S. Kamienny
G. Stevens

Special values of $L$-functions attached to $X_1(N)$

Compositio Mathematica, tome 49, n° 1 (1983), p. 121-142

<http://www.numdam.org/item?id=CM_1983__49_1_121_0>
SPECIAL VALUES OF L-FUNCTIONS ATTACHED TO $X_1(N)$

S. Kamienny and G. Stevens

Introduction

The conjectures of Birch and Swinnerton-Dyer have stimulated recent interest in the arithmetic properties of special values of $L$-functions. B. Mazur [7] has proven a weak analog of these conjectures for the Jacobian of the modular curve $X_0(N)$, $N$ prime. Crucial in his work are formulae for "universal special values" modulo the Eisenstein ideal.

In the present paper we show how Mazur's formulae can be extended to $X_1(N)$. We construct a cohomology class $\varphi \in H^1(X_1(N); A)$ and produce formulae for its "special values" in Theorem 3.4. Using a generalization by E. Friedman [2] of a result of L. Washington [12] we prove a nonvanishing result for these "special values" in Theorem 4.2.

The aim of §§5 and 6 is to use Theorem 3.4 to prove congruences involving the special values $L(f, \chi, 1)$ attached to a weight two cusp form $f$ on $X_1(N)$. This goal has not been entirely achieved. In Theorem 6.3 we use a technical assumption of local freeness of the homology of $X_1(N)$. For $X_0(N)$ Mazur [6] was able to prove this condition. Unfortunately his proof does not work in our case.

Finally, §8 illustrates these results with the example $X_1(13)$.

§1. Universal special values of $L$-functions

Let $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ be the extended upper half plane. Let $N > 3$ be prime and

$$\Gamma = \Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) | c \equiv 0, \ a \equiv d \equiv \pm 1 \ (\text{mod } N) \right\}.$$
Denote by \( Y = Y_1(N) \) the open Riemann surface \( \Gamma \backslash \mathcal{H} \) and by \( X = X_1(N) \) the associated compactification \( \Gamma \backslash \mathcal{H}^* \). The finite set \( X \backslash Y = \Gamma \backslash \mathbb{P}^1(\mathbb{Q}) \) will be denoted by cusps.

For an arbitrary set \( S \) let \( \text{Div}^0(S) \) be the group of formal finite sums of degree 0 supported on \( S \), i.e.

\[
\text{Div}^0(S) = \left\{ \sum_{s \in S} a_s \cdot \{s\} | a_s \in \mathbb{Z}, \text{almost all } a_s = 0, \text{ and } \sum a_s = 0 \right\}.
\]

The following is a slight variation of a definition given by Mazur ([7], II §1).

**Definition 1.1:** The universal modular symbol attached to \( \Gamma \) is the homomorphism

\[
\text{Univ}: \text{Div}^0(\mathbb{P}^1(\mathbb{Q})) \to H_1(X, \text{cusps}; \mathbb{Z})
\]

defined by \( \text{Univ}(\{r\} - \{s\}) = \) the relative homology class represented by the projection to \( X \) of the oriented geodesic in \( \mathcal{H}^* \) joining \( s \) to \( r \).

The group \( \text{Div}^0(\text{cusps}) \) is naturally identified with the reduced homology group \( \check{H}_0(\text{cusps}; \mathbb{Z}) \). Let \( Z \) be the kernel of the natural projection \( \text{Div}^0(\mathbb{P}^1(\mathbb{Q})) \to \text{Div}^0(\text{cusps}) \). Then the vertical arrows in the following commutative diagram of exact sequences are surjective.

\[
\begin{array}{cccccc}
0 & \to Z & \xrightarrow{\text{Univ}} & \text{Div}^0(\mathbb{P}^1(\mathbb{Q})) & \to & \text{Div}^0(\text{cusps}) & \to 0 \\
\downarrow \text{Univ} & & \downarrow \text{Univ} & & \downarrow \text{Univ} & & \\
0 & \to H_1(X; Z) & \to H_1(X, \text{cusps}; Z) & \to \check{H}_0(\text{cusps}; Z) & \to 0
\end{array}
\]

(1.2)

Let \( \chi: \mathbb{Z} \to \mathbb{C} \) be a nontrivial primitive Dirichlet character of conductor \( m \) prime to \( N \). Let \( \mathbb{Z}[\chi] \) be the ring generated over \( \mathbb{Z} \) by the values of \( \chi \). Then

\[
\sum_{a=0}^{m-1} \left\{ \frac{a}{m} \right\} \otimes \bar{\chi}(a) \in \text{Div}^0(\mathbb{P}^1(\mathbb{Q})) \otimes \mathbb{Z}[\chi]
\]

defines an element of \( \mathbb{Z} \otimes \mathbb{Z}[\chi] \) since the rational numbers \( \frac{a}{m}, (a, m) = 1 \) are \( \Gamma \)-equivalent.

Following Mazur [7] we define:

**Definition 1.3:** The universal special value of the \( L \)-function twisted
by $\chi$ is the homology class

$$A(\chi) = \text{Univ} \left( \sum_{a=0}^{m-1} \left\{ \frac{a}{m} \right\} \otimes \bar{\chi}(a) \right) \in H_1(X; \mathbb{Z}[\chi]).$$

This terminology is justified by the next proposition. Let $f(z) = \sum_{n=1}^{\infty} a_n q^n$, $q = e^{2\pi i z}$, be a weight two cusp form on $\Gamma$, and let $\omega(f)$ be the holomorphic 1-form on $X$ whose pullback to $\mathcal{X}$ is $f \cdot \frac{dq}{q} = 2\pi if(z)dz$. Let $\varphi_f \in H^1(X; \mathbb{C})$ be the cohomology class represented by $\omega(f)$. The $L$-function of $f$ twisted by $\chi$ is defined by the Dirichlet series

$$L(f, \chi, s) = \sum_{n=1}^{\infty} a_n \chi(n)n^{-s}$$

which converges absolutely for $\text{Re}(s) > 3/2$ and continues to an entire function on $\mathbb{C}$.

**Proposition 1.4:** Let $\chi \neq 1$ be a primitive Dirichlet character of conductor $m$ prime to $N$. Then

$$\tau(\bar{\chi})L(f, \chi, 1) = A(\chi) \cap \varphi_f,$$

where $\tau(\bar{\chi}) = \sum_{a=0}^{m-1} \bar{\chi}(a)e^{2\pi ia/m}$ is the usual Gauss sum, and $\cap$ denotes cap product:

$$\cap : H_1(X; \mathbb{C}) \times H^1(X; \mathbb{C}) \to \mathbb{C}.$$ 

*For a proof see Birch [1].* 

We will adopt Mazur's point of view and shift the emphasis from the special values $L(f, \chi, 1)$ to the universal special values $A(\chi)$.

We can define "special values of $L$-functions" attached to an arbitrary cohomology class $\varphi \in H^1(X; A)$ with values in any abelian group $A$. Suppose we have a $\mathbb{Z}[\chi]$-module $A[\chi]$ together with a $\mathbb{Z}[\chi]$-homomorphism

$$A \otimes_{\mathbb{Z}} \mathbb{Z}[\chi] \to A[\chi].$$

Then cap product with $\varphi$ defines a homomorphism

$$\cap \varphi : H_1(X; \mathbb{Z}[\chi]) \to A[\chi].$$
DEFINITION 1.5: The “special value” associated to the pair $\varphi, \chi$ is

$$A(\varphi, \chi) \triangleq A(\chi) \cap \varphi \in A[\chi].$$

§ 2. Modular units and Dedekind sums

For $x \in \mathbb{R}$ define the Bernoulli functions by

$$\mathbb{B}_1(x) = \begin{cases} (x - [x]) - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

and

$$\mathbb{B}_2(x) = (x - [x])^2 - (x - [x]) + \frac{1}{6},$$

where $[x]$ denotes the largest integer $\leq x$. Then $\mathbb{B}_1, \mathbb{B}_2$ define functions $\mathbb{R}/\mathbb{Z} \to \mathbb{R}$.

The Siegel units $g_x, x = (x, y) \in (\mathbb{Z}/N\mathbb{Z})^2$, may be defined by their $q$-expansions:

$$g_{(x, y)}(z) = q^{1/2}{\mathbb{B}_2\left(\frac{x}{N}\right)} \prod_{m \equiv x(N) \atop m > 0} (1 - e^{2\pi iy/N} \cdot q^{m/N}) \prod_{m \equiv -x(N) \atop m > 0} (1 - e^{-2\pi iy/N} \cdot q^{m/N})$$

where $\mathbb{B}_2\left(\frac{x}{N}\right)$ is the periodic second Bernoulli function and $q = e^{2\pi iz}$.

The special function $g_0$ is the square of the Dedekind $\eta$-function. The functions $g_x^{12N}, x \neq 0$ are modular functions of level $N$ which vanish nowhere on $\mathcal{H}$.

The functions $g_x$ have been studied in great detail. For example, Kubert and Lang use them to describe cuspidal groups of modular curves (see [4]).

The transformation formulae for the Siegel units are well-known (see e.g. Schoeneberg [9] where they are referred to as the Dedekind functions). We summarize the results in the next proposition.

**Proposition 2.1:** There is a choice of the logarithms $\log(g_x(z)), x \in (\mathbb{Z}/N\mathbb{Z})^2$ such that for $\gamma \in SL_2(\mathbb{Z})$

$$\pi_x(\gamma) \triangleq (\log(g_x(\gamma z)) - \log(g_x(z)))$$
is independent of \( z \in \mathcal{H} \) and:

1. \( \pi_x(y) = \frac{1}{12N^2} \mathbb{Z}; \)

2. For \( \alpha, \beta \in \text{SL}_2(\mathbb{Z}) \),

\[
\pi_x(\alpha \beta) = \pi_x(\alpha) + \pi_{x\alpha}(\beta);
\]

3. If \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}), \ c \geq 0, \) then

\[
\pi_x(\gamma) = \begin{cases} 
\frac{a}{2c} \mathbb{B}_2 \left( \frac{x}{N} \right) + \frac{d}{2c} \mathbb{B}_2 \left( \frac{ax + cy}{N} \right) - s \left( \frac{a}{N}, \frac{x}{N}, \frac{y}{N} \right) & \text{if } c > 0 \\
\frac{b}{2d} \mathbb{B}_2 \left( \frac{x}{N} \right) & \text{if } c = 0,
\end{cases}
\]

where \( s(\ ) \) is the generalized Dedekind sum of Rademacher ([8], pp. 534–543).

\[
s \left( \frac{a}{N}, \frac{x}{N}, \frac{y}{N} \right) = \sum_{v=0}^{c-1} \mathbb{B}_1 \left( \frac{v}{c} + \frac{x}{N} \right) \cdot \mathbb{B}_1 \left( a \cdot \frac{v}{c} + \frac{x}{N} \right). \]

Let \( \varepsilon \) be an even primitive Dirichlet character of conductor \( N \). Then we may define a homomorphism \( \Gamma_0(N) \rightarrow \mathbb{C}^* \), which we also denote by \( \varepsilon \), by

\[
\varepsilon : \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma_0(N) \mapsto \varepsilon(d).
\]

Define a map \( \Psi : \Gamma_0(N) \rightarrow \mathbb{Q}[\varepsilon] \) by

\[
\Psi(\alpha) = \sum_{x \in \mathbb{Z} / N\mathbb{Z}^*} \varepsilon(x) \cdot \pi_{(0,x)}(\alpha).
\]

Then \( \psi \) satisfies the following two relations for \( \alpha, \beta \in \Gamma_0(N) \):

\[
\begin{align*}
(a) & \quad \Psi(\alpha \beta) = \Psi(\alpha) + \bar{\varepsilon}(\alpha) \Psi(\beta), \\
(b) & \quad \Psi(\alpha \beta \alpha^{-1}) = \bar{\varepsilon}(\alpha) \Psi(\beta).
\end{align*}
\]

Relation (a) expresses that \( \Psi \) is a crossed homomorphism.
The crossed homomorphism $\Psi$ may be expressed in terms of the "twisted" Dedekind sum $D_\epsilon : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{Q}[\epsilon]$ defined by

$$D_\epsilon(\infty) = 0$$

$$D_\epsilon \left( \frac{a}{b} \right) = \sum_{v=0}^{b-1} \mathbb{B}_1 \left( \frac{v}{b} \right) \cdot \mathbb{B}_{1,\epsilon} \left( \frac{Nav}{b} \right) \quad (2.3)$$

for $(a, b) = 1, b > 0$. The following identities hold for $r \in \mathbb{P}^1(\mathbb{Q})$:

(a) $D_\epsilon(-r) = -D_\epsilon(r)$,
(b) $D_\epsilon(r + 1) = D_\epsilon(r) \quad (2.4)$

A simple calculation shows that we could also have defined $D_\epsilon$ by the formula

(c) $D_\epsilon \left( \frac{a}{b} \right) = \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^*} \varepsilon(x) \cdot s \left( a, b; 0, \frac{x}{N} \right)$.

**Proposition 2.5**: Let $a = (a, b) \in \Gamma_0(N)$, then

$$\Psi(x) = -D_\epsilon \left( \frac{a}{Nc} \right)$$

$$= -D_\epsilon \left( \frac{b}{d} \right) - \frac{c}{2d} \cdot \mathbb{B}_{1,\epsilon}.$$  

**Proof**: The first equality is a direct calculation. If $c = 0$ then

$$\Psi(x) = \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^*} \varepsilon(x) \cdot \frac{b}{12d} = 0$$

and $-D_\epsilon \left( \frac{a}{Nc} \right) = -D_\epsilon(\infty) = 0$. If $c > 0$, then

$$\Psi(x) = \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^*} \varepsilon(x) \left[ \frac{a + d}{12Nc} - s(a, Nc; 0, \frac{x}{N}) \right]$$

$$= -\sum_{x} \varepsilon(x) \cdot s \left( a, Nc; 0, \frac{x}{N} \right)$$

$$= -D_\epsilon \left( \frac{a}{Nc} \right).$$
To prove the second equality let \( \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \), and note that \( \pi_{(x,0)}(\sigma) = 0 \) for \( x \in (\mathbb{Z}/N\mathbb{Z}) \). Hence for \( \alpha \in \Gamma_0(N) \), \( \pi_{(0,x)}(\alpha) = \pi_{(0,x)}(\alpha \sigma^2) = \pi_{(0,x)}(\alpha \sigma) + \pi_{(0,x)}(\sigma^2) = \pi_{(0,x)}(\alpha \sigma) \). We have then

\[
\begin{align*}
\Psi(\alpha) &= \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^*} \varepsilon(x) \cdot \pi_{(0,x)}(\alpha \sigma) \\
&= \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^*} \varepsilon(x) \left[ \frac{b}{12d} - \frac{Nc}{2d} \mathbb{B}_2 \left( \frac{dx}{N} \right) - s(b,d;0,x/N) \right] \\
&= -\varepsilon(d) \cdot \frac{c}{2d} \cdot \mathbb{B}_2, - D\left( \frac{b}{d} \right). \quad \square
\end{align*}
\]

§3. Congruences for universal special values

Let \( \langle \mathcal{P} \rangle \) be the normal subgroup of \( \Gamma = \Gamma_1(N) \) generated by the set \( \mathcal{P} \) of parabolic elements. The isomorphisms

\[
\pi_1(Y) \cong \Gamma/\{ \pm 1 \},
\]

\[
\pi_1(X) \cong \Gamma/\langle \mathcal{P} \rangle
\]

induce isomorphisms

\[
H^1(X; A) \cong \text{Hom}(\Gamma/\langle \mathcal{P} \rangle; A)
\]

\[
H^1(Y; A) \cong \text{Hom}(\Gamma/\{ \pm 1 \}; A)
\]

where \( A \) is any abelian group.

Let \( \Psi : \Gamma \to \mathbb{Q}[\varepsilon] \) be the homomorphism obtained by restriction to \( \Gamma \) of the crossed homomorphism of §2.

**Proposition 3.2:** (1) \( \Psi(\langle \mathcal{P} \rangle) \) is the principal fractional ideal, \( b \), in \( \mathbb{Q}[\varepsilon] \) generated by \( \frac{1}{2} \mathbb{B}_2, \varepsilon \);

(2) \( \Psi(\Gamma) \subseteq b + \mathbb{Z}[\varepsilon] \).

**Proof:** (1) A parabolic element of \( \Gamma_1(N) \) is conjugate in \( \Gamma_0(N) \) to a power of one of the matrices

\[
\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \pm \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}.
\]
So by (2.2) $\Psi(\langle S \rangle)$ is generated as a $\mathbb{Z}[\varepsilon]$-module by

$$\theta\left(\begin{array}{l} 1 \\ 1 \\ 0 \\ 1 \end{array}\right) = 0$$

and

$$\theta\left(\begin{array}{l} 1 \\ 0 \\ N \\ 1 \end{array}\right) = \frac{1}{2} \mathbb{B}_{2,\varepsilon}.$$

(2) We use the following lemma.

**Lemma 3.3:** For $a, b \in \mathbb{Z}$, $b \cdot D_\varepsilon\left(\frac{a}{b}\right) \in \mathbb{Z}[\varepsilon].$ \hfill \Box

Let $\alpha \in \Gamma$, $\alpha = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix}$. By Proposition 2.5

$$\Psi(\alpha) = - D_\varepsilon\left(\frac{a}{Nc}\right) = \frac{-c}{2d} \cdot \mathbb{B}_{2,\varepsilon} - D_\varepsilon\left(\frac{b}{d}\right).$$

So by 3.3

$$Nc \cdot \Psi(\alpha) = - Nc \cdot D_\varepsilon\left(\frac{a}{Nc}\right) \in \mathbb{Z}[\varepsilon]$$

and

$$d \cdot \Psi(\alpha) = - c \cdot \frac{1}{2} \mathbb{B}_{2,\varepsilon} - d \cdot D_\varepsilon\left(\frac{b}{d}\right)$$

$$\in b + \mathbb{Z}[\varepsilon].$$

But $(Nc, d) = 1$ then implies $\Psi(\alpha) \in b + \mathbb{Z}[\varepsilon]$.

**Proof of Lemma 3.3:** We may assume $(a, b) = 1$ and $b > 0$. Then

$$b \cdot D_\varepsilon\left(\frac{a}{b}\right) = \sum_{v=0}^{b-1} b \cdot B_1\left(\frac{v}{b}\right) \cdot B_1,\varepsilon\left(\frac{N\alpha v}{b}\right)$$

$$= \left(\sum_{v=0}^{b-1} v \cdot B_1,\varepsilon\left(\frac{N\alpha v}{b}\right)\right) - \left(\frac{b}{2} \sum_{v=0}^{b-1} B_1,\varepsilon\left(\frac{N\alpha v}{b}\right)\right).$$

Using the distribution law for the Bernoulli functions, the second term can be simplified to $- \frac{b}{2} \cdot \bar{\alpha}(b) \cdot B_{1,\varepsilon}$ which vanishes since $\varepsilon$ is an even character.
We use the following identity for \( r \in \mathbb{Q}^+ \):

\[
\mathbb{B}_{1, \epsilon}(r) = \begin{cases} 
- \sum_{0 \leq a < r \atop a \in \mathbb{Z}} \epsilon(a) & \text{if } r \notin \mathbb{Z}; \\
\frac{1}{2} \epsilon(r) - \sum_{0 \leq a < r \atop a \in \mathbb{Z}} \epsilon(a) & \text{if } r \in \mathbb{Z}.
\end{cases}
\]

Hence, if \( N \nmid b \) each summand of

\[
\sum_{v=0}^{b-1} v \cdot \mathbb{B}_{1, \epsilon} \left( \frac{Na^v}{b} \right)
\]

is in \( \mathbb{Z}[\epsilon] \). If \( b = Nm \) then, calculating modulo \( \mathbb{Z}[\epsilon] \), the only nonzero terms are those for which \( m \mid v \). So the above expression simplifies to

\[
\equiv \sum_{k=0}^{N-1} (mk) \cdot \mathbb{B}_{1, \epsilon}(ak) \\
\equiv \frac{1}{2} m \sum_{k=0}^{N-1} k \cdot \epsilon(ak) \equiv \frac{m \cdot N \cdot \epsilon(a)}{2} \cdot \mathbb{B}_{1, \epsilon} \\
\equiv 0 \pmod{\mathbb{Z}[\epsilon]}.
\]

By the last proposition, \( \Psi \) induces a homomorphism

\[
\Phi : \Gamma \to (\mathbb{Z}[\epsilon] + b)/b
\]

which vanishes on \( \langle \mathcal{P} \rangle \). Let \( A = \Psi(\Gamma)/b \) be the image of \( \Phi \); then by (3.1) \( \Phi \) corresponds to a cohomology class

\[
\varphi \in H^1(X; A).
\]

In Theorem 3.4 we compute the special values of \( \varphi \).

Let \( \chi \) be an odd primitive Dirichlet character of conductor \( m \) prime to \( N \), and set

\[
A[\chi] = (\mathbb{Z}[\epsilon] + b) \cdot \mathbb{Z}[\chi]/b \cdot \mathbb{Z}[\chi].
\]

There is a natural \( \mathbb{Z}[\chi] \)-homomorphism

\[
A \otimes_{\mathbb{Z}} \mathbb{Z}[\chi] \to A[\chi]
\]
with respect to which we define the special values $\Lambda(\varphi, \chi) \in A[\chi]$ as in 1.5.

**Theorem 3.4:** With $\chi, m$ as above

$$\Lambda(\varphi, \chi) \equiv \chi(N) \cdot \bar{a}(m) \cdot B_{1, \chi} \cdot (\frac{1}{2} B_{2, \varepsilon} - B_{1, \varepsilon}) \pmod{b \cdot Z[\chi]}.$$  

**Proof:** For each $b \in (Z/mZ)^*$ choose an element

$$\gamma_b = \left( \begin{array}{c} a \cdot b \\ Nc \end{array} \right) \in \Gamma_0(N)$$

such that $\gamma_b \cdot \{0\} = \left\{ \frac{b}{m} \right\}$. Then $\gamma_b \cdot \gamma_1^{-1} \in \Gamma_1(N)$ and $\gamma_b \cdot \gamma_1^{-1} \cdot \{ \frac{1}{m} \} = \left\{ \frac{b}{m} \right\}$. So

$$\Lambda(\varphi, \chi) = \Lambda(\chi) \cap \varphi$$

$$= \text{Univ} \left( \sum_{b=1}^{m-1} \left( \left\{ \frac{b}{m} \right\} - \left\{ \frac{1}{m} \right\} \right) \otimes \tilde{\chi}(b) \right) \cap \varphi$$

$$= \sum_{b=1}^{m-1} \tilde{\chi}(b) \cdot \Psi(\gamma_b \cdot \gamma_1^{-1})$$

$$= \sum_{b=1}^{m-1} \tilde{\chi}(b) \cdot \Psi(\gamma_b) \pmod{b \cdot Z[\chi]}.$$

By Proposition 2.5

$$\Psi(\gamma_b) = R(\gamma_b) - D_{\varepsilon}(b/m)$$

where

$$R(\gamma_b) = -\bar{a}(m) \cdot \frac{c}{m} \cdot \frac{1}{2} B_{2, \varepsilon}.$$  

Since $Nbc \equiv -1 \pmod{m}$, $\tilde{\chi}(b) = \chi(-Nc)$, hence

$$\sum_{b=1}^{m-1} \tilde{\chi}(b) \cdot R(\gamma_b) \equiv \chi(N) \bar{a}(m) \cdot B_{1, \chi} \cdot \frac{1}{2} B_{2, \varepsilon} \pmod{b \cdot Z[\chi]}.$$
We also have
\[
\sum_{b=1}^{m-1} \overline{x}(b) \cdot D_{\epsilon} \left( \frac{b}{m} \right) = \sum_{\nu=1}^{m-1} \mathbb{B}_1 \left( \frac{\nu}{m} \right) \cdot \sum_{b=1}^{m-1} \overline{x}(b) \cdot \mathbb{B}_{1,\epsilon} \left( \frac{Nb}{m} \right)
\]
\[
= \left( \sum_{\nu=1}^{m-1} \chi(\nu) \cdot \mathbb{B}_1 \left( \frac{\nu}{m} \right) \right) \cdot \left( \sum_{b=1}^{m-1} \overline{x}(b) \cdot \mathbb{B}_{1,\epsilon} \left( \frac{Nb}{m} \right) \right)
\]
(since $\chi$ is primitive)
\[
= \chi(N)\overline{\alpha}(m)\mathbb{B}_{1,\epsilon} \cdot \mathbb{B}_{1,\epsilon}\overline{\chi}.
\]

§4. A nonvanishing theorem

We will need the following result, which was proved for prime $m$ by L. Washington [12] and for general $m$ by E. Friedman [2].

Let

**THEOREM 4.1:** Let $\mathfrak{p}$ be a prime of $\mathbb{Q}$, and $m > 0$ be an integer prime to $\mathfrak{p}$. Let $L$ be a finite abelian extension of $\mathbb{Q}$. Then the number of odd Dirichlet characters $\chi$ of $\text{Gal}(L(\mathfrak{p}^m)/\mathbb{Q})$ such that

\[
\frac{1}{2} \mathbb{B}_{1,\chi} \equiv 0 \pmod{\mathfrak{p}}
\]

is finite. \(\square\)

The following result is an almost immediate consequence.

**THEOREM 4.2:** Let $2 \notin \mathcal{P} \subseteq \mathbb{Z}[\epsilon]$ be an odd prime containing $b \cap \mathbb{Z}[\epsilon]$, and let $m > 0$ be an integer prime to $\mathcal{P}$. The number of odd characters $\chi$ of $\text{Gal}(\mathbb{Q}(\mathfrak{p}^m)/\mathbb{Q})$ such that

\[
A(\varphi, \chi) \equiv 0 \pmod{\mathcal{P} + b \cdot \mathbb{Z}[\chi]}
\]

is finite. \(\square\)

**PROOF:** Let $\mathfrak{p}$ be an extension of $\mathcal{P}$ to $\mathbb{Q}$. For $\chi$ as in the theorem, $\mathbb{B}_{1,\chi}$ is $\mathfrak{p}$-integral. Hence

\[
A(\varphi, \chi) \equiv \chi(N) \cdot \overline{\alpha}(m) \mathbb{B}_{1,\epsilon} \cdot \mathbb{B}_{1,\epsilon} \overline{\chi} \pmod{\mathfrak{p}}.
\]
The result follows from 4.1. □

For an abelian group $M$ let $M_{(2)} = M \otimes \mathbb{Z}[\frac{1}{2}]$.

**Corollary 4.3:**

1. $(\Psi(\Gamma))_{(2)} = (b + \mathbb{Z}[\epsilon])_{(2)}$;
2. $A_{(2)} \cong \left( \frac{\mathbb{Z}[\epsilon]}{b \cap \mathbb{Z}[\epsilon]} \right)_{(2)}$

**Proof:** By 3.2, $b \subseteq \Psi(\Gamma) \subseteq b + \mathbb{Z}[\epsilon]$. We will show $\mathbb{Z}[\epsilon] \cap \Psi(\Gamma)_{(2)} = \mathbb{Z}[\epsilon]$. Suppose this is not the case. Then there is a prime ideal $\mathcal{P} \subseteq \mathbb{Z}[\epsilon]$ not containing 2 for which

$$
\mathcal{P} \supseteq \mathbb{Z}[\epsilon] \cap \Psi(\Gamma).
$$

Then for every odd primitive Dirichlet character $\chi$ of conductor prime to $N$, we have $A(\varphi, \chi) \equiv 0 \pmod{(\mathcal{P} + b)\mathbb{Z}[\chi]}$ contradicting Theorem 4.2. □

§5. Hecke operators and the Eisenstein ideal

The Hecke operators $U_N, T_l, \langle a \rangle (l \neq N \text{ prime}, a \in \mathbb{Z}/N\mathbb{Z}^*)$ act in the standard fashion on the homology groups in diagram (1.2). Let $i$ be the involution of $\mathcal{H}^*$ defined by $z \mapsto -\overline{z}$. Then $i$ descends to an involution of $X$ which induces an involution, which we also call $i$, on the second row of (1.2). The operators $U_N, T_l, \langle a \rangle, i$ are mutually commutative.

We now define analogous operators on the first row of (1.2). Clearly the group ring $\mathbb{Z}[GL_2(\mathbb{Q})]$ acts on $\text{Div}^0(\mathbb{P}^1(\mathbb{Q}))$. For each prime $p$ let

$$
U_p = \sum_{k=0}^{p-1} \begin{pmatrix} 1 & k \\ 0 & p \end{pmatrix} \in \mathbb{Z}[GL_2(\mathbb{Q})].
$$

For $a \in (\mathbb{Z}/N\mathbb{Z})^*$ let $\langle a \rangle = \sigma_a$ where $\sigma_a \in \Gamma_0(N)$ is any element satisfying the congruence

$$
\sigma_a = \begin{pmatrix} * & * \\ 0 & a \end{pmatrix} \pmod{N}.
$$

For a prime $l \neq N$ set

$$
T_l = U_l + \sigma_l \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$
Finally, let

\[ t = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \]

Then \( U_N, T_l, \langle a \rangle, n(l \neq N, a \in (\mathbb{Z}/N\mathbb{Z})^*) \) define operators on \( \text{Div}^0(\mathbb{P}^1(\mathbb{Q})) \) which preserve the subgroup \( \mathbb{Z} \) and hence also act on \( \text{Div}^0(\text{cusps}) \).

All arrows of (1.2) commute with these operators.

Recall the Dedekind sum \( D_\epsilon : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{Q}[\epsilon] \). For a prime \( p \) define \( (D_\epsilon | U_p) : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{Q}[\epsilon] \) by

\[ (D_\epsilon | U_p)(r) = \sum_{k=0}^{p-1} D_\epsilon \left( \begin{pmatrix} 1 & k \\ 0 & p \end{pmatrix} \right) \cdot r, \quad r \in \mathbb{P}^1(\mathbb{Q}). \]

If \( l \neq N \) is prime define \( D_\epsilon | S_{l, \epsilon} \) by

\[ (D_\epsilon | S_{l, \epsilon})(r) = \bar{\epsilon}(l) \cdot D_\epsilon \left( \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot r, \quad r \in \mathbb{P}^1(\mathbb{Q}) \]

and let \( D_\epsilon | T_{l, \epsilon} = D_\epsilon | U_l + D_\epsilon | S_{l, \epsilon} \). Also define \( (D_\epsilon | l) \) by

\[ (D_\epsilon | l)(r) = D_\epsilon \left( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot r. \]

**Lemma 5.1:**

1. For an arbitrary prime \( p \) and \( r \in \mathbb{P}^1(\mathbb{Q}) \)

\[ (D_\epsilon | U_p)(r) = (\bar{\epsilon}(p) + p) \cdot D_\epsilon(r) - \bar{\epsilon}(p) \cdot D_\epsilon \left( \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right) \cdot r. \]

2. For \( l \neq N \) prime,

\[ D_\epsilon | T_{l, \epsilon} = (\bar{\epsilon}(l) + l) \cdot D_\epsilon, \]

3. \( D_\epsilon | U_N = N \cdot D_\epsilon, \)

4. \( D_\epsilon | l = -D_\epsilon. \)

**Proof:** (1) is an amusing though lengthy calculation. (2) and (3) follow from (1) by letting \( p = l \neq N, p = N \), respectively. (4) is a restatement of 2.4(a).

Let \( \mathbb{T} \subseteq \text{End}(H_1(X; \mathbb{Z})) \) be the commutative algebra generated over \( \mathbb{Z} \) by the operators \( U_N, T_l, \langle a \rangle \). By Poincaré duality \( \mathbb{T} \) also acts on \( H^1(X; B) \) for any abelian group \( B \).
PROPOSITION 5.2: Let $\varphi \in H^1(X; A)$ be the cohomology class of $3$. Then

\begin{enumerate}
\item $\varphi \langle a \rangle = \bar{\alpha}(a) \cdot \varphi$ for $a \in (\mathbb{Z}/N\mathbb{Z})^*$;
\item $\varphi | T_l = (\bar{\alpha}(l) + l) \cdot \varphi$ for $l \neq N$ prime;
\item $\varphi | U_N = N \cdot \varphi$;
\item $\varphi | _i = - \varphi$.
\end{enumerate}

PROOF: Let $\gamma \in \Gamma$ and $[\gamma] \in H_1(X; \mathbb{Z})$ be the corresponding homology class.

\begin{enumerate}
\item $[\gamma] \cap (\varphi \langle a \rangle) = [\sigma_a^\gamma] \cap \varphi = \Phi(\sigma_a^\gamma \sigma_a^{-1}) = \bar{\alpha}(a) \cdot \Phi(\gamma) = [\gamma] \cap (\bar{\alpha}(a) \cdot \varphi)$.
\item Since $[\gamma] = \text{Univ}(\gamma \cdot \{i \infty\} - \{i \infty\})$ we have

$[\gamma] \cap (\varphi | T_l) = \text{Univ}(T_l \cdot (\gamma \cdot \{i \infty\} - \{i \infty\})) \cap \varphi$

$= (\bar{\alpha}(l) + l) \cdot (D_\gamma | T_{i,e})(\gamma \cdot \{i \infty\} - D_e(\gamma \cdot \{i \infty\})))$

$= (\bar{\alpha}(l) + l) \cdot (\bar{\alpha}(a) \cdot \varphi)$

(5.1(2))
\end{enumerate}

The proofs of (3), (4) are similar to that of (2) using 5.1(3), (4). □

We see now that $A$ inherits the structure of $\mathbb{T}[i]$-module by letting the operators $U_N$, $T_l$, $\langle a \rangle$, $i$ act as $N$, $\bar{\alpha}(l) + l$, $\bar{\alpha}(a)$, $-1$ respectively. With respect to this structure the homomorphism

$\cap \varphi : H_1(X; \mathbb{Z}) \to A$

is a $\mathbb{T}[i]$-homomorphism.

We will refer to the ideal $I = \text{Ann}_T(A) \subseteq \mathbb{T}$ as the Eisenstein ideal. By Corollary 4.3 $A(2)$ is a cyclic $\mathbb{T}(2)$-module with a canonical generator. Hence there are isomorphisms

$$(\mathbb{T}/I)_{(2)} \cong A_{(2)} \cong (\mathbb{Z}[e]/b \cap Z[\varepsilon])_{(2)}.$$ (5.3)

The isomorphisms provide a one-to-one correspondence between odd primes $P \subseteq \mathbb{T}$ for which $P \equiv 1$ and odd primes $\mathcal{P} \subseteq \mathbb{Z}[\varepsilon]$ for which $\mathcal{P} \equiv b \cap Z[\varepsilon]$. If $P \leftrightarrow \mathcal{P}$ then we have

$A_P \cong A_{\mathcal{P}} \cong \mathbb{Z}[\varepsilon]_{\mathcal{P}}/b_{\mathcal{P}}$.

REMARK 5.4: Let $a \in (\mathbb{Z}/N\mathbb{Z})^*$ be a primitive element and suppose $\alpha(a)$ is a primitive $d$-th root of unity where $d | (N - 1)$. Let $F_d(T) \in \mathbb{Z}[T]$ be the irreducible monic polynomial whose roots are the primitive $d$-th roots of 1. Then

$$\langle U_N - N; T_l - \langle l \rangle - l, l \neq N; F_d(\langle a \rangle) \rangle \subseteq I.$$

It would be interesting to know if these two ideals are in fact equal.
§6. Congruences for special values of $L$-functions

A theorem of Shimura ([10], Theorem 3.51) asserts that the space $\mathcal{S}_2(\Gamma)$ of weight 2 cusp forms over $\Gamma$ is a free $\mathbb{T} \oplus \mathbb{C}$-module of rank 1. Hence there is a one-to-one correspondence

$$
\begin{align*}
\left\{ \begin{array}{l}
\text{normalized} \\
\text{weight two} \\
\text{parabolic} \\
\text{$\mathbb{T}$-eigenforms,} \\
f
\end{array} \right\} & \leftrightarrow \left\{ \begin{array}{l}
\text{homomorphisms} \\
\mathbb{T} \xrightarrow{h_f} \mathbb{C}
\end{array} \right\}
\end{align*}
$$

For an eigenform $f$, the homomorphism $h_f$ satisfies

$$
f|_{\mathfrak{p}} = h_f(\mathfrak{p}) \cdot f \quad \text{for} \quad \mathfrak{p} \in \mathbb{T}.
$$

Let $\mathcal{P}(f) = \ker(h_f)$ and $\mathcal{O}(f) = \text{image}(h_f)$. Then $\mathcal{P}(f)$ is a minimal prime ideal in $\mathbb{T}$ and $\mathcal{O}(f)$ is the ring generated by the eigenvalues. The ring $\mathcal{O}(f)$ is a possibly nonmaximal order in its quotient field.

Write $H$ for $H_1(X; \mathbb{Z})$, and let

$$
H^- = H/(1 + i)H.
$$

Another consequence of Shimura’s result is:

**Proposition 6.2:** $H^- \otimes_2 \mathbb{Q}$ is a free rank 1 $\mathbb{T} \otimes \mathbb{Q}$-module.

**Proof:** The pairing

$$
(H \otimes \mathbb{C}) \times \mathcal{S}_2(\Gamma) \to \mathbb{C}
$$

$$(\gamma, f) \mapsto \int_{(1-i)f} \omega(f)
$$

factors through $H^- \otimes \mathbb{C}$ to give a nondegenerate pairing

$$
(H^- \otimes \mathbb{C}) \times \mathcal{S}_2(\Gamma) \to \mathbb{C}.
$$

By Shimura’s theorem $(H^- \otimes \mathbb{C})$ is a free rank 1 $\mathbb{T} \otimes \mathbb{C}$-module. □

Let $R$ be either $\mathbb{T}$ or $\mathbb{Z}[\varepsilon]$. If $M$ is an $R$-module and $P \subseteq R$ is a prime ideal we will write $M_p \cong M \otimes_R R_p$ for the localization of $M$ at $P$.

In the next theorem we will be concerned with prime ideals $P \subseteq \mathbb{T}$ for which $H^-_P$ is a free rank 1 $\mathbb{T}_P$-module. By the last proposition this in-
cludes almost all primes \( P \). Whether in general there exist primes \( P \supseteq I \) satisfying this condition we do not know.

**Theorem 6.3:** Let \( 2 \notin P \supseteq I \) be a prime ideal in \( \mathbb{T} \) for which \( H_P^- \) is a free rank 1 \( \mathbb{T}_p \)-module. Let \( \mathcal{P} \subseteq \mathbb{Z}[\mathfrak{e}] \) be the corresponding prime obtained from 5.3. Let \( f \) be a normalized weight 2 parabolic \( \mathbb{T} \)-eigenform on \( \Gamma_1(N) \) such that \( \mathcal{P}(f) \subseteq P \) and let

\[
\pi : \mathcal{O}(f) \to A/PA \cong \mathbb{Z}[\mathfrak{e}]/\mathcal{P}
\]

be the natural projection.

Then there is a period \( \Omega \in \mathbb{C}^* \) such that for all primitive odd quadratic Dirichlet characters \( \chi \) of conductor \( m \) prime to \( N \), and to \( \mathcal{P} \)

\[
(1) \quad A_f(\chi) \overset{\text{def}}{=} \frac{\pi(\chi) \cdot L(f, \chi, 1)}{\Omega} \in \mathcal{O}(f)
\]

and

\[
(2) \quad \pi(A_f(\chi)) \equiv \chi(N)\mathbb{B}_1, \chi \cdot \mathbb{B}_1, \epsilon_\chi \pmod{\mathcal{P}}.
\]

**Proof:** Let \( j : H \to H_P^- \) be the natural map. Then the composition

\[
H \xrightarrow{\wedge} A \longrightarrow A/PA
\]

factors through \( j \) to give a homomorphism

\[
\Phi_P^- : H_P^- \to A/PA.
\]

Let \( \gamma_0 \in H \) be such that \( \gamma_0 \cap \varphi \equiv 1 \pmod{P \cdot A} \). Then \( j(\gamma_0) \) generates \( H_P^- \) as a \( \mathbb{T}_p \)-module. Let

\[
\Omega = \frac{1}{2} \int_{(1 - i)\gamma_0} \omega(f) \in \mathbb{C}^*.
\]

The homomorphism \( \Phi_f : H \to \mathbb{C} \) defined by

\[
\Phi_f(\gamma) = \frac{1}{2\Omega} \int_{(1 - i)\gamma} \omega(f)
\]

has its image in the quotient field of \( \mathcal{O}(f) \) and factors through \( H \to H^- \). Since \( \Phi_f(\gamma_0) = 1 \), image \( (\Phi_f) \subseteq \mathcal{O}(f)_p \). By modifying \( \Omega \) we may assume image \( (\Phi_f) \subseteq \mathcal{O}(f) \) and \( \Phi_f(\gamma_0) \equiv 1 \pmod{P \cdot \mathcal{O}(f)} \). This gives rise to a \( \mathbb{T} \)-
The following diagram commutes

For an odd quadratic character $\chi$ as in the theorem we have $\iota \cdot \Lambda(\chi) = -\Lambda(\chi)$. Hence

$$\Lambda_f(\chi) = \frac{1}{\Omega} (\Lambda(\chi) \cap \varphi_f) = \Phi_f(\Lambda(\chi)) \in \mathcal{O}(f).$$

Also

$$\pi(\Lambda_f(\chi)) = \pi \circ \Phi_f(\Lambda(\chi)) = \Phi_f \circ j(\Lambda(\chi))$$

$$\equiv \Lambda(\varphi, \chi) \pmod{P}.$$ 

The result follows from Theorem 3.2 because $m$ is prime to $P$ and hence $B_{1, \chi}$ is $P$-integral.

§7. Compatibility with the conjecture of Birch and Swinnerton-Dyer

In [7], Mazur gives a descent argument which shows that his congruence formulae are compatible with the conjecture of Birch and Swinnerton-Dyer. A generalization of this descent has been given by the first author [3]. We review the results here.

Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\piinz}$ be a normalized weight two newform on $\Gamma_1(N)$, $N$ prime, and let $K(f) = \mathbb{Q}(a_n|n = 1, 2, \ldots)$ be the field generated by the Fourier coefficients. Shimura [10] has shown how one can associate to $f$ a simple abelian variety quotient $A_f/\mathbb{Q}$ of the Jacobian of $X_1(N)$. For a Dirichlet character $\chi$ the $L$-function of $A_f/\mathbb{Q}$ can be expressed as a product:

$$L(A_f, \chi, s) = \prod_{\sigma} L(f_{\sigma}, \chi, s)$$

where $\sigma$ ranges through the imbeddings $\sigma : K(f) \hookrightarrow \mathbb{C}$, and $f_{\sigma}$ is defined
Now suppose \( \chi \) is an odd quadratic character and let \( K_\chi \) be the associated imaginary quadratic field. By a theorem of Shimura ([11], Theorem 1 (iii)),

\[
L(f, \chi, 1) = 0 \Leftrightarrow L(f, \overline{\chi}, 1) = 0.
\]

Hence the conjecture of Birch and Swinnerton-Dyer predicts:

\[
L(f, \chi, 1) = 0 \Leftrightarrow \text{rk}(A(K_\chi)) = \text{rk}(A(\mathbb{Q})).
\] (7.1)

The following theorem is proved in [3].

**Theorem 7.2:** Let \( P \supseteq I \) be a locally principal prime ideal in \( \mathbb{T} \) of residual characteristic \( p \not\equiv 2 \cdot \text{ord}(\varepsilon) \). If \( K_\chi = \mathbb{Q}(\sqrt{-3}) \) assume \( p \neq 3 \). Let \( \mathcal{P} \subseteq \mathbb{Z}[\varepsilon] \) be the prime associated to \( P \) by (5.3). Then the following implication holds:

\[
\mathbb{B}_{1, \chi} \cdot \mathbb{B}_{1, \chi^e} \not\equiv 0 \pmod{\mathcal{P}}
\]

\[
\Rightarrow \text{rk}(A_f(K_\chi)) = 0.
\]

If this theorem is combined with Theorem 6.3, we obtain a weak form of one implication of (7.1).

In [7] Mazur shows that if \( f \) is a newform on \( \Gamma_0(N) \), \( \chi \) is an odd quadratic character, and \( L(f, \chi, 1) = 0 \) for trivial reasons (i.e. because of the sign of the functional equation), then \( \text{rk}(A_f(K_\chi)) \geq 1 \). In fact he explicitly constructs a point of infinite order, namely, the Birch-Heegner point. In our case, where \( f \) is a newform on \( \Gamma_1(N) \) with nontrivial Nebentypus, the functional equation relates \( L(f, \chi, 1) \) to \( L(f, \overline{\chi}, 1) \) where

\[
f(z) = \sum_{n=1}^{\infty} \tilde{a}_n e^{2\pi inz}
\]

is distinct from \( f(z) \). Hence there are no "trivial" zeroes. The construction of Birch-Heegner points also does not generalize.
§8. An example: $X_1(13)$

The second author has used an algorithm due to Birch [1] and Manin [5] to compute the universal modular symbol ([5], [7]) for $\Gamma = \Gamma_1(13)$. The results of this calculation show that the curve $X = X_1(13)$ has genus two and that precisely two Nebentypus characters occur in the character decomposition

$$\mathcal{S}_2(\Gamma) = \bigoplus_{\varepsilon} \mathcal{S}_2(\Gamma, \varepsilon)$$

of the space of weight two cusp forms.

Let $\varepsilon : \mathbb{Z} \rightarrow \mathbb{C}$ be the Dirichlet character of conductor 13 satisfying $\varepsilon(7) = \omega = e^{2\pi i/6}$. Then

$$\dim_{\mathbb{C}}(\mathcal{S}_2(\Gamma, \varepsilon)) = \dim_{\mathbb{C}}(\mathcal{S}_2(\Gamma, \bar{\varepsilon})) = 1.$$ 

Let $f \in \mathcal{S}_2(\Gamma, \bar{\varepsilon})$ be the normalized element. Then $f$ is a parabolic $\mathbb{T}$-eigenform with character $\bar{\varepsilon}$. The homomorphism $h_f$ of (6.1) defines an isomorphism

$$h_f : \mathbb{T} \xrightarrow{\sim} \mathbb{Z}[\omega].$$

A direct calculation shows $\frac{1}{2}B_{2,\varepsilon} = \frac{2 \cdot (3 + 2\omega)}{(1 + 3\omega)}$. Let $\mathcal{P} \subseteq \mathbb{Z}[\omega]$ be the prime ideal of norm 19 generated by $3 + 2\omega$, and $P = h_f^{-1}(\mathcal{P}) \subseteq \mathbb{T}$. By (5.3) the Eisenstein ideal satisfies $I \subseteq P$. But $h_f(T_2) = (-1 - \omega), h_f(\langle 2 \rangle) = \omega$, so

$$3 + 2\omega = h_f(2 + \langle 2 \rangle - T_2) \in h_f(I).$$

Hence $P = I$, and

$$A \cong (\mathbb{Z}[\omega]/\mathcal{P}) \cong (\mathbb{Z}/19\mathbb{Z}).$$

The question in Remark 5.4 has an affirmative answer in this case.

Since $\mathbb{T}$ is a Dedekind domain, Proposition 6.2 shows that $H_P$ is a free rank 1 $\mathbb{T}_P$-module. So Theorem 6.3 applies.

In the following pages we display the values
where $\Omega \in \mathbb{C}^*$ has been chosen so that

1. $\Lambda_f(\chi) \in \mathbb{Z}[\omega]$;
2. $\Lambda_f(\chi) \equiv 3 \cdot \chi(13) \cdot \overline{a(m_\chi)} \cdot \mathbb{B}_{1, x} \cdot \mathbb{B}_{1, \epsilon_2} (\mod \mathcal{P})$.

The character $\chi$ ranges through the imaginary quadratic characters associated to the fields $\mathbb{Q}(-P_\chi)$ where $0 < p_\chi \leq 3001$ is prime. These numbers have been computed using an algorithm due to Birch [1] and Manin [5].

$X_1(13)$: Algebraic parts, $\Lambda_f(\chi)$, of the special value of the $L$-function twisted by odd quadratic characters, $\chi$. $\Lambda_f(\chi) = a + bw$, $m = (1 + \sqrt{-3})/2$.

<table>
<thead>
<tr>
<th>$p_\chi$</th>
<th>$a$</th>
<th>$b$</th>
<th>$p_\chi$</th>
<th>$a$</th>
<th>$b$</th>
<th>$p_\chi$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>2</td>
<td>-2</td>
<td>5</td>
<td>0</td>
<td>-2</td>
<td>7</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>13</td>
<td>10</td>
<td>-10</td>
<td>17</td>
<td>8</td>
<td>-8</td>
<td>19</td>
<td>0</td>
<td>-14</td>
</tr>
<tr>
<td>29</td>
<td>0</td>
<td>0</td>
<td>31</td>
<td>-4</td>
<td>0</td>
<td>37</td>
<td>-4</td>
<td>0</td>
</tr>
<tr>
<td>43</td>
<td>0</td>
<td>-22</td>
<td>47</td>
<td>4</td>
<td>0</td>
<td>53</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>61</td>
<td>-4</td>
<td>4</td>
<td>67</td>
<td>38</td>
<td>-38</td>
<td>71</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>79</td>
<td>-8</td>
<td>0</td>
<td>83</td>
<td>28</td>
<td>0</td>
<td>89</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>101</td>
<td>20</td>
<td>0</td>
<td>103</td>
<td>-4</td>
<td>0</td>
<td>107</td>
<td>14</td>
<td>-14</td>
</tr>
<tr>
<td>113</td>
<td>8</td>
<td>-8</td>
<td>127</td>
<td>10</td>
<td>-10</td>
<td>131</td>
<td>-4</td>
<td>0</td>
</tr>
<tr>
<td>139</td>
<td>0</td>
<td>-50</td>
<td>149</td>
<td>56</td>
<td>-56</td>
<td>151</td>
<td>-8</td>
<td>0</td>
</tr>
<tr>
<td>163</td>
<td>0</td>
<td>-50</td>
<td>167</td>
<td>2</td>
<td>-2</td>
<td>173</td>
<td>2</td>
<td>-2</td>
</tr>
<tr>
<td>181</td>
<td>0</td>
<td>-62</td>
<td>191</td>
<td>0</td>
<td>-10</td>
<td>193</td>
<td>-8</td>
<td>0</td>
</tr>
<tr>
<td>199</td>
<td>0</td>
<td>-6</td>
<td>211</td>
<td>22</td>
<td>-22</td>
<td>223</td>
<td>18</td>
<td>-18</td>
</tr>
<tr>
<td>229</td>
<td>0</td>
<td>-168</td>
<td>233</td>
<td>0</td>
<td>-8</td>
<td>239</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>251</td>
<td>0</td>
<td>-18</td>
<td>257</td>
<td>16</td>
<td>0</td>
<td>263</td>
<td>6</td>
<td>-6</td>
</tr>
<tr>
<td>271</td>
<td>14</td>
<td>-14</td>
<td>277</td>
<td>20</td>
<td>-20</td>
<td>281</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>293</td>
<td>52</td>
<td>-52</td>
<td>307</td>
<td>48</td>
<td>0</td>
<td>311</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>317</td>
<td>0</td>
<td>22</td>
<td>331</td>
<td>0</td>
<td>-74</td>
<td>337</td>
<td>0</td>
<td>-94</td>
</tr>
<tr>
<td>349</td>
<td>-82</td>
<td>0</td>
<td>353</td>
<td>80</td>
<td>0</td>
<td>359</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>373</td>
<td>28</td>
<td>-28</td>
<td>379</td>
<td>58</td>
<td>-58</td>
<td>383</td>
<td>0</td>
<td>6</td>
</tr>
<tr>
<td>397</td>
<td>158</td>
<td>-158</td>
<td>401</td>
<td>36</td>
<td>0</td>
<td>409</td>
<td>40</td>
<td>-40</td>
</tr>
<tr>
<td>421</td>
<td>0</td>
<td>-134</td>
<td>431</td>
<td>6</td>
<td>-6</td>
<td>433</td>
<td>116</td>
<td>-116</td>
</tr>
<tr>
<td>443</td>
<td>52</td>
<td>0</td>
<td>449</td>
<td>14</td>
<td>-14</td>
<td>457</td>
<td>20</td>
<td>0</td>
</tr>
<tr>
<td>463</td>
<td>-8</td>
<td>0</td>
<td>467</td>
<td>16</td>
<td>0</td>
<td>479</td>
<td>14</td>
<td>-14</td>
</tr>
<tr>
<td>491</td>
<td>70</td>
<td>-70</td>
<td>499</td>
<td>-52</td>
<td>0</td>
<td>503</td>
<td>0</td>
<td>-14</td>
</tr>
<tr>
<td>521</td>
<td>0</td>
<td>-58</td>
<td>523</td>
<td>22</td>
<td>-22</td>
<td>541</td>
<td>0</td>
<td>-94</td>
</tr>
<tr>
<td>557</td>
<td>76</td>
<td>0</td>
<td>563</td>
<td>2</td>
<td>0</td>
<td>569</td>
<td>14</td>
<td>0</td>
</tr>
<tr>
<td>577</td>
<td>0</td>
<td>-26</td>
<td>587</td>
<td>18</td>
<td>-18</td>
<td>593</td>
<td>0</td>
<td>66</td>
</tr>
<tr>
<td>601</td>
<td>-40</td>
<td>0</td>
<td>607</td>
<td>0</td>
<td>-14</td>
<td>613</td>
<td>144</td>
<td>0</td>
</tr>
<tr>
<td>619</td>
<td>-116</td>
<td>0</td>
<td>631</td>
<td>-26</td>
<td>0</td>
<td>641</td>
<td>12</td>
<td>-12</td>
</tr>
<tr>
<td>647</td>
<td>2</td>
<td>-2</td>
<td>653</td>
<td>22</td>
<td>0</td>
<td>659</td>
<td>0</td>
<td>-54</td>
</tr>
<tr>
<td>673</td>
<td>-40</td>
<td>0</td>
<td>677</td>
<td>0</td>
<td>8</td>
<td>683</td>
<td>0</td>
<td>18</td>
</tr>
<tr>
<td>701</td>
<td>0</td>
<td>-72</td>
<td>709</td>
<td>84</td>
<td>-84</td>
<td>719</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>733</td>
<td>0</td>
<td>-206</td>
<td>739</td>
<td>150</td>
<td>-150</td>
<td>743</td>
<td>14</td>
<td>-14</td>
</tr>
<tr>
<td>757</td>
<td>-62</td>
<td>0</td>
<td>761</td>
<td>42</td>
<td>-42</td>
<td>769</td>
<td>-122</td>
<td>0</td>
</tr>
<tr>
<td>787</td>
<td>0</td>
<td>-106</td>
<td>797</td>
<td>58</td>
<td>-58</td>
<td>809</td>
<td>28</td>
<td>0</td>
</tr>
<tr>
<td>821</td>
<td>50</td>
<td>0</td>
<td>823</td>
<td>0</td>
<td>-50</td>
<td>827</td>
<td>44</td>
<td>0</td>
</tr>
<tr>
<td>839</td>
<td>0</td>
<td>-18</td>
<td>853</td>
<td>0</td>
<td>-150</td>
<td>857</td>
<td>0</td>
<td>-18</td>
</tr>
<tr>
<td>863</td>
<td>8</td>
<td>0</td>
<td>877</td>
<td>88</td>
<td>-88</td>
<td>881</td>
<td>-84</td>
<td>0</td>
</tr>
<tr>
<td>887</td>
<td>18</td>
<td>-18</td>
<td>907</td>
<td>18</td>
<td>-18</td>
<td>911</td>
<td>-4</td>
<td>0</td>
</tr>
</tbody>
</table>
### Special values of $L$-functions attached to $X_1(N)$

<table>
<thead>
<tr>
<th>$p_x$</th>
<th>$a$</th>
<th>$b$</th>
<th>$p_x$</th>
<th>$a$</th>
<th>$b$</th>
<th>$p_x$</th>
<th>$a$</th>
<th>$b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>929</td>
<td>44</td>
<td>-44</td>
<td>937</td>
<td>0</td>
<td>-178</td>
<td>941</td>
<td>0</td>
<td>-24</td>
</tr>
<tr>
<td>953</td>
<td>82</td>
<td>-82</td>
<td>967</td>
<td>24</td>
<td>0</td>
<td>971</td>
<td>0</td>
<td>-26</td>
</tr>
<tr>
<td>983</td>
<td>4</td>
<td>0</td>
<td>991</td>
<td>2</td>
<td>-2</td>
<td>997</td>
<td>-24</td>
<td>24</td>
</tr>
<tr>
<td>1013</td>
<td>0</td>
<td>-18</td>
<td>1019</td>
<td>40</td>
<td>0</td>
<td>1021</td>
<td>148</td>
<td>-148</td>
</tr>
<tr>
<td>1033</td>
<td>384</td>
<td>-384</td>
<td>1039</td>
<td>0</td>
<td>-16</td>
<td>1049</td>
<td>84</td>
<td>-84</td>
</tr>
<tr>
<td>1061</td>
<td>0</td>
<td>-114</td>
<td>1063</td>
<td>38</td>
<td>-38</td>
<td>1069</td>
<td>-18</td>
<td>0</td>
</tr>
<tr>
<td>1091</td>
<td>-8</td>
<td>0</td>
<td>1093</td>
<td>0</td>
<td>-74</td>
<td>1097</td>
<td>16</td>
<td>0</td>
</tr>
<tr>
<td>1109</td>
<td>32</td>
<td>-32</td>
<td>1117</td>
<td>0</td>
<td>-224</td>
<td>1123</td>
<td>-20</td>
<td>0</td>
</tr>
<tr>
<td>1151</td>
<td>0</td>
<td>-26</td>
<td>1153</td>
<td>96</td>
<td>-96</td>
<td>1163</td>
<td>0</td>
<td>-6</td>
</tr>
<tr>
<td>1181</td>
<td>98</td>
<td>0</td>
<td>1187</td>
<td>0</td>
<td>-62</td>
<td>1193</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>1213</td>
<td>100</td>
<td>-100</td>
<td>1217</td>
<td>0</td>
<td>118</td>
<td>1223</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>1231</td>
<td>0</td>
<td>-26</td>
<td>1237</td>
<td>-8</td>
<td>0</td>
<td>1249</td>
<td>0</td>
<td>-282</td>
</tr>
<tr>
<td>1277</td>
<td>56</td>
<td>0</td>
<td>1279</td>
<td>-88</td>
<td>0</td>
<td>1283</td>
<td>54</td>
<td>0</td>
</tr>
<tr>
<td>1291</td>
<td>0</td>
<td>-178</td>
<td>1297</td>
<td>12</td>
<td>0</td>
<td>1301</td>
<td>0</td>
<td>24</td>
</tr>
<tr>
<td>1307</td>
<td>0</td>
<td>30</td>
<td>1319</td>
<td>0</td>
<td>2</td>
<td>1321</td>
<td>0</td>
<td>-146</td>
</tr>
<tr>
<td>1361</td>
<td>16</td>
<td>-16</td>
<td>1367</td>
<td>26</td>
<td>-26</td>
<td>1373</td>
<td>88</td>
<td>0</td>
</tr>
<tr>
<td>1399</td>
<td>-28</td>
<td>0</td>
<td>1409</td>
<td>0</td>
<td>2</td>
<td>1423</td>
<td>0</td>
<td>-42</td>
</tr>
<tr>
<td>1429</td>
<td>0</td>
<td>-54</td>
<td>1433</td>
<td>72</td>
<td>0</td>
<td>1439</td>
<td>0</td>
<td>-14</td>
</tr>
<tr>
<td>1451</td>
<td>16</td>
<td>0</td>
<td>1453</td>
<td>-96</td>
<td>0</td>
<td>1459</td>
<td>34</td>
<td>-34</td>
</tr>
<tr>
<td>1481</td>
<td>0</td>
<td>-14</td>
<td>1483</td>
<td>40</td>
<td>0</td>
<td>1487</td>
<td>52</td>
<td>0</td>
</tr>
<tr>
<td>1493</td>
<td>152</td>
<td>0</td>
<td>1499</td>
<td>0</td>
<td>-10</td>
<td>1511</td>
<td>14</td>
<td>-14</td>
</tr>
<tr>
<td>1531</td>
<td>22</td>
<td>-22</td>
<td>1543</td>
<td>0</td>
<td>-38</td>
<td>1549</td>
<td>116</td>
<td>0</td>
</tr>
<tr>
<td>1559</td>
<td>0</td>
<td>0</td>
<td>1567</td>
<td>0</td>
<td>-42</td>
<td>1571</td>
<td>78</td>
<td>-78</td>
</tr>
<tr>
<td>1583</td>
<td>22</td>
<td>-22</td>
<td>1597</td>
<td>118</td>
<td>0</td>
<td>1601</td>
<td>-30</td>
<td>0</td>
</tr>
<tr>
<td>1609</td>
<td>-224</td>
<td>0</td>
<td>1613</td>
<td>0</td>
<td>-72</td>
<td>1619</td>
<td>0</td>
<td>22</td>
</tr>
<tr>
<td>1627</td>
<td>298</td>
<td>-298</td>
<td>1637</td>
<td>0</td>
<td>0</td>
<td>1657</td>
<td>356</td>
<td>-356</td>
</tr>
<tr>
<td>1667</td>
<td>62</td>
<td>-62</td>
<td>1669</td>
<td>0</td>
<td>-126</td>
<td>1693</td>
<td>32</td>
<td>0</td>
</tr>
<tr>
<td>1699</td>
<td>0</td>
<td>-218</td>
<td>1709</td>
<td>74</td>
<td>-74</td>
<td>1721</td>
<td>0</td>
<td>-90</td>
</tr>
<tr>
<td>1733</td>
<td>-8</td>
<td>8</td>
<td>1741</td>
<td>0</td>
<td>-110</td>
<td>1747</td>
<td>32</td>
<td>0</td>
</tr>
<tr>
<td>1759</td>
<td>0</td>
<td>-6</td>
<td>1777</td>
<td>170</td>
<td>-170</td>
<td>1783</td>
<td>82</td>
<td>-82</td>
</tr>
<tr>
<td>1789</td>
<td>0</td>
<td>-158</td>
<td>1801</td>
<td>504</td>
<td>-504</td>
<td>1811</td>
<td>0</td>
<td>-86</td>
</tr>
<tr>
<td>1831</td>
<td>-38</td>
<td>38</td>
<td>1847</td>
<td>28</td>
<td>0</td>
<td>1861</td>
<td>-76</td>
<td>0</td>
</tr>
<tr>
<td>1871</td>
<td>-12</td>
<td>0</td>
<td>1873</td>
<td>0</td>
<td>-206</td>
<td>1877</td>
<td>0</td>
<td>90</td>
</tr>
<tr>
<td>1889</td>
<td>84</td>
<td>-84</td>
<td>1901</td>
<td>-26</td>
<td>0</td>
<td>1907</td>
<td>0</td>
<td>-46</td>
</tr>
<tr>
<td>1931</td>
<td>0</td>
<td>-150</td>
<td>1933</td>
<td>82</td>
<td>82</td>
<td>1949</td>
<td>0</td>
<td>-210</td>
</tr>
<tr>
<td>1973</td>
<td>116</td>
<td>0</td>
<td>1979</td>
<td>58</td>
<td>85</td>
<td>1987</td>
<td>542</td>
<td>-542</td>
</tr>
<tr>
<td>1997</td>
<td>0</td>
<td>2</td>
<td>1999</td>
<td>-34</td>
<td>34</td>
<td>2003</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>2017</td>
<td>-52</td>
<td>0</td>
<td>2027</td>
<td>24</td>
<td>0</td>
<td>2029</td>
<td>0</td>
<td>22</td>
</tr>
<tr>
<td>2053</td>
<td>0</td>
<td>-218</td>
<td>2063</td>
<td>0</td>
<td>2</td>
<td>2069</td>
<td>114</td>
<td>0</td>
</tr>
<tr>
<td>2083</td>
<td>138</td>
<td>-138</td>
<td>2087</td>
<td>0</td>
<td>-2</td>
<td>2089</td>
<td>104</td>
<td>-104</td>
</tr>
<tr>
<td>2111</td>
<td>4</td>
<td>0</td>
<td>2113</td>
<td>438</td>
<td>-438</td>
<td>2129</td>
<td>88</td>
<td>0</td>
</tr>
<tr>
<td>2137</td>
<td>0</td>
<td>-304</td>
<td>2141</td>
<td>60</td>
<td>-60</td>
<td>2143</td>
<td>118</td>
<td>-118</td>
</tr>
<tr>
<td>2161</td>
<td>-320</td>
<td>0</td>
<td>2179</td>
<td>-228</td>
<td>0</td>
<td>2203</td>
<td>0</td>
<td>-6</td>
</tr>
<tr>
<td>2213</td>
<td>16</td>
<td>0</td>
<td>2221</td>
<td>-412</td>
<td>0</td>
<td>2237</td>
<td>0</td>
<td>26</td>
</tr>
<tr>
<td>2243</td>
<td>0</td>
<td>58</td>
<td>2251</td>
<td>-10</td>
<td>10</td>
<td>2267</td>
<td>240</td>
<td>0</td>
</tr>
<tr>
<td>2273</td>
<td>202</td>
<td>0</td>
<td>2281</td>
<td>-132</td>
<td>132</td>
<td>2287</td>
<td>-52</td>
<td>0</td>
</tr>
<tr>
<td>2297</td>
<td>16</td>
<td>-16</td>
<td>2309</td>
<td>0</td>
<td>-62</td>
<td>2311</td>
<td>-18</td>
<td>18</td>
</tr>
<tr>
<td>2339</td>
<td>0</td>
<td>0</td>
<td>2341</td>
<td>0</td>
<td>-266</td>
<td>2347</td>
<td>0</td>
<td>18</td>
</tr>
<tr>
<td>2357</td>
<td>282</td>
<td>-282</td>
<td>2371</td>
<td>128</td>
<td>0</td>
<td>2377</td>
<td>418</td>
<td>0</td>
</tr>
<tr>
<td>2383</td>
<td>0</td>
<td>2</td>
<td>2389</td>
<td>-336</td>
<td>0</td>
<td>2393</td>
<td>0</td>
<td>-34</td>
</tr>
<tr>
<td>2411</td>
<td>0</td>
<td>-138</td>
<td>2417</td>
<td>106</td>
<td>0</td>
<td>2423</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>2441</td>
<td>0</td>
<td>0</td>
<td>2447</td>
<td>2</td>
<td>-2</td>
<td>2459</td>
<td>38</td>
<td>-38</td>
</tr>
<tr>
<td>2473</td>
<td>58</td>
<td>0</td>
<td>2477</td>
<td>38</td>
<td>-38</td>
<td>2503</td>
<td>0</td>
<td>-26</td>
</tr>
<tr>
<td>2531</td>
<td>0</td>
<td>-74</td>
<td>2539</td>
<td>0</td>
<td>-86</td>
<td>2543</td>
<td>44</td>
<td>0</td>
</tr>
</tbody>
</table>


$$
\begin{array}{cccc|cccc|cccc}
\hline
p_x & a & b & p_x & a & b & p_x & a & b & p_x & a & b \\
2551 & -46 & 46 & 2557 & 288 & -288 & 2579 & 12 & 0 & 2591 & 0 & -38 \\
2593 & 642 & -642 & 2609 & 32 & -32 & 2617 & 40 & -40 & 2621 & 0 & -24 \\
2633 & -64 & 64 & 2647 & 24 & 0 & 2657 & 0 & -32 & 2659 & 0 & -98 \\
2663 & 10 & -10 & 2671 & 0 & -114 & 2677 & 0 & -200 & 2683 & 0 & 0 \\
2687 & 0 & 10 & 2689 & 204 & 0 & 2693 & 204 & 0 & 2699 & 68 & 0 \\
2707 & 282 & -282 & 2711 & 0 & -38 & 2713 & 462 & -462 & 2719 & 2 & -2 \\
2729 & 0 & -24 & 2731 & -172 & 0 & 2741 & 172 & 0 & 2749 & 294 & -294 \\
2753 & 202 & 0 & 2767 & 6 & -6 & 2777 & 0 & -58 & 2789 & -16 & 16 \\
2791 & 0 & -50 & 2797 & 86 & 0 & 2801 & 64 & -64 & 2803 & 12 & 0 \\
2819 & 238 & -238 & 2833 & 0 & 34 & 2837 & 116 & 0 & 2843 & 0 & -82 \\
2851 & 0 & -250 & 2857 & -104 & 0 & 2861 & 0 & -98 & 2879 & 0 & -38 \\
2887 & -44 & 0 & 2897 & 214 & 0 & 2903 & 0 & -6 & 2909 & -72 & 0 \\
2917 & 0 & -242 & 2927 & 6 & -6 & 2939 & 112 & 0 & 2953 & -106 & 0 \\
2957 & 152 & -152 & 2963 & 108 & 0 & 2969 & 0 & -126 & 2971 & 0 & -374 \\
2999 & 0 & -10 & 3001 & -324 & 0 & & & & & & \\
\hline
\end{array}
$$

REFERENCES


(Oblatum 17-VIII-1981 & 22-II-1982)

S. Kamienny
Department of Mathematics
University of California, Berkeley
Berkeley, California
U.S.A.

G. Stevens
Department of Mathematics
Rutgers University
New Brunswick, N.J. 08907
U.S.A.