega_M(Q + x)| \leq \\ &\leq \int |\Omega_M(Q_{1\pm\varepsilon} + x - y) - \Omega_M(Q + x)| \phi_\varepsilon(y) dy. \end{aligned}$$

As  $y \in \text{supp } \phi_\varepsilon$  implies that the symmetric difference between  $Q_{1\pm\varepsilon} + x - y$  and  $Q + x$  is included in  $\{x; 1 - 2\varepsilon \leq |x_i| \leq 1 + 2\varepsilon\}$  and  $\Omega_M$  has a bounded density, we get

$$|\phi_\varepsilon * \chi_{Q_{1\pm\varepsilon}} * \Omega_M(x) - \Omega_M(Q + x)| \leq C\varepsilon. \tag{20}$$

This completes the proof of the theorem as (19) and (20) implies (4).

REMARK 4: It can be seen from the proof that the theorem is true for any measure that satisfies  $\lim_{|t| \rightarrow \infty} \inf |1 - f(t)| > 0$ . Also  $Q$  can be an arbitrary parallelepiped and the estimate is uniform for  $Q$  in bounded sets.

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