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THE NON RATIONALITY OF THE GENERIC ENRIQUES' THREFOILD

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The Enriques threefold, i.e. the hypersurface of $\mathbb{P}^4$ having as hyperplane sections the classical Enriques surfaces (i.e. the surfaces of degree 6 in $\mathbb{P}^3$, passing through the edges of a tetrahedron), was studied classically by several Authors.

Fano suggested that it was not unirational ([10] p. 94), but Roth proved that it was unirational and, in order to prove the non-rationality, he gave an argument involving the Severi torsion. This point was in disagreement with Serre [12], where it is shown that a non singular unirational variety cannot have torsion. Tyrrel [13] pointed out that Roth's argument was not correct because of the existence of some not ordinary singular points.

In this note we find, for a generic Enriques threefold $V$, a non singular model $\tilde{V}'$ containing an open set $W$, which is a conic bundle (in the sense of [1]) over a suitable surface, with a complete non singular curve $\Delta$ of genus 5 as curve of the degenerate conics. By analyzing $\tilde{V}' - W$ explicitly we prove, as in the case of standard conics bundles, that the Chow group $A^2(\tilde{V}')$ is isomorphic to the Prym variety $Prym(\tilde{A}/\Delta)$.

At the end, since $\Delta$ has genus 5 and so is not included in th. 4.9 of [1], we need some careful analysis about its halfcanonical series, to conclude that $Prym(\tilde{A}/\Delta)$ is not a Jacobian of a curve and therefore $V$ is not rational.

In the complex projective space $\mathbb{P}^4$ of homogeneous coordinates $(x_0 : x_1 : x_2 : x_3 : x_4)$ we consider the irreducible generic hypersurface $V$ of equation:

(*) Lavoro eseguito nell'ambito del G.N.S.A.G.A. del C.N.R.
In particular, \( c_i \neq 0, i = 1, 2, 3, 4 \).

It is known ([11] p. 44, [13] p. 897) that its generic hyperplane section is an Enriques surface and that \( V \) gets the following singularities:

(i) six double planes \( \pi_{ij} \) of equations \( x_i = x_j = 0 \), \( 1 \leq i < j \leq 4 \),

(ii) four triple lines of equations \( x_i = x_j = x_k = 0 \), \( 1 \leq i < j < k \leq 4 \),

(iii) one quadruple point at \( 0(1,0,0,0,0) \) and other two non ordinary quadruple points on each triple line.

It is also known that \( V \) is unirational ([10] p. 97).

In order to prove that \( V \) is non rational, we consider the following rational map:

\[
\varphi : \mathbb{P}^4(x_0 : x_1 : x_2 : x_3 : x_4) - \rightarrow \mathbb{P}^3(x : y : z : t)
\]
given by

\[
x : y : z : t = x_1x_3 : x_1x_4 : x_2x_3 : x_2x_4.
\]

\( \varphi \) is not defined over the planes \( \pi_{12} \) and \( \pi_{34} \), moreover the image of \( \varphi \) is the quadric surface \( Q \subseteq \mathbb{P}^3 \) of equation \( xt = yz \).

**Lemma 1:** For all \( q \in Q \), let \( E_q = \varphi^{-1}(q) \) be the inverse image of \( q \). The Zariski closure of \( E_q \) is a plane in \( \mathbb{P}^4 \) passing through the point \( 0(1,0,0,0,0) \) and intersecting each plane \( \pi_{12} \) and \( \pi_{34} \) along a line.

In other words, \( Q \) parametrizes the planes in \( \mathbb{P}^4 \) cutting these two fixed planes along a line.

**Proof:** Let \( q = (x, y, z, t) \in Q \) and \( p \in E_q \). Since \( p \) doesn’t belong to the planes \( \pi_{12} \) and \( \pi_{34} \), there exist only two hyperplanes passing through \( p \) and containing one of them. Their equations are precisely:

\[
\begin{cases}
\alpha x_1 - \beta x_2 = 0 \\
\gamma x_3 - \delta x_4 = 0
\end{cases}
\]

\((*)\)

where

\[
\alpha : \beta = : \tilde{t} : \tilde{y} = \tilde{z} : \tilde{x}
\]
and

\[
\gamma : \delta = : \tilde{y} : \tilde{x} = \tilde{t} : \tilde{z}.
\]

So the equations \((*)\) define exactly the Zariski closure of \( E_q \).
It follows immediately:

**Lemma 2:** The following (not linearly independent) equations in $\mathbb{P}^4 \times \mathbb{P}^3$

\[
\begin{align*}
xt &= yz \\
zx_1 - xx_2 &= 0 \\
tx_1 - yx_2 &= 0 \\
yx_3 - xx_4 &= 0 \\
tx_3 - zx_4 &= 0
\end{align*}
\]

define the Zariski closure $\Gamma_\varphi$ of the graph of $\varphi$.

**Note.** $\Gamma_\varphi$ can be obtained by blowing up $\mathbb{P}^4$ along the ideal of the planes $\pi_{12}$ and $\pi_{34}$.

**Lemma 3:** The equations of lemma 2, together with the following ones:

\[
\begin{align*}
x^2(c_4x_2^2 + c_2x_4^2) + t^2(c_1x_3^2 + c_3x_2^2) \\
\quad + xt(x_0^2 + x_0 \sum_{i=1}^{4} a_i x_i + \sum_{i,j=1}^{4} b_{ij} x_i x_j) = 0 \\
y^2(c_3x_2^2 + c_2x_3^2) + z^2(c_4x_1^2 + c_1x_4^2) \\
\quad + yz(x_0^2 + x_0 \sum_{i=1}^{4} a_i x_i + \sum_{i,j=1}^{4} b_{ij} x_i x_j) = 0
\end{align*}
\]

define the strict transform $V'$ of $V$ in $\Gamma_\varphi$.

**Proof:** Immediate.

**Remark:** Let $\pi : V' \to Q$ be the restriction to $V'$ of the canonical projection. For the fibre $\pi^{-1}(q)$ of a point $q \in Q$ we have three possibilities:

(1) if all coordinates of $q$ are different from zero, $\pi^{-1}(q)$ is a (possibly degenerate) conic. In fact it is the residual conic cut out on $V$ by the plane $E_q$, apart from the two (double) lines lying on the plane $\pi_{12}$ and $\pi_{34}$.

In the above notations, if $E_q$ has equations

\[
\begin{align*}
\alpha x_1 - \beta x_2 &= 0 \\
\gamma x_3 - \delta x_4 &= 0
\end{align*}
\]
(α, β, γ, δ: fixed), on $E_q$ we may assume homogeneous coordinates $v : u : r$

such that, for a point $p \in E_q$

\[
\begin{align*}
x_0 &= v \\
x_1 &= \beta u \\
x_2 &= \alpha u \\
x_3 &= \delta r \\
x_4 &= \gamma r
\end{align*}
\]

In this coordinate system the conic $\pi^{-1}(q)$ has equation:

\[
\begin{align*}
\alpha \beta \gamma \delta \{v^2 + v[(a_1 \beta + a_2 \alpha)u + (a_3 \delta + a_4 \gamma)r] \\
+ (b_{11} \beta^2 + b_{12} \alpha \beta + b_{22} \alpha^2)u^2 \\
+ (b_{33} \delta^2 + b_{34} \gamma \delta + b_{44} \gamma^2)r^2 \\
+ (b_{13} \beta \delta + b_{14} \beta \gamma + b_{23} \alpha \delta + b_{24} \alpha \gamma)ur \\
+ (c_1 x^2 \gamma^2 \delta^2 + c_2 \beta^2 \gamma \delta^2)r^2 \\
+ (c_3 x^2 \beta^2 \gamma^2 + c_4 x^2 \beta^2 \delta^2)u^2 \} = 0.
\end{align*}
\]

(2) if exactly two coordinates are zero, then $\pi^{-1}(q)$ is a double line.

(3) if three coordinates are zero, $\pi^{-1}(q) = E_q$.

(2) and (3) follow immediately from the equations.

We want to prove that $V$ is birationally equivalent to a conic bundle.

Let $X = (1,0,0,0)$, $Y = (0,1,0,0)$, $Z = (0,0,1,0)$, $T = (0,0,0,1)$ be the four points of the case (3), and blow $Q$ up in these points, or, equivalently, take the strict transform $G$ of $Q$ in the blowing up of $\mathbb{P}^3$ along the two lines of equations $x = t = 0$ and $y = z = 0$. So we realize $G$ in

\[
\mathbb{P}^1(\lambda : \mu) \times \mathbb{P}^1(v : \rho) \times \mathbb{P}^3(x : y : z : t)
\]

by the equations

\[
\begin{align*}
\lambda x - \mu t &= 0 \\
n y - \rho z &= 0 \\
x t - y z &= 0.
\end{align*}
\]

If $\varepsilon : G \to Q$ denotes the structure map, by base-change we obtain a birational morphism $\tilde{\varepsilon} : \tilde{\Gamma}_\varphi = \Gamma_\varphi \times G \to \Gamma_\varphi$ and a structure map $\tilde{\pi} : \tilde{\Gamma}_\varphi \to G$.

The strict transform $\tilde{V}$ of $V'$ in $\tilde{\Gamma}_\varphi$ has equations in

\[
\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^4
\]

\[
\begin{align*}
\lambda x - \mu t &= 0 \\
n y - \rho z &= 0 \\
x t - y z &= 0
\end{align*}
\]

(**)
where

\[ F = x_0^2 + x_0 \sum_{i=1}^{4} a_i x_i + \sum_{i,j=1}^{4} b_{ij} x_i x_j. \]

It follows that \( \tilde{\pi} : \tilde{V} \to G \) is a "conic bundle" birationally equivalent to \( V \).

In fact, if \( g \in G \) \( \varepsilon(g) = X \) (or \( Y, Z, T \)) it follows from those equations that \( \tilde{\pi}^{-1}(g) \) is still a conic, precisely, if \( \varepsilon(g) = X \), it is:

\[
\begin{align*}
 y &= z = t = \lambda = x_2 = x_4 \\
 &= \rho^2 c_2 x_3^2 + \nu^2 c_4 x_1^2 + \nu \rho [x_0^2 + x_0(a_1 x_1 + a_3 x_3)] \\
 &+ (b_{11} x_1^2 + b_{13} x_1 x_3 + b_{33} x_3^2) = 0
\end{align*}
\]

Nevertheless, we'll see that \( \tilde{V} \) still gets some singularities.

First of all, we want to study the locus of the degenerate conics. We have the following

**Proposition 1:** The locus of the degenerate conics for \( \tilde{\pi} : \tilde{V} \to G \) is given by:

- a non singular curve \( \Delta \) parametrizing the conics of rank 2,
- four lines (disjoint from \( \Delta \) and not intersecting each other), parametrizing the double lines.

**Proof:** At first we study \( \pi : V' \to Q \).

The condition for a conic \( \pi^{-1}(q) \) in order to be degenerate is the following:

\[
\begin{align*}
&\alpha^2 \beta^2 \gamma^2 \delta^2 \{4[\gamma \delta(b_{11} \beta^2 + b_{12} \alpha \beta + b_{22} \alpha^2) + \alpha \beta(c_3 \gamma^2 + c_4 \delta^2)] \\
&\cdot [\alpha \beta(b_{33} \delta^2 + b_{34} \gamma \delta + b_{44} \gamma^2) + \gamma \delta(c_1 \alpha^2 + c_2 \beta^2)] + \alpha \beta \gamma \delta(a_1 \beta + a_2 \alpha) \\
&\cdot (a_3 \delta + a_4 \gamma)(b_{13} \beta \delta + b_{14} \beta \gamma + b_{23} \alpha \delta + b_{24} \alpha \gamma) - \alpha \beta (a_3 \delta + a_4 \gamma)^2 \\
&\cdot [\gamma \delta(b_{11} \beta^2 + b_{12} \alpha \beta + b_{22} \alpha^2) + \alpha \beta(c_3 \gamma^2 + c_4 \delta^2)] - \gamma \delta(a_1 \beta + a_2 \alpha)^2 \\
&\cdot [\alpha \beta(b_{33} \delta^2 + b_{34} \gamma \delta + b_{44} \gamma^2) + \gamma \delta(c_1 \alpha^2 + c_2 \beta^2)] \\
&- \alpha \beta \gamma \delta(b_{13} \beta \delta + b_{14} \beta \gamma + b_{23} \alpha \delta + b_{24} \alpha \gamma)^2 = 0.
\end{align*}
\]
Therefore the curve of the degenerate conics on $Q$ is given by:

(i) four lines, parametrizing the double lines

$$
\begin{align*}
    z &= t = 0 & \text{(for } \alpha = 0) \\
    x &= y = 0 & \text{(for } \beta = 0) \\
    y &= t = 0 & \text{(for } \gamma = 0) \\
    x &= z = 0 & \text{(for } \delta = 0)
\end{align*}
$$

(ii) the curve $C \subseteq Q$ of type $(4,4)$ of equations

$$
xt - yz = 0 \\
4[(b_{11}xy + b_{12}xt + b_{22}zt + c_3yt + c_4xz) \\
\times (b_{33}xz + b_{34}xt + b_{44}yt + c_1zt + c_2xy)] \\
+ xt(a_1a_3x + a_1a_4y + a_2a_3z + a_2a_4t) \\
\times (b_{13}x + b_{14}y + b_{23}z + b_{24}t) \\
- (a_3^2xz + 2a_3a_4xt + a_4^2yt) \\
\times (b_{11}xy + b_{12}xt + b_{22}zt + c_3yt + c_4xz) \\
- (a_1^2xy + 2a_1a_2xt + a_2^2zt) \\
\times (b_{33}xz + b_{34}xt + b_{44}yt + c_1zt + c_2xy) \\
- xt(b_{13}x + b_{14}y + b_{23}z + b_{24}t)^2 = 0
$$

$C$ is the complete intersection of $Q$ and a quartic surface $R$.

A simple direct computation shows that $X, Y, Z, T$ are ordinary double points of $C$, and $C$ is not tangent to the four fundamental lines lying on $Q$.

Moreover, we may see that $\pi^{-1}(q)$ has rank 2, $\forall q \in C - \{X, Y, Z, T\}$ (it follows from considerations on the minors of order 2 in the discriminant of $\pi^{-1}(q)$). Hence $C$ must be non singular in $q$ (cf. [1] prop. 1.2).

Therefore, for a generic $V$, the strict transform $\Lambda$ of $C$ in the blowing up $\varepsilon : G \to Q$ is non singular and doesn’t intersect the strict transform of the four fundamental lines of $Q$.

Now, in order to examine the singularities of $\check{V}$, we denote by $\check{H}$ the section of $\check{F}_\phi$ with $x_0 = 0$.

$\check{H}$ is birationally equivalent to the hyperplane $H$ of $\mathbb{P}^4$ of equation $x_0 = 0$. By projecting from the point $0(1,0,0,0,0)$, we obtain a rational map $\eta : V \to H$, which is $2 - 1$ outside the double planes of $V$. By base-change we get a fibre-diagram
LEMMA 4: The ramification locus $R_\eta$ of $\tilde{\eta}$ has equations on $\tilde{V}$

\[
\lambda \mu (2x_0 + \sum_{i=1}^{4} a_i x_i) = 0
\]
\[
\nu \rho (2x_0 + \sum_{i=1}^{4} a_i x_i) = 0
\]

PROOF: The restriction of $\tilde{\eta}$ to $k_g = \tilde{\eta}^{-1}(g) \subseteq E_g$ coincides with the projection of $k_g$ on the "line at infinity" of $E_g$. So we get the ramification points of $\tilde{\eta}|_{k_g}$ by intersecting $k_g$ with the polar line of 0 to $k_g$, or, equivalently, with the polar hyperplane of 0 to the quadric hypersurfaces obtained by fixing the coordinates of $g$ in the equations (***) of $\tilde{V}$.

In this way we find exactly the required equations.

COROLLARY: $R_\eta$ is the union of the following sections of $\tilde{V}$:

\[
\tilde{V} \cap \{ \lambda = \nu = 0 \} = A_{\lambda \nu}
\]
\[
\tilde{V} \cap \{ \lambda = \rho = 0 \} = A_{\lambda \rho}
\]
\[
\tilde{V} \cap \{ \mu = \nu = 0 \} = A_{\mu \nu}
\]
\[
\tilde{V} \cap \{ \mu = \rho = 0 \} = A_{\mu \rho}
\]
\[
\tilde{V} \cap \{ 2x_0 + \sum_{i=1}^{4} a_i x_i \} = B
\]

PROPOSITION 2: $A_{ij} (i, j = \lambda, \mu, \nu, \rho)$ is a smooth quadric surface. $B$ is a smooth Enriques surface.

PROOF: 1) Let's consider, for example, $A_{\lambda \nu}$. Its equations in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^4$ are given by:

\[
\lambda = \nu = t = z = x_2 = x_3 = x_4 = 0,
\]

so they determine a smooth surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$:

\[
A_{\lambda \nu} = \{ t = z = 0 \} \times \{ x_2 = x_3 = x_4 = 0 \}.
\]
2) $B$ is an Enriques surface.

In fact the ramification divisor of the map $\eta: V \rightarrow H$ has equation, in the coordinates $(x_1 : x_2 : x_3 : x_4)$, the discriminant of the polynomial of degree two in $x_0$ defining $V$ in $\mathbb{P}^4$, i.e.

$$x_1x_2x_3x_4\left[\frac{\sum_{i=1}^{4} a_i x_i}{2} - 4\sum_{i,j=1}^{4} b_{ij} x_i x_j - 4c_1x_2^2x_3^2x_4^2 - 4c_3x_4^2x_2^2x_3^2 - 4c_4x_2^2x_3^2x_4^2\right] = 0.$$  

The expression contained in the square brackets is the canonical equation of an Enriques surface in $\mathbb{P}^3$, which is birationally equivalent to $B$ by means of $\eta$.

It is possible to verify on the equations that the blowing up defining $\tilde{V}$ induces a desingularization of this Enriques surface, but we can also observe directly that $\tilde{\pi}|_B: B \rightarrow G$ is a double covering with ramification divisor

$$R_\pi = (A_{x_1} + A_{x_2} + A_{x_3} + A_{x_4}) \cdot B + \Delta'$$

where $\tilde{\pi}(\Delta') = \Delta$ is the irreducible smooth curve of $G$ studied in prop. 1. So $A_{x_1} \cdot B = L_{x_1}$ is a line of equations

$$\lambda = v = t = z = x_2 = x_3 = x_4 = 2x_0 + \sum_{i=1}^{4} a_i x_i = 0$$

and

$$\tilde{\pi}(L_{x_1}) = L_{x_1} = \{t = z = 0\}$$

is a fundamental line on $G$ parametrizing the conics of rank 1. It follows that $R_\pi$ is a (reducible) smooth curve, and therefore $B$ is non singular.

**Proposition 3:** $\tilde{V}$ is non singular, except for four couples of lines contained in $A_{x_1}, A_{x_2}, A_{x_3}, A_{x_4}$, having equations

$$\lambda = v = t = z = x_2 = x_3 = x_4 = (x_0^2 + a_1 x_0 x_4 + b_{11}x_1^2) = 0$$

$$\lambda = \rho = y = t = x_1 = x_2 = x_4 = (x_0^2 + a_3 x_0 x_3 + b_{33}x_3^2) = 0$$

$$\mu = v = x = z = x_1 = x_2 = x_3 = (x_0^2 + a_4 x_0 x_4 + b_{44}x_4^2) = 0$$

$$\mu = \rho = x = y = x_1 = x_3 = x_4 = (x_0^2 + a_2 x_0 x_2 + b_{22}x_2^2) = 0.$$  

Moreover all these points are ordinary double points.
PROOF: Let $D = A_{\lambda v} + A_{\lambda \rho} + A_{\mu v} + A_{\mu \rho}$. Then $\tilde{\eta}: \tilde{V} - D \longrightarrow \tilde{H} - D$ is a double covering with smooth ramification locus, therefore it is non singular and all singular points of $\tilde{V}$ are necessarily belonging to $D$.

It suffices to consider one of the connected components of $D$, for example $A_{\lambda v}$, and to argue locally. So, taking the open set $U \subseteq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3 \times \mathbb{P}^4$ where $\mu = \rho = x = x_0 = 1$ and assuming affine coordinates $(\lambda, v, y, z, t, x_1, x_2, x_3, x_4)$, we easily see that the tangent space at any point $p = (0, 0, \tilde{y}, 0, 0, \tilde{x}_1, 0, 0, 0) \in A_{\lambda v} \cap U$ to $\tilde{V} \cap U$ is given by the following (not linearly independent) equations:

$$
\begin{align*}
t - \tilde{y}z &= 0 \\
\lambda - t &= 0 \\
v\tilde{y} - z &= 0 \\
z\tilde{x}_1 - x_2 &= 0 \\
t\tilde{x}_1 - \tilde{y}x_2 &= 0 \\
\tilde{y}x_3 - x_4 &= 0 \\
\lambda(1 + a_1\tilde{x}_1 + b_{11}\tilde{x}_1^2) &= 0 \\
v(1 + a_1\tilde{x}_1 + b_{11}\tilde{x}_1^2) &= 0
\end{align*}
$$

Since $\dim T_{p, p} \geq 3$, at most six of them are linearly independent. Two different cases are possible:

1) $1 + a_1\tilde{x}_1 + b_{11}\tilde{x}_1^2 \neq 0$

Then $\lambda = v = t = z = x_2 = \tilde{y}x_3 - x_4 = 0$ are six independent equations, so $\dim T_{p, p} = 3$ and $\tilde{V}$ is non singular at $p$.

2) $1 + a_1\tilde{x}_1 + b_{11}\tilde{x}_1^2 = 0$.

In this case the last two equations are identically zero, and the first six are related by the (unique) relation

$$
\tilde{x}_1(t - \tilde{y}z) = (t\tilde{x}_1 - \tilde{y}x_2) - \tilde{y}(\tilde{x}_1z - x_2)
$$

(Note that $\tilde{x}_1 \neq 0$). So $\dim T_{p, p} = 4$.

To determine the tangent cone at $p$ to $\tilde{V}$, we assume as local parameters at $p$ to $\tilde{F}_{p}$ (which is a non singular four-dimensional variety), for example, $v, x_3, y' = y - \tilde{y}, x_1 = x_1 - \tilde{x}_1$ and we obtain a term of lower degree of the kind

$$
Av^2 + Bvx' + Cvx_3 + c_2x_3^2 = 0 \quad (\text{***})
$$
where

\[ A = c_4 x_1^2 + c_3 x_1^2 y^2 + a_2 x_1 y + b_{12} x_1^2 y \]
\[ B = a_1 + 2b_{11} x_1 \]
\[ C = a_3 + a_4 y + b_{13} x_1 + b_{14} y x_1 \]

It is a quadric cone over the conic (***) with discriminant

\[ H = c_2 (a_1 + 2b_{11} x_1)^2 \]

(remember that \( c_2 \neq 0 \)). For a generic \( V, H \neq 0 \), so \( p \) is an ordinary double point. It is immediate to see that from \( H = 0 \) it follows that the solutions of the equation 2) (and so either the two corresponding lines on \( V \) and the two quadruple points on the line \( x_2 = x_3 = x_4 = 0 \) on \( V \)) coincide.

**Corollary:** A non singular model for \( V \) is given by the strict transform \( V' \) of \( V \) in the blowing up of \( \Gamma_\phi \) along these eight singular lines.

**Proof:** It can be done directly in the above local coordinates. In particular, for each line blown up we get an exceptional quadric.

**Remark:** In conclusion, we have got a non singular model \( V' \) of \( V \) and a map \( \tilde{f} : V' \to G \) whose fibres are:

i) a non singular conic if \( g \notin A \cup \{ L_{ij} \} \)

ii) two different lines if \( g \in A \)

iii) a double line and two conics if \( g \in L_{ij} \).

Therefore the inverse image \( \bar{V} \) of the four foundamental lines \( L_{ij} \) is a union of quadrics (the \( A_{ij} \)'s and the exceptional ones).

\( W = \bar{V}' - \bar{V} \) is isomorphic to \( \bar{V} - \cup A_{ij} \), so that it is a non singular conic bundle over \( G - \cup L_{ij} \).

\( \Delta \) is the complete non singular curve of degenerate conics of the bundle.

We can construct in a standard way (see [1] 1.5) a double covering \( q : \tilde{\Delta} \to \Delta \) such that every point \( t \in \tilde{\Delta} \) parametrizes one of the two lines contained in the conic \( k_{q(t)} \). Let us call this line \( L(t) \) and look to it as an element of \( C^2(W) \). By similar arguments as in [1] 3.1, and considering also [1] 3.1.9 one can prove the following

**Proposition 4:** The map \( t \mapsto L(t) \) extends to a surjective homomorphism
\[ \varphi : J(\tilde{A}) \to A^2(W) \]

whose kernel is \( q^*J(\Delta) \). Taking the quotient, we obtain an isomorphism

\[ \psi : P = \text{Prym}(\tilde{A}/\Delta) \to A^2(W). \]

**Corollary:** \( P \xrightarrow{\sim} A^2(V') \).

**Proof:** Let \( \tilde{Y} \) be the desingularization of \( Y \). We have the exact sequence

\[ A^1(\tilde{Y}) \to A^2(\tilde{V}') \to A^2(W) \to 0 \]

and \( A^1(\tilde{Y}) = 0 \) since \( Y \) is the union of quadric surfaces. We observe that this is a group isomorphism.

**Proposition 5:** \( P \) is the algebraic representative of \( A^2(\tilde{V}') \) (cfr. [1] def. 3.2.3.), and the principal polarization \( \mathcal{H} \) of \( P \) is the incidence polarization relative to \( X \) ([1] def. 3.4.2.).

**Proof:** It can be shown by the same arguments as [1] Prop. 3.3 and [1] Prop. 3.5.

**Lemma 5:** Let \( C \) be a canonical curve in \( \mathbb{P}^4 \) which is a complete intersection:

1. \( C \) has a half-canonical \( g_4^1 \) if and only if \( C \) is contained in a quadric \( U \) of rank three;
2. the unique ruling of two-planes of \( U \) cuts out the half-canonical \( g_4^1 \) on \( C \).

**Proof:** (1) Let us suppose that \( C \) has a half-canonical \( g_4^1 \); if \( D \) is an effective divisor belonging to the \( g_4^1 \) then, by Riemann–Roch Theorem and the hypothesis \( 2D \sim K \), it follows

\[ h^\circ(D) = 2 = h^\circ(K - D). \]

Since \( K \) is cut out by the hyperplane sections, \( h^\circ(K - D) = 2 \) means that \( \text{Supp} \, D \subset \pi \) where \( \pi \) is a two-plane; now we can take another
effective divisor $D'$ which is linearly equivalent to $D$ and, without loss of
generality, we can assume $\text{Supp } D \cap \text{Supp } D' = \emptyset$ (if not $C$ will have a $g_3^1$
and will not be a complete intersection). Let $\pi'$ be the two-plane con-
taining $\text{Supp } D'$: since $D + D' \sim K$ it follows that $\pi \cup \pi'$ is contained in
an hyperplane $H$ of $\mathbb{P}^4$ and that $\pi, \pi'$ intersect along a line $u$. Moreover
there is only one net of quadric surfaces in $H$ passing through
$\text{Supp } D \cup \text{Supp } D'$, so that they are sections by $H$ of the quadric hyper-
surfaces through $C$. Since $\pi \cup \pi'$ belongs to the net, there is a quadric
hypersurface $U$ containing $C \cup \pi \cup \pi'$.

$U$ is singular because it contains some 2-plane, then its rank may be
equal to 3 or 4. If $U$ had rank 4 then $U$ will be a cone over a quadric
surface $S$ in $\mathbb{P}^3$. In this case $\pi$ will be a plane through the vertex of the
cone and a line $l$ of $S$; then $D$ will be cut out on $C$ by the two-planes
through the vertex of $U$ and a line in the same ruling of $l$; which is
absurd since $\pi'$ belongs clearly to the other ruling of two-planes of $U$.
Then $U$ must have rank 3 and its singular locus has to be the line
$u = \pi \cap \pi'$.

Viceversa and (2) follow easily by the above arguments.

**Lemma 6:** Let $V$ be a generic Enriques' threefold, then a canonical model
of $A$ is the complete intersection in $\mathbb{P}^4$ of three quadric hypersurfaces
$Q_1, Q_2, Q_3$ where $Q_1, Q_2$ have rank 3 and $Q_3$ is generic.

**Proof:** We considered a singular model $C$ of $A$ given by the following
equations in $\mathbb{P}^3 (x: y: z: t)$, (see Prop. 1):

\[ xt - zy = 0 \]
\[ F_4(x, y, z, t) = 4^4 \cdot [(b_{11}xy + b_{12}xt + b_{22}zt + c_3yt + c_4xz) \]
\[ \times (b_{33}xz + b_{34}xt + b_{44}yt + c_1zt + c_2xy)] \]
\[ + xt \cdot [(a_1a_4y + a_2a_3z)(b_{13}x + b_{14}y + b_{23}z + b_{24}t) \]
\[ - (b_{14}y + b_{23}z)^2 + zy[(a_1a_3x + a_2a_4t) \]
\[ \times (b_{13}x + b_{14}y + b_{23}z + b_{24}t) - (b_{13}x + b_{24}t)^2] \]
\[ - 2(b_{13}x + b_{24}t)(b_{14}y + b_{23}z)xt - (a_3^2xz + 2a_3a_4xt + a_4^2yt) \]
\[ \times (b_{11}xy + b_{12}xt + b_{22}zt + c_3yt + c_4xz) \]
\[ - (a_1^2xy + 2a_1a_2xt + a_2^2zt) \]
\[ \times (b_{33}xz + b_{34}xt + b_{44}yt + c_1zt + c_2xy) = 0. \]

$C$ is of type $(4, 4)$ in the quadric surface $Q = \{ xt - zy = 0 \}$ and has
four ordinary double points: $X(1:0:0:0), Y(0:1:0:0), Z(0:0:1:0),
T(0:0:0:1)$. The linear system of quadric surfaces in $\mathbb{P}^3$ containing the
four double points and distinct from $Q$ cuts out on $C$ the canonical system. Moreover it defines a rational morphism

$$\Phi: \mathbb{P}^3 \longrightarrow \mathbb{P}^4$$

which desingularizes $C$ and embeds it canonically in $\mathbb{P}^4$. Indeed the equations of $\Phi$ can be defined by setting

$$u_0 = xy, \quad u_1 = xz, \quad u_2 = xt, \quad u_3 = yt, \quad u_4 = zt$$

where $(u_0: u_1: u_2: u_3: u_4)$ are projective coordinates in $\mathbb{P}^4$. Then the strict transform of $Q$ is the intersection of the two quadric hypersurfaces of rank three:

$$Q_1: u_2^2 - u_0u_4 = 0, \quad Q_2: u_2^2 - u_1u_3 = 0.$$

Moreover the quartic form $F_4(x, y, z, t)$ can also be written as a quadratic form $F(xy, xz, xt, yt, zt)$ in $xy, xz, xt, yt, zt$. It follows immediately that the strict transform $A'$ of $C$ in $\mathbb{P}^4$ has equations:

$$u_2^2 - u_1u_3 = u_2^2 - u_0u_4 = 0$$

$$F(u_0, u_1, u_2, u_3, u_4) = 0$$

where $F(u_0, u_1, u_2, u_3, u_4)$ is a quadratic form.

Then the affine space of the coefficients of the equation of $V$ maps on the affine space of the coefficients of a quadratic form $F \in \mathbb{C}[u_0, u_1, u_2, u_3, u_4]$. One can compute directly that this map is of maximal rank and surjective.

From this fact we can argue that $A'$ is the complete intersection of $\mathbb{P}^4$ of $Q_1, Q_2$ and a third generic quadratic hypersurface $Q_3$. In particular $A'$ is smooth and canonically embedded in $\mathbb{P}^4$.

**Corollary 3**: Let $V$ be a generic Enriques' threefold:

(i) $A$ is not hyperelliptic, trigonal, nor elliptic-hyperelliptic;
(ii) $A$ has two half-canonical $g_1^4$'s $L_1, L_2$;
(iii) $A$ does not contain a half-canonical divisor $N$ such that $N \not\sim L_i$, $(i = 1, 2)$, $h^0(N) \neq 0$, $h^0(N)$ even.

**Proof**: (i) follows from the proof of the above lemma and from the fact that $Q_3$ is generic. (ii) follows from Lemma 5. Now we show (iii): by (i) and Lemma 6 $A$ is the base locus of a net $\Sigma$ of quadric hypersurfaces.
containing the 2 quadrics $Q_1, Q_2$ of rank 3. Moreover, being $Q_3$ generic, \( \Sigma \) is generic in the family of nets as above, so that \( \Sigma \) does not contain a third quadric of rank 3 different from $Q_1, Q_2$. Then, by Lemma 5, \( A \) cannot carry a half-canonical divisor \( N \) with \( h^\sigma(N) = 2 \) and \( N \not\sim L_i \). In the end, if \( h^\sigma(N) = 4 \), \( A \) will be clearly elliptic or rational which is absurd.

**REMARK:** By the corollary above \( A \) is generic among the curves of genus 5 having 2 and only 2 half-canonical $g^1_4$'s. Thinking of \( A \) as a singular curve of type (4,4) in the quadric $Q$ (see Prop. 1) these $g^1_4$'s arise by intersection with the two rulings of lines in $Q$.

Let us consider now the étale double covering of $A$:

$$q: \overline{A} \to A$$

(see the remark before Prop. 4); we will compute the semiperiod giving such a covering.

We have seen (see the remark before Prop. 1) that there is a birational morphism of $V$ with a (singular) conic bundle $\mathcal{V}$ on the surface $G$. $G$ is the blowing up

$$\varepsilon: G \to Q$$

of the quadric surface $Q = \{xt - yz = 0\}$ in the four fundamental points of $\mathbb{P}^3(x: y: z: t)$. Let $\pi: \mathcal{V} \to G$ be the map fibering $\mathcal{V}$ in conics; \( \forall g \in G \) the conic $\pi^{-1}(g)$ is obtained, via the birational morphism from $V$ to $\mathcal{V}$, from a conic $K_g$ in $V$ contained in a 2-plane $E_g$ meeting both the 2-plane $\pi_{12} = \{x_1 = x_2 = 0\}$, $\pi_{34} = \{x_3 = x_4 = 0\}$ along a line (Lemma 1).

**LEMMA 7:** The locus in $G$:

$$\{g \in G/K_g \cap (E_g \cap \pi_{12}) \text{ is exactly one point} \}$$

is given by:

(i) a non singular elliptic curve $\nabla \subset G$ which is the strict transform, via $\varepsilon: G \to Q$, of a quartic elliptic curve in $Q$ passing through the four fundamental points of $\varepsilon^{-1}$;

(ii) two rational curves $l_1, l_2$ which are the strict transforms of the lines $\{y = t = 0\}$, $\{x = z = 0\}$ belonging to the same ruling in $Q$.

**PROOF:** The equation of a conic $K_g \subset E_g = \{ax_1 - \beta x_2 = \gamma x_3 - \delta x_4$
= 0} \subset \mathbb{P}^4 is given in the remark following Lemma 3. The coefficients of such a equation depend on \((x : \beta) \times (y : \delta)\), the projective coordinates on \(E_g\) are \((u : v : r)\) and the line \(E_g \cap \pi_{12}\) is given by setting \(u = 0\). It turns out easily that, if \(K_g\) satisfies the required condition, then \((x : \beta) \times (y : \delta)\) annihilates the following equation:

\[
(x\beta y^2 \delta^2) \cdot [x\beta (a_3 \delta + a_4 y)^2 - 4x\beta (b_{33} \delta^2 + b_{34} \gamma \delta + b_{44} \gamma^2)] = 0.
\]

With the same notations of Lemma 1 we have \(\alpha : \beta = t : y = z : x; \gamma : \delta = y : x = t : z\) so that the set of zeroes of the second factor of the above equation becomes the locus in \(\mathbb{P}^3(x : y : z : t)\):

\[
xt - yz = 0
\]
\[
(a_3^2 - 4b_{33})xz + (a_4^2 - 4b_{44})ty
\]
\[
+ 2(a_3a_4 - 2b_{34})xt - 4c_1 tz - 4c_2 xy = 0.
\]

If \(V\) is generic this is clearly a smooth quartic elliptic curve in \(Q\), passing through the four fundamental points of \(\mathbb{P}^3\), that is through the fundamental points of \(\varepsilon^{-1}\); this shows (i). To show (ii) we observe that the fibers of \(\tilde{\pi}\) on \(l_2(l_2')\) are double lines (see Prop. 1) and that these double lines arise, by the birational morphism quoted above, from the line \(\{x_3 = x_4 = x_1 = 0\}\) counted twice. This one meets \(\pi_{12}\) twice in the point \((1:0:0:0:0)\) and this shows (ii); moreover it is clear from the geometric situation that the locus we are considering cannot have other components.

**LEMMA 8:** We have on \(G\):

(i) \((\varepsilon(V), l_2) = (\varepsilon(V), l_2') = 0\)

(ii) \(\Delta\) and \(V\) does not meet along the four exceptional divisors of \(G\)

(iii) \((\Delta, V) = 8\) and, for every \(p \in \Delta \cap \nabla, i(p; \Delta \cap \nabla) = 2\).

**PROOF:** \(\varepsilon(\Delta), \varepsilon(V), \varepsilon(l_2), \varepsilon(l_2')\) pass all through the four fundamental points of \(\varepsilon^{-1}\), since \(V\) is generic it is clear from the equation of \(\varepsilon(V)\) written in Lemma 7 that \(\varepsilon(l_2), \varepsilon(l_2')\) are not tangent to \(\varepsilon(V)\); in the same way one can also see that, for every fundamental point 0, the tangent line in 0 to \(\varepsilon(V)\) cannot be a component of the tangent cone to \(\varepsilon(\Delta)\) in 0. This shows (i) and (ii). Now we have on \(Q:\) \((\varepsilon(\Delta), \varepsilon(V)) = 16\); moreover \(\varepsilon(V)\) meets the four singular points of \(\varepsilon(\Delta)\) and these are also the fundamental ones for \(\varepsilon^{-1}\). Then, by (ii), \((\Delta, V) = 8\).
Another direct computation shows that $i(p; A \cap V) = 2$ for every $p \in A \cap V$.

Let us consider now the double covering:

$$f: \tilde{\mathcal{G}} \to G$$

branched over $\nabla \cup l_2 \cup l'_2$. $\tilde{\mathcal{G}}$ is smooth since $\nabla \cup l_2 \cup l'_2$ is smooth. Moreover the open set $\tilde{\mathcal{G}} - (l_2 \cup l'_2)$ parametrizes the couples $(g, x)$ where $g \in G$ and $x \in K_g \cap \pi_{12}$. It follows that $f^{-1}(A)$ parametrizes the lines being components of the degenerate conics $K_g$ of rank 2. Then $f^{-1}(A)$ is a (singular) model of $\tilde{\mathcal{G}}$. Indeed $f^{-1}(A)$ is singular exactly in the four points of the set $f^{-1}(A \cap V)$: this can be obtained, with a local computation, by observing that, for every such a point $x$, $i(f(x); A \cap V) = 2$ and $V$ belongs to the branch locus of $f$.

Clearly we have the commutative diagram:

$$\begin{array}{ccc}
\tilde{\mathcal{G}} & \xrightarrow{q} & A \\
\downarrow \nu & & \downarrow \nu \\
f^{-1}(A) & \xrightarrow{f| f^{-1}(A)} & A
\end{array}$$

where $\nu$ is the normalization morphism.

Let us call $L_1$ a divisor on $A$ belonging to the halfcanonical $g_4^1$ cut out on $\mathfrak{g}(A)$ by the lines of $Q$ not in the ruling of $\mathfrak{g}(l_2)$; let us call $L_2$ a divisor in the other half-canonical $g_4^1$ of $A$, (see corollary 3), we have the following

**Proposition 6:** If $\{p_1, p_2, p_3, p_4\} = A \cap V$ and $D = p_1 + p_2 + p_3 + p_4$ on $A$ then

$$\eta = D - L_1$$

is the semiperiod giving the étale double covering $q: \tilde{\mathcal{G}} \to A$.

**Proof:** On $G$ we have $V \sim 2l_1 + l_2 + l'_2$ where $l_1$ is the (global) transform of a line of $Q$ not in the ruling of $\mathfrak{g}(l_2)$.

Then $O_d(V - 2l_1 - l_2 - l'_2) \cong O_d(2D - 2L_1) \cong O_d$ so that $\eta = D - L_1$ is a semiperiod.

Observe now that $\tilde{\mathcal{G}}$ is a (smooth) rational surface: let $m$ be the transform of a generic line $\mathfrak{g}(m) \sim \mathfrak{g}(l_2)$; since $(m, V + l_2 + l'_2) = 2$ then $f: f^{-1}(m) \to m$ is a double covering of $\mathbb{P}^1$ branched on two points. It follows that $\tilde{\mathcal{G}}$ carries a pencil of rational curves so that, by Noether’s theorem, it is a rational surface.
Since $\mathbf{V} + l_2 + l'_2$ is the branch locus of $f$ it turns out that

$$2f^{-1}(\mathbf{V}) - l_2 - l'_2 \sim f^*(2l_1) \sim 2f^*(l_1)$$

and, since Pic $\tilde{G}$ has no torsion (being $\tilde{G}$ rational),

$$f^{-1}(\mathbf{V} - l_2 - l'_2) \sim f^*(l_1).$$

By setting $A_s = f^{-1}(A)$ we have:

$$O_{3}(f^{-1}(\mathbf{V} - l_2 - l'_2) - f^*(l_1)) \cong O_{A_s}$$

that is:

$$O_{A_s} \cong O_{3}(\nu f^*(D - L_1)) \cong O_{3}(q^*\eta).$$

Then $q^*\eta$ is trivial on $A$: this happens if and only if $q: A \to A$ is given by $\eta$.

**Remark:** $\eta \not\sim L_1 - L_2$: since $2D$ is cut out on $\mathcal{C}(\mathbf{A})$ by an elliptic curve of type $(2, 2)$ on $Q$, it follows that $\text{Supp} \ D$ cannot be contained in a line of $Q$, so that $D \not\sim L_i$. This shows also that $\eta$ cannot be trivial.

**Corollary 4:** $\eta \sim D' - L_2$ where $2D'$ is cut out on $\Delta$ by a smooth elliptic curve $\mathbf{V}'$ parametrizing the conics $K_g$ of rank $\geq 2$ such that $K_g \cap \pi_{34}$ is exactly one point.

**Proof:** Exactly as to show $\eta \sim D - L_1$: it suffices to substitute $\pi_{12}$ with $\pi_{34}$ and $l_2, l'_2$ with the corresponding rational curves $l_1, l'_1$ strict transforms of the lines $\{z = t = 0\}, \{x = y = 0\}$.

**Corollary 5:** If $V$ is generic, on $\Delta$ there is no effective even theta characteristic $N$ such that $h^\circ(N + \eta)$ is even.

Moreover $h^\circ(L_i + \eta) = 1$.

**Proof:** If $V$ is generic on $\Delta$ there are only two effective even theta characteristics: namely $L_1, L_2$, (see Corollary 3). Since $\eta \sim D - L_1 \sim D' - L_2$ it follows that $L_i + \eta$ is effective so that $h^\circ(L_i + \eta) \neq 0$. Now we cannot have $h^\circ(L_i + \eta) > 2$ unless $\Delta$ is elliptic or rational which is absurd, nor $h^\circ(L'_i + \eta) = 2$ since $L_1 + \eta \not\sim L_2$. Then $h^\circ(L_i + \eta) = 1.$
PROPOSITION 7: A generic Enriques' threefold $V$ is not rational.

PROOF: Let us consider the étale double covering $q: \tilde{V} \to V$: the Prym variety associated to $q$ is an abelian variety $P$ with principal polarization $\mathcal{P}$. Moreover $P$ is the algebraic representant of $A^2(\tilde{V})$ and $\mathcal{P}$ is the incidence polarization (see Prop. 5). Then, by [1] Prop. 4.6, it suffices to show that $(P, \mathcal{P})$ as a principally polarized abelian variety, is not isomorphic to a product of jacobians of curves.

To get this result we observe that, by Corollary 3, $\Delta$ cannot be hyperelliptic, trigonal nor elliptic-hyperelliptic. Moreover $\Delta$ has 2 and only 2 even effective theta characteristics: $L_1, L_2$. By Proposition 6 and Corollary 4 $q$ is given by $\eta \sim D - L_1 \sim D' - L_2$; and by Corollary 5 $\Delta$ cannot carry an even effective theta characteristic $N$ such that $h^0(N + \eta)$ is even. Then it follows from [6] Theorem 7 (d) pag. 344 that $(P, \mathcal{P})$ cannot be a jacobian nor a product of jacobians of curves.

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