COMPOSITIO MATHEMATICA

S. KAMIENNY
G. STEVENS

Special values of L-functions attached to $X_1(N)$

Compositio Mathematica, tome 49, nº 1 (1983), p. 121-142
<http://www.numdam.org/item?id=CM_1983__49_1_121_0>

Compositio Mathematica 49 (1983) 121–142

SPECIAL VALUES OF L-FUNCTIONS ATTACHED TO $X_1(N)$

S. Kamienny and G. Stevens

Introduction

The conjectures of Birch and Swinnerton-Dyer have stimulated recent interest in the arithmetic properties of special values of L-functions. B. Mazur [7] has proven a weak analog of these conjectures for the Jacobian of the modular curve $X_0(N)$, N prime. Crucial in his work are formulae for "universal special values" modulo the Eisenstein ideal.

In the present paper we show how Mazur's formulae can be extended to $X_1(N)$. We construct a cohomology class $\varphi \in H^1(X_1(N); A)$ and produce formulae for its "special values" in Theorem 3.4. Using a generalization by E. Friedman [2] of a result of L. Washington [12] we prove a nonvanishing result for these "special values" in Theorem 4.2.

The aim of §§5 and 6 is to use Theorem 3.4 to prove congruences involving the special values $L(f, \chi, 1)$ attached to a weight two cusp form f on $X_1(N)$. This goal has not been entirely achieved. In Theorem 6.3 we use a technical assumption of local freeness of the homology of $X_1(N)$. For $X_0(N)$ Mazur [6] was able to prove this condition. Unfortunately his proof does not work in our case.

Finally, §8 illustrates these results with the example $X_1(13)$.

§1. Universal special values of L-functions

Let $\mathscr{H}^* = \mathscr{H} \cup \mathbb{P}^1(\mathbb{Q})$ be the extended upper half plane. Let $N > 3$ be prime and

$$\Gamma = \Gamma_1(N) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \,|\, c \equiv 0,\ a \equiv d \equiv \pm 1 \ (\operatorname{mod} N)\right\}.$$

Denote by $Y = Y_1(N)$ the open Riemann surface $\Gamma\backslash\mathscr{H}$ and by $X = X_1(N)$ the associated compactification $\Gamma\backslash\mathscr{H}^*$. The finite set $X\backslash Y = \Gamma\backslash\mathbb{P}^1(\mathbb{Q})$ will be denoted by *cusps*.

For an arbitrary set S let $\mathrm{Div}^0(S)$ be the group of formal finite sums of degree 0 supported on S, i.e.

$$\mathrm{Div}^0(S) = \left\{\sum_{s\in S} a_s\cdot\{s\}\,|\,a_s\in\mathbb{Z},\ \text{almost all } a_s = 0, \text{ and } \sum_s a_s = 0\right\}.$$

The following is a slight variation of a definition given by Mazur ([7], II §1).

DEFINITION 1.1: The universal modular symbol attached to Γ is the homomorphism

$$Univ: \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \to H_1(X, cusps; \mathbb{Z})$$

defined by $Univ(\{r\} - \{s\})$ = the relative homology class represented by the projection to X of the oriented geodesic in $\mathscr{H}^*$ joining s to r. □

The group $\mathrm{Div}^0(cusps)$ is naturally identified with the reduced homology group $\tilde{H}_0(cusps; \mathbb{Z})$. Let Z be the kernel of the natural projection $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) \twoheadrightarrow \mathrm{Div}^0(cusps)$. Then the vertical arrows in the following commutative diagram of exact sequences are surjective.

$$\begin{array}{ccccccc} 0 & \to Z & \hookrightarrow & \mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q})) & \to & \mathrm{Div}^0(cusps) & \to 0 \\ & \downarrow{\scriptstyle Univ} & & \downarrow{\scriptstyle Univ} & & \downarrow\wr & \\ 0 \to & H_1(X;\mathbb{Z}) & \hookrightarrow & H_1(X, cusps; \mathbb{Z}) & \to & \tilde{H}_0(cusps;\mathbb{Z}) & \to 0 \end{array} \tag{1.2}$$

Let $\chi: \mathbb{Z} \to \mathbb{C}$ be a nontrivial primitive Dirichlet character of conductor m prime to N. Let $\mathbb{Z}[\chi]$ be the ring generated over $\mathbb{Z}$ by the values of χ. Then

$$\sum_{a=0}^{m-1}\left\{\frac{a}{m}\right\}\otimes\bar{\chi}(a)\in\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))\otimes\mathbb{Z}[\chi]$$

defines an element of $Z\otimes\mathbb{Z}[\chi]$ since the rational numbers $\frac{a}{m}$, $(a, m) = 1$ are Γ-equivalent.

Following Mazur [7] we define:

DEFINITION 1.3: The universal special value of the L-function twisted by χ is the homology class

$$\Lambda(\chi) = Univ\left(\sum_{a=0}^{m-1} \left\{\frac{a}{m}\right\} \otimes \bar{\chi}(a)\right) \in H_1(X; \mathbb{Z}[\chi]). \qquad \square$$

This terminology is justified by the next proposition. Let $f(z) = \sum_{n=1}^{\infty} a_n q^n$, $q = e^{2i\pi z}$, be a weight two cusp form on Γ, and let $\omega(f)$ be the holomorphic 1-form on X whose pullback to $\mathscr{H}$ is $f \cdot \dfrac{dq}{q} = 2\pi i f(z)\, dz$. Let $\varphi_f \in H^1(X; \mathbb{C})$ be the cohomology class represented by $\omega(f)$. The L-function of f twisted by χ is defined by the Dirichlet series

$$L(f, \chi, s) = \sum_{n=1}^{\infty} a_n \chi(n) n^{-s}$$

which converges absolutely for $\mathrm{Re}(s) > 3/2$ and continues to an entire function on $\mathbb{C}$.

PROPOSITION 1.4: *Let $\chi \neq 1$ be a primitive Dirichlet character of conductor m prime to N. Then*

$$\tau(\bar{\chi}) L(f, \chi, 1) = \Lambda(\chi) \cap \varphi_f,$$

where $\tau(\bar{\chi}) = \sum_{a=0}^{m-1} \bar{\chi}(a)\, e^{2\pi i a/m}$ is the usual Gauss sum, and $\cap$ denotes cap product:

$$\cap : H_1(X; \mathbb{C}) \times H^1(X; \mathbb{C}) \to \mathbb{C}. \qquad \square$$

For a proof see Birch [1].

We will adopt Mazur's point of view and shift the emphasis from the special values $L(f, \chi, 1)$ to the universal special values $\Lambda(\chi)$.

We can define "special values of L-functions" attached to an arbitrary cohomology class $\varphi \in H^1(X; A)$ with values in any abelian group A. Suppose we have a $\mathbb{Z}[\chi]$-module $A[\chi]$ together with a $\mathbb{Z}[\chi]$-homomorphism

$$A \otimes_{\mathbb{Z}} \mathbb{Z}[\chi] \to A[\chi].$$

Then cap product with φ defines a homomorphism

$$\cap\, \varphi : H_1(X; \mathbb{Z}[\chi]) \to A[\chi].$$

DEFINITION 1.5: The "special value" associated to the pair φ, χ is

$$\Lambda(\varphi, \chi) \overset{\mathrm{dfn}}{=} \Lambda(\chi) \cap \varphi \in A[\chi]. \qquad \square$$

§2. Modular units and Dedekind sums

For $x \in \mathbb{R}$ define the Bernoulli functions by

$$\mathbb{B}_1(x) = \begin{cases} (x - [x]) - \frac{1}{2} & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

and

$$\mathbb{B}_2(x) = (x - [x])^2 - (x - [x]) + \tfrac{1}{6},$$

where $[x]$ denotes the largest integer $\leq x$. Then $\mathbb{B}_1, \mathbb{B}_2$ define functions $\mathbb{R}/\mathbb{Z} \to \mathbb{R}$.

The Siegel units g_x, $x = (x, y) \in (\mathbb{Z}/N\mathbb{Z})^2$, may be defined by their q-expansions:

$$g_{(x,y)}(z) = q^{\frac{1}{2}\mathbb{B}_2\left(\frac{x}{N}\right)} \cdot \prod_{\substack{m \equiv x(N) \\ m > 0}} (1 - e^{2\pi i y/N} \cdot q^{m/N})$$

$$\cdot \prod_{\substack{m \equiv -x(N) \\ m > 0}} (1 - e^{-2\pi i y/N} \cdot q^{m/N})$$

where $\mathbb{B}_2\left(\dfrac{x}{N}\right)$ is the periodic second Bernoulli function and $q = e^{2\pi i z}$. The special function g_0 is the square of the Dedekind η-function. The functions g_x^{12N}, $x \neq 0$ are modular functions of level N which vanish nowhere on $\mathscr{H}$.

The functions g_x have been studied in great detail. For example, Kubert and Lang use them to describe cuspidal groups of modular curves (see [4]).

The transformation formulae for the Siegel units are well-known (see e.g. Schoeneberg [9] where they are referred to as the Dedekind functions). We summarize the results in the next proposition.

PROPOSITION 2.1: *There is a choice of the logarithms* $\log(g_x(z))$, $x \in (\mathbb{Z}/N\mathbb{Z})^2$ *such that for* $\gamma \in SL_2(\mathbb{Z})$

$$\pi_x(\gamma) \overset{\mathrm{dfn}}{=} (\log(g_x(\gamma z)) - \log(g_{x\gamma}(z)))$$

is independent of $z \in \mathscr{H}$ *and:*

(1) $\pi_x(\gamma) \in \dfrac{1}{12N^2}\mathbb{Z}$;

(2) *For* α, $\beta \in SL_2(\mathbb{Z})$,

$$\pi_x(\alpha\beta) = \pi_x(\alpha) + \pi_{x\alpha}(\beta);$$

(3) *If* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, $c \geq 0$, *then*

$$\pi_x(\gamma) = \begin{cases} \dfrac{a}{2c}\mathbb{B}_2\left(\dfrac{x}{N}\right) + \dfrac{d}{2c}\mathbb{B}_2\left(\dfrac{ax+cy}{N}\right) - s\left(a, c; \dfrac{x}{N}, \dfrac{y}{N}\right) & \text{if } c > 0 \\ \dfrac{b}{2d}\mathbb{B}_2\left(\dfrac{x}{N}\right) & \text{if } c = 0, \end{cases}$$

where $s(\ )$ *is the generalized Dedekind sum of Rademacher* ([8], *pp.* 534–543)

$$s\left(a, c; \frac{x}{N}, \frac{y}{N}\right) = \sum_{v=0}^{c-1} \mathbb{B}_1\left(\frac{v + \frac{x}{N}}{c}\right) \cdot \mathbb{B}_1\left(a \cdot \frac{v + \frac{x}{N}}{c} + \frac{y}{N}\right). \qquad \square$$

Let ε be an even primitive Dirichlet character of conductor N. Then we may define a homomorphism $\Gamma_0(N) \to \mathbb{C}^*$, which we also denote by ε, by

$$\varepsilon : \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma_0(N) \mapsto \varepsilon(d).$$

Define a map $\Psi : \Gamma_0(N) \to \mathbb{Q}[\varepsilon]$ by

$$\Psi(\alpha) = \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^*} \varepsilon(x) \cdot \pi_{(0,x)}(\alpha).$$

Then ψ satisfies the following two relations for α, $\beta \in \Gamma_0(N)$:

(a) $\Psi(\alpha\beta) = \Psi(\alpha) + \bar{\varepsilon}(\alpha)\Psi(\beta)$,

(b) $\Psi(\alpha\beta\alpha^{-1}) = \bar{\varepsilon}(\alpha)\Psi(\beta)$. (2.2)

Relation (a) expresses that Ψ is a crossed homomorphism.

The crossed homomorphism Ψ may be expressed in terms of the "twisted" Dedekind sum $D_\varepsilon: \mathbb{P}^1(\mathbb{Q}) \to \mathbb{Q}[\varepsilon]$ defined by

$$D_\varepsilon(\infty) = 0$$
$$D_\varepsilon\left(\frac{a}{b}\right) = \sum_{v=0}^{b-1} \mathbb{B}_1\left(\frac{v}{b}\right) \cdot \mathbb{B}_{1,\varepsilon}\left(\frac{Nav}{b}\right) \tag{2.3}$$

for $(a, b) = 1, b > 0$. The following identities hold for $r \in \mathbb{P}^1(\mathbb{Q})$:

(a) $D_\varepsilon(-r) = -D_\varepsilon(r)$,
(b) $D_\varepsilon(r + 1) = D_\varepsilon(r)$. (2.4)

A simple calculation shows that we could also have defined D_ε by the formula

(c) $D_\varepsilon\left(\frac{a}{b}\right) = \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^*} \varepsilon(x) \cdot s\left(a, b; 0, \frac{x}{N}\right)$.

PROPOSITION 2.5: *Let* $\alpha = \begin{pmatrix} a & b \\ Nc & d \end{pmatrix} \in \Gamma_0(N)$, *then*

$$\begin{aligned} \Psi(\alpha) &= -D_\varepsilon\left(\frac{a}{Nc}\right) \\ &= -D_\varepsilon\left(\frac{b}{d}\right) - \bar{\varepsilon}(d) \cdot \frac{c}{2d} \cdot \mathbb{B}_{2,\varepsilon}. \end{aligned}$$

PROOF: The first equality is a direct calculation. If $c = 0$ then

$$\Psi(\alpha) = \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^*} \varepsilon(x) \cdot \frac{b}{12d} = 0$$

and $-D_\varepsilon\left(\frac{a}{Nc}\right) = -D_\varepsilon(\infty) = 0$. If $c > 0$, then

$$\begin{aligned} \Psi(\alpha) &= \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^*} \varepsilon(x)\left[\frac{a + d}{12Nc} - s\left(a, Nc; 0, \frac{x}{N}\right)\right] \\ &= -\sum_x \varepsilon(x) \cdot s\left(a, Nc; 0, \frac{x}{N}\right) \\ &= -D_\varepsilon\left(\frac{a}{Nc}\right). \end{aligned}$$

To prove the second equality let $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and note that $\pi_{(x,0)}(\sigma) = 0$ for $x \in (\mathbb{Z}/N\mathbb{Z})$. Hence for $\alpha \in \Gamma_0(N)$, $\pi_{(0,x)}(\alpha) = \pi_{(0,x)}(\alpha\sigma^2) = \pi_{(0,x)}(\alpha\sigma) + \pi_{(0,x)\alpha\sigma}(\sigma) = \pi_{(0,x)}(\alpha\sigma)$. We have then

$$\begin{aligned} \Psi(\alpha) &= \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^*} \varepsilon(x) \cdot \pi_{(0,x)}(\alpha\sigma) \\ &= \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^*} \varepsilon(x) \left[\frac{b}{12d} - \frac{Nc}{2d} \mathbb{B}_2\left(\frac{dx}{N}\right) - s\left(b, d; 0, \frac{x}{N}\right) \right] \\ &= -\bar{\varepsilon}(d) \cdot \frac{c}{2d} \cdot \mathbb{B}_{2,\varepsilon} - D_\varepsilon\left(\frac{b}{d}\right). \end{aligned}$$

$\square$

§3. Congruences for universal special values

Let $\langle \mathscr{P} \rangle$ be the normal subgroup of $\Gamma = \Gamma_1(N)$ generated by the set $\mathscr{P}$ of parabolic elements. The isomorphisms

$$\pi_1(Y) \cong \Gamma/\{\pm 1\},$$
$$\pi_1(X) \cong \Gamma/\langle \mathscr{P} \rangle$$

induce isomorphisms

$$\begin{tikzcd} H^1(X; A) \arrow[r, leftrightarrow, "\sim"] \arrow[d, hook] & \mathrm{Hom}(\Gamma/\langle \mathscr{P} \rangle; A) \arrow[d, hook] \\ H^1(Y; A) \arrow[r, leftrightarrow, "\sim"] & \mathrm{Hom}(\Gamma/\{\pm 1\}; A) \end{tikzcd}$$

where A is any abelian group.

Let $\Psi : \Gamma \to \mathbb{Q}[\varepsilon]$ be the homomorphism obtained by restriction to Γ of the crossed homomorphism of §2.

PROPOSITION 3.2: (1) *$\Psi(\langle \mathscr{P} \rangle)$ is the principal fractional ideal,* $\mathfrak{b}$, *in* $\mathbb{Q}[\varepsilon]$ *generated by* $\frac{1}{2}\mathbb{B}_{2,\varepsilon}$;

(2) $\Psi(\Gamma) \subseteq \mathfrak{b} + \mathbb{Z}[\varepsilon]$.

PROOF: (1) A parabolic element of $\Gamma_1(N)$ is conjugate in $\Gamma_0(N)$ to a power of one of the matrices

$$\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \pm \begin{pmatrix} 1 & 0 \\ N & 1 \end{pmatrix}.$$

So by (2.2) $\Psi(\langle\mathscr{P}\rangle)$ is generated as a $\mathbb{Z}[\varepsilon]$-module by

$$\Psi\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = 0$$

and

$$\Psi\begin{pmatrix}1 & 0\\ N & 1\end{pmatrix} = \tfrac{1}{2}\mathbb{B}_{2,\varepsilon}.$$

(2) We use the following lemma.

LEMMA 3.3: *For* $a, b \in \mathbb{Z}$, $b \cdot D_\varepsilon\left(\frac{a}{b}\right) \in \mathbb{Z}[\varepsilon]$. $\square$

Let $\alpha \in \Gamma$, $\alpha = \begin{pmatrix} a & b\\ Nc & d\end{pmatrix}$. *By Proposition* 2.5

$$\Psi(\alpha) = -D_\varepsilon\left(\frac{a}{Nc}\right) = \frac{-c}{2d}\cdot\mathbb{B}_{2,\varepsilon} - D_\varepsilon\left(\frac{b}{d}\right).$$

So by 3.3

$$Nc\cdot\Psi(\alpha) = -Nc\cdot D_\varepsilon\left(\frac{a}{Nc}\right) \in \mathbb{Z}[\varepsilon]$$

and

$$\begin{aligned} d\cdot\Psi(\alpha) &= -c\cdot\tfrac{1}{2}\mathbb{B}_{2,\varepsilon} - d\cdot D_\varepsilon\left(\frac{b}{d}\right)\\ &\in \mathfrak{b} + \mathbb{Z}[\varepsilon]. \end{aligned}$$

But $(Nc, d) = 1$ *then implies* $\Psi(\alpha) \in \mathfrak{b} + \mathbb{Z}[\varepsilon]$. $\square$

PROOF OF LEMMA 3.3: We may assume $(a, b) = 1$ and $b > 0$. Then

$$\begin{aligned} b\cdot D_\varepsilon\left(\frac{a}{b}\right) &= \sum_{v=0}^{b-1} b\cdot\mathbb{B}_1\left(\frac{v}{b}\right)\cdot\mathbb{B}_{1,\varepsilon}\left(\frac{Nav}{b}\right)\\ &= \left(\sum_{v=0}^{b-1} v\cdot\mathbb{B}_{1,\varepsilon}\left(\frac{Nav}{b}\right)\right) - \left(\frac{b}{2}\sum_{v=0}^{b-1}\mathbb{B}_{1,\varepsilon}\left(\frac{Nav}{b}\right)\right). \end{aligned}$$

Using the distribution law for the Bernoulli functions, the second term can be simplified to $-\frac{b}{2}\cdot\bar{\varepsilon}(b)\cdot\mathbb{B}_{1,\varepsilon}$ which vanishes since ε is an even character.

We use the following identity for $r \in \mathbb{Q}^+$:

$$\mathbb{B}_{1,\varepsilon}(r) = \begin{cases} -\sum\limits_{\substack{0 \le a < r \\ a \in \mathbb{Z}}} \varepsilon(a) & \text{if } \ r \notin \mathbb{Z}; \\ \tfrac{1}{2}\varepsilon(r) - \sum\limits_{\substack{0 \le a < r \\ a \in \mathbb{Z}}} \varepsilon(a) & \text{if } \ r \in \mathbb{Z}. \end{cases}$$

Hence, if $N \nmid b$ each summand of

$$\sum_{v=0}^{b-1} v \cdot \mathbb{B}_{1,\varepsilon}\left(\frac{Nav}{b}\right)$$

is in $\mathbb{Z}[\varepsilon]$. If $b = Nm$ then, calculating modulo $\mathbb{Z}[\varepsilon]$, the only nonzero terms are those for which $m \mid v$. So the above expression simplifies to

$$\begin{aligned} &\equiv \sum_{k=0}^{N-1} (mk) \cdot \mathbb{B}_{1,\varepsilon}(ak) \\ &\equiv \tfrac{1}{2}m \sum_{k=0}^{N-1} k \cdot \varepsilon(ak) \equiv \frac{m \cdot N \cdot \varepsilon(a)}{2} \cdot \mathbb{B}_{1,\varepsilon} \\ &\equiv 0 \pmod{\mathbb{Z}[\varepsilon]}. \end{aligned}$$

□

By the last proposition, Ψ induces a homomorphism

$$\Phi : \Gamma \to (\mathbb{Z}[\varepsilon] + \mathfrak{b})/\mathfrak{b}$$

which vanishes on $\langle \mathscr{P} \rangle$. Let $A = \Psi(\Gamma)/\mathfrak{b}$ be the image of Φ; then by (3.1) Φ corresponds to a cohomology class

$$\varphi \in H^1(X; A).$$

In Theorem 3.4 we compute the special values of φ.

Let χ be an odd primitive Dirichlet character of conductor m prime to N, and set

$$A[\chi] = (\mathbb{Z}[\varepsilon] + \mathfrak{b}) \cdot \mathbb{Z}[\chi]/\mathfrak{b} \cdot \mathbb{Z}[\chi].$$

There is a natural $\mathbb{Z}[\chi]$-homomorphism

$$A \otimes_{\mathbb{Z}} \mathbb{Z}[\chi] \to A[\chi]$$

with respect to which we define the special values $\Lambda(\varphi,\chi)\in A[\chi]$ as in 1.5.

THEOREM 3.4: *With χ, m as above*

$$\Lambda(\varphi,\chi)\equiv\chi(N)\cdot\bar{\varepsilon}(m)\cdot\mathbb{B}_{1,\chi}\cdot(\tfrac{1}{2}\mathbb{B}_{2,\varepsilon}-\mathbb{B}_{1,\varepsilon\bar{\chi}})(\operatorname{mod}\mathfrak{b}\cdot\mathbb{Z}[\chi]).$$

PROOF: For each $b\in(\mathbb{Z}/m\mathbb{Z})^*$ choose an element

$$\gamma_b=\begin{pmatrix} a & b\\ Nc & m\end{pmatrix}\in\Gamma_0(N)$$

such that $\gamma_b\cdot\{0\}=\left\{\dfrac{b}{m}\right\}$. Then $\gamma_b\cdot\gamma_1^{-1}\in\Gamma_1(N)$ and $\gamma_b\cdot\gamma_1^{-1}\cdot\left\{\dfrac{1}{m}\right\}=\left\{\dfrac{b}{m}\right\}$. So

$$\begin{aligned}\Lambda(\varphi,\chi)&=\Lambda(\chi)\cap\varphi\\&=Univ\left(\sum_{b=1}^{m-1}\left(\left\{\frac{b}{m}\right\}-\left\{\frac{1}{m}\right\}\right)\otimes\bar{\chi}(b)\right)\cap\varphi\\&\equiv\sum_{b=1}^{m-1}\bar{\chi}(b)\cdot\Psi(\gamma_b\cdot\gamma_1^{-1})\\&\equiv\sum_{b=1}^{m-1}\bar{\chi}(b)\cdot\Psi(\gamma_b)\quad(\operatorname{mod}\mathfrak{b}\cdot\mathbb{Z}[\chi]).\end{aligned}$$

By Proposition 2.5

$$\Psi(\gamma_b)=R(\gamma_b)-D_\varepsilon(b/m)$$

where

$$R(\gamma_b)=-\bar{\varepsilon}(m)\cdot\frac{c}{m}\cdot\tfrac{1}{2}\mathbb{B}_{2,\varepsilon}.$$

Since $Nbc\equiv-1\ (\operatorname{mod}m)$, $\bar{\chi}(b)=\chi(-Nc)$, hence

$$\sum_{b=1}^{m-1}\bar{\chi}(b)\cdot R(\gamma_b)\equiv\chi(N)\bar{\varepsilon}(m)\mathbb{B}_{1,\chi}\cdot\tfrac{1}{2}\mathbb{B}_{2,\varepsilon}\quad(\operatorname{mod}\mathfrak{b}\cdot\mathbb{Z}[\chi]).$$

We also have

$$\sum_{b=1}^{m-1} \bar{\chi}(b)\cdot D_\varepsilon\left(\frac{b}{m}\right) = \sum_{\nu=0}^{m-1} \mathbb{B}_1\left(\frac{\nu}{m}\right)\cdot \sum_{b=1}^{m-1} \bar{\chi}(b)\cdot \mathbb{B}_{1,\varepsilon}\left(\frac{Nb\nu}{m}\right)$$
$$= \left(\sum_{\nu=1}^{m-1} \chi(\nu)\cdot \mathbb{B}_1\left(\frac{\nu}{m}\right)\right)\cdot\left(\sum_{b=1}^{m-1} \bar{\chi}(b)\cdot \mathbb{B}_{1,\varepsilon}\left(\frac{Nb}{m}\right)\right)$$

(since χ is primitive)

$$= \chi(N)\bar{\varepsilon}(m)\mathbb{B}_{1,\chi}\cdot \mathbb{B}_{1,\varepsilon\bar{\chi}}. \qquad \square$$

§4. A nonvanishing theorem

We will need the following result, which was proved for prime m by L. Washington [12] and for general m by E. Friedman [2].

Let

$$\mu_{m^\infty} = \{\zeta \in \mathbb{C}^* \mid \zeta^{m^k} = 1 \text{ for some } k \geq 0\}.$$

THEOREM 4.1: Let $\mathfrak{P}$ be a prime of $\bar{\mathbb{Q}}$, and $m > 0$ be an integer prime to $\mathfrak{P}$. Let L be a finite abelian extension of $\mathbb{Q}$. Then the number of odd Dirichlet characters χ of $\mathrm{Gal}(L(\mu_{m^\infty})/\mathbb{Q})$ such that

$$\tfrac{1}{2}\mathbb{B}_{1,\chi} \equiv 0 \pmod{\mathfrak{P}}$$

is finite. $\square$

The following result is an almost immediate consequence.

THEOREM 4.2: Let $2 \notin \mathscr{P} \subseteq \mathbb{Z}[\varepsilon]$ be an odd prime containing $\mathfrak{b} \cap \mathbb{Z}[\varepsilon]$, and let $m > 0$ be an integer prime to $\mathscr{P}$. The number of odd characters χ of $\mathrm{Gal}(\mathbb{Q}(\mu_{m^\infty})/\mathbb{Q})$ such that

$$\Lambda(\varphi,\chi) \equiv 0 \pmod{(\mathscr{P} + \mathfrak{b})\cdot \mathbb{Z}[\chi]}$$

is finite. $\square$

PROOF: Let $\mathfrak{P}$ be an extension of $\mathscr{P}$ to $\bar{\mathbb{Q}}$. For χ as in the theorem, $\mathbb{B}_{1,\chi}$ is $\mathfrak{P}$-integral. Hence

$$\Lambda(\varphi,\chi) \equiv \chi(N)\cdot \bar{\varepsilon}(m)\mathbb{B}_{1,\chi}\cdot \mathbb{B}_{1,\varepsilon\bar{\chi}} \pmod{\mathfrak{P}}.$$

The result follows from 4.1. $\square$

For an abelian group M let $M_{(2)} = M \otimes \mathbb{Z}[\frac{1}{2}]$.

COROLLARY 4.3: (1) $(\Psi(\Gamma))_{(2)} = (\mathfrak{b} + \mathbb{Z}[\varepsilon])_{(2)}$;

(2) $A_{(2)} \cong \left(\mathbb{Z}[\varepsilon] \big/ \mathfrak{b} \cap \mathbb{Z}[\varepsilon]\right)_{(2)}$

PROOF: By 3.2, $\mathfrak{b} \subseteq \Psi(\Gamma) \subseteq \mathfrak{b} + \mathbb{Z}[\varepsilon]$. We will show $\mathbb{Z}[\varepsilon] \cap \Psi(\Gamma)_{(2)} = \mathbb{Z}[\varepsilon]$. Suppose this is not the case. Then there is a prime ideal $\mathscr{P} \subseteq \mathbb{Z}[\varepsilon]$ not containing 2 for which

$$\mathscr{P} \supseteq \mathbb{Z}[\varepsilon] \cap \Psi(\Gamma).$$

Then for every odd primitive Dirichlet character χ of conductor prime to N, we have $\Lambda(\varphi, \chi) \equiv 0 \pmod{(\mathscr{P} + \mathfrak{b})\mathbb{Z}[\chi]}$ contradicting Theorem 4.2. $\square$

§5. Hecke operators and the Eisenstein ideal

The Hecke operators U_N, T_l, $\langle a \rangle$ ($l \neq N$ prime, $a \in (\mathbb{Z}/N\mathbb{Z})^*$) act in the standard fashion on the homology groups in diagram (1.2). Let ι be the involution of $\mathscr{H}^*$ defined by $z \mapsto -\bar{z}$. Then ι descends to an involution of X which induces an involution, which we also call ι, on the second row of (1.2). The operators U_N, T_l, $\langle a \rangle, \iota$ are mutually commutative.

We now define analogous operators on the first row of (1.2). Clearly the group ring $\mathbb{Z}[GL_2(\mathbb{Q})]$ acts on $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$. For each prime p let

$$U_p = \sum_{k=0}^{p-1} \begin{pmatrix} 1 & k \\ 0 & p \end{pmatrix} \in \mathbb{Z}[GL_2(\mathbb{Q})].$$

For $a \in (\mathbb{Z}/N\mathbb{Z})^*$ let $\langle a \rangle = \sigma_a$ where $\sigma_a \in \Gamma_0(N)$ is any element satisfying the congruence

$$\sigma_a = \begin{pmatrix} * & * \\ 0 & a \end{pmatrix} \pmod{N}.$$

For a prime $l \neq N$ set

$$T_l = U_l + \sigma_l \cdot \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}.$$

Finally, let

$$\iota = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then U_N, T_l, $\langle a \rangle$, $\iota(l \neq N,\ a \in (\mathbb{Z}/N\mathbb{Z})^*)$ define operators on $\mathrm{Div}^0(\mathbb{P}^1(\mathbb{Q}))$ which preserve the subgroup Z and hence also act on $\mathrm{Div}^0(\mathit{cusps})$.

All arrows of (1.2) commute with these operators.

Recall the Dedekind sum $D_\varepsilon : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{Q}[\varepsilon]$. For a prime p define $(D_\varepsilon | U_p) : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{Q}[\varepsilon]$ by

$$(D_\varepsilon | U_p)(r) = \sum_{k=0}^{p-1} D_\varepsilon\left(\begin{pmatrix} 1 & k \\ 0 & p \end{pmatrix} \cdot r\right), \quad r \in \mathbb{P}^1(\mathbb{Q}).$$

If $l \neq N$ is prime define $D_\varepsilon | S_{l,\varepsilon}$ by

$$(D_\varepsilon | S_{l,\varepsilon})(r) = \bar{\varepsilon}(l) \cdot D_\varepsilon\left(\begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \cdot r\right), \quad r \in \mathbb{P}^1(\mathbb{Q})$$

and let $D_\varepsilon | T_{l,\varepsilon} = D_\varepsilon | U_l + D_\varepsilon | S_{l,\varepsilon}$. Also define $(D_\varepsilon | \iota)$ by

$$(D_\varepsilon | \iota)(r) = D_\varepsilon\left(\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cdot r\right).$$

LEMMA 5.1:

(1) *For an arbitrary prime p and $r \in \mathbb{P}^1(\mathbb{Q})$*

$$(D_\varepsilon | U_p)(r) = (\bar{\varepsilon}(p) + p) \cdot D_\varepsilon(r) - \bar{\varepsilon}(p) \cdot D_\varepsilon\left(\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \cdot r\right),$$

(2) *For $l \neq N$ prime,*

$$D_\varepsilon | T_{l,\varepsilon} = (\bar{\varepsilon}(l) + l) \cdot D_\varepsilon,$$

(3) $D_\varepsilon | U_N = N \cdot D_\varepsilon$,

(4) $D_\varepsilon | \iota = -D_\varepsilon$.

PROOF: (1) is an amusing though lengthy calculation. (2) and (3) follow from (1) by letting $p = l \neq N$, $p = N$, respectively. (4) is a restatement of 2.4(a). □

Let $\mathbb{T} \subseteq \mathrm{End}(H_1(X; \mathbb{Z}))$ be the commutative algebra generated over $\mathbb{Z}$ by the operators U_N, T_l, $\langle a \rangle$. By Poincaré duality $\mathbb{T}$ also acts on $H^1(X; B)$ for any abelian group B.

Proposition 5.2: *Let $\varphi \in H^1(X; A)$ be the cohomology class of* 3. *Then*
(1) $\varphi | \langle a \rangle = \bar{\varepsilon}(a) \cdot \varphi$ *for* $a \varepsilon (\mathbb{Z}/N\mathbb{Z})^*$;
(2) $\varphi | T_l = (\bar{\varepsilon}(l) + l) \cdot \varphi$ *for* $l \neq N$ *prime*;
(3) $\varphi | U_N = N \cdot \varphi$;
(4) $\varphi | \iota = -\varphi$.

Proof: Let $\gamma \in \Gamma$ and $[\gamma] \in H_1(X; \mathbb{Z})$ be the corresponding homology class.

(1) $[\gamma] \cap (\varphi | \langle a \rangle) = [\sigma_a \gamma \sigma_a^{-1}] \cap \varphi = \Phi(\sigma_a \gamma \sigma_a^{-1}) = \bar{\varepsilon}(a) \cdot \Phi(\gamma) = [\gamma] \cap (\bar{\varepsilon}(a) \cdot \varphi)$.

(2) Since $[\gamma] = Univ(\gamma \cdot \{i\infty\} - \{i\infty\})$ we have

$$\begin{aligned} [\gamma] \cap (\varphi | T_l) &= Univ(T_l \cdot (\gamma \cdot \{i\infty\} - \{i\infty\})) \cap \varphi \\ &= (D_\varepsilon | T_{l,\varepsilon})(\gamma \cdot \{i\infty\}) - (D_\varepsilon | T_{l,\varepsilon})(\{i\infty\}) \\ &= (\bar{\varepsilon}(l) + l) \cdot (D_\varepsilon(\gamma \cdot \{i\infty\}) - D_\varepsilon(\{i\infty\})) && (5.1(2)) \\ &= (\bar{\varepsilon}(l) + l) \cdot ([\gamma] \cap \varphi). \end{aligned}$$

The proofs of (3), (4) are similar to that of (2) using 5.1(3), (4). □

We see now that A inherits the structure of $\mathbb{T}[\iota]$-module by letting the operators U_N, T_l, $\langle a \rangle$, ι act as N, $\bar{\varepsilon}(l) + l$, $\bar{\varepsilon}(a)$, -1 respectively. With respect to this structure the homomorphism

$$\cap \varphi : H_1(X; \mathbb{Z}) \to A$$

is a $\mathbb{T}[\iota]$-homomorphism.

We will refer to the ideal $I = \mathrm{Ann}_{\mathbb{T}}(A) \subseteq \mathbb{T}$ as the Eisenstein ideal. By Corollary 4.3 $A_{(2)}$ is a cyclic $\mathbb{T}_{(2)}$-module with a canonical generator. Hence there are isomorphisms

$$(\mathbb{T}/I)_{(2)} \cong A_{(2)} \cong (\mathbb{Z}[\varepsilon]/\mathfrak{b} \cap \mathbb{Z}[\varepsilon])_{(2)}. \tag{5.3}$$

The isomorphisms provide a one-to-one correspondence between odd primes $P \subseteq \mathbb{T}$ for which $P \supseteq \mathrm{I}$ and odd primes $\mathscr{P} \subseteq \mathbb{Z}[\varepsilon]$ for which $\mathscr{P} \supseteq \mathfrak{b} \cap \mathbb{Z}[\varepsilon]$. If $P \leftrightarrow \mathscr{P}$ then we have

$$A_P \cong A_{\mathscr{P}} \cong \mathbb{Z}[\varepsilon]_{\mathscr{P}}/\mathfrak{b}_{\mathscr{P}}.$$

Remark 5.4: Let $a \in (\mathbb{Z}/N\mathbb{Z})^*$ be a primitive element and suppose $\varepsilon(\alpha)$ is a primitive d-th root of unity where $d | (N-1)$. Let $F_d(T) \in \mathbb{Z}[T]$ be the irreducible monic polynomial whose roots are the primitive d-th roots of 1. Then

$$\langle U_N - N; T_l - \langle l \rangle - l, l \neq N; F_d(\langle a \rangle) \rangle \subseteq I.$$

It would be interesting to know if these two ideals are in fact equal.

§6. Congruences for special values of L-functions

A theorem of Shimura ([10], Theorem 3.51) asserts that the space $\mathscr{S}_2(\Gamma)$ of weight 2 cusp forms over Γ is a free $\mathbb{T} \oplus \mathbb{C}$-module of rank 1. Hence there is a one-to-one correspondence

$$\left\{\begin{array}{c}\text{normalized}\\ \text{weight two}\\ \text{parabolic}\\ \mathbb{T}\text{-eigenforms,}\\ f\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{homomorphisms}\\ \\ \\ \mathbb{T} \xrightarrow{h_f} \mathbb{C}\end{array}\right\}$$

For an eigenform f, the homomorphism h_f satisfies

$$f|\alpha = h_f(\alpha)\cdot f \quad \text{for} \quad \alpha \in \mathbb{T}.$$

Let $\mathscr{P}(f) = \ker(h_f)$ and $\mathcal{O}(f) = \text{image}(h_f)$. Then $\mathscr{P}(f)$ is a minimal prime ideal in $\mathbb{T}$ and $\mathcal{O}(f)$ is the ring generated by the eigenvalues. The ring $\mathcal{O}(f)$ is a possibly nonmaximal order in its quotient field.

Write H for $H_1(X;\mathbb{Z})$, and let

$$H^- = H/(1 + \iota)H.$$

Another consequence of Shimura's result is:

Proposition 6.2: *$H^- \otimes_{\mathbb{Z}} \mathbb{Q}$ is a free rank 1 $\mathbb{T} \otimes \mathbb{Q}$-module.*

Proof: The pairing

$$(H \otimes \mathbb{C}) \times \mathscr{S}_2(\Gamma) \to \mathbb{C}$$

$$(\gamma, f) \mapsto \int_{(1-\iota)\gamma} \omega(f)$$

factors through $H^- \otimes \mathbb{C}$ to give a nondegenerate pairing

$$(H^- \otimes \mathbb{C}) \times \mathscr{S}_2(\Gamma) \to \mathbb{C}.$$

By Shimura's theorem $(H^- \otimes \mathbb{C})$ is a free rank 1 $\mathbb{T} \otimes \mathbb{C}$-module. □

Let R be either $\mathbb{T}$ or $\mathbb{Z}[\varepsilon]$. If M is an R-module and $P \subseteq R$ is a prime ideal we will write $M_p \cong M \otimes_R R_p$ for the localization of M at P.

In the next theorem we will be concerned with prime ideals $P \subseteq \mathbb{T}$ for which H_p^- is a free rank 1 $\mathbb{T}_p$-module. By the last proposition this includes almost all primes P. Whether in general there exist primes $P \supseteq I$ satisfying this condition we do not know.

THEOREM 6.3: *Let $2 \notin P \supseteq I$ be a prime ideal in $\mathbb{T}$ for which H_p^- is a free rank 1 $\mathbb{T}_p$-module. Let $\mathscr{P} \subseteq \mathbb{Z}[\varepsilon]$ be the corresponding prime obtained from 5.3. Let f be a normalized weight 2 parabolic $\mathbb{T}$-eigenform on $\Gamma_1(N)$ such that $\mathscr{P}(f) \subseteq P$ and let*

$$\pi : \mathcal{O}(f) \twoheadrightarrow A/PA \cong \mathbb{Z}[\varepsilon]/\mathscr{P}$$

be the natural projection.

Then there is a period $\Omega \in \mathbb{C}^*$ such that for all primitive odd quadratic Dirichlet characters χ of conductor m prime to N, and to $\mathscr{P}$

$$(1) \quad \Lambda_f(\chi) \overset{\mathrm{dfn}}{=} \frac{\tau(\chi) \cdot L(f, \chi, 1)}{\Omega} \in \mathcal{O}(f)$$

and

$$(2) \quad \pi(\Lambda_f(\chi)) \equiv \chi(N)\bar{\varepsilon}(m)\mathbb{B}_{1,\chi} \cdot \mathbb{B}_{1,\varepsilon\chi} \pmod{\mathscr{P}}.$$

PROOF: Let $j : H \to H_P^-$ be the natural map. Then the composition

$$H \xrightarrow{\cap \varphi} A \longrightarrow A/PA$$

factors through j to give a homomorphism

$$\Phi_P^- : H_P^- \to A/PA.$$

Let $\gamma_0 \in H$ be such that $\gamma_0 \cap \varphi \equiv 1 \pmod{P \cdot A}$. Then $j(\gamma_0)$ generates H_P^- as a $\mathbb{T}_p$-module. Let

$$\Omega = \frac{1}{2} \int_{(1-\iota)\gamma_0} \omega(f) \in \mathbb{C}^*.$$

The homomorphism $\Phi_f : H \to \mathbb{C}$ defined by

$$\Phi_f(\gamma) = \frac{1}{2\Omega} \int_{(1-\iota)\gamma} \omega(f)$$

has its image in the quotient field of $\mathcal{O}(f)$ and factors through $H \twoheadrightarrow H^-$. Since $\Phi_f(\gamma_0) = 1$, image $(\Phi_f) \subseteq \mathcal{O}(f)_P$. By modifying Ω we may assume image $(\Phi_f) \subseteq \mathcal{O}(f)$ and $\Phi_f(\gamma_0) \equiv 1 \pmod{P \cdot \mathcal{O}(f)}$. This gives rise to a $\mathbb{T}$-

homomorphism

$$\Phi_f^- : H^- \to \mathcal{O}(f).$$

The following diagram commutes

$$\begin{tikzcd} H \arrow[r, "j"] & H_P^- \arrow[r, two heads, "\Phi_P^-"] \arrow[dr, "(\Phi_f^-)_P"'] & A/PA \\ & & \pi_P \arrow[u, "\mathcal{O}(f)_P"'] \end{tikzcd}$$

For an odd quadratic character χ as in the theorem we have $\iota \cdot \Lambda(\chi) = -\Lambda(\chi)$. Hence

$$\Lambda_f(\chi) = \frac{1}{\Omega}(\Lambda(\chi) \cap \varphi_f) = \Phi_f(\Lambda(\chi)) \in \mathcal{O}(f).$$

Also

$$\begin{aligned} \pi(\Lambda_f(\chi)) = \pi \circ \Phi_f(\Lambda(\chi)) &= \Phi_P^- \circ j(\Lambda(\chi)) \\ &\equiv \Lambda(\varphi, \chi) \pmod{\mathscr{P}}. \end{aligned}$$

The result follows from Theorem 3.2 because m is prime to $\mathscr{P}$ and hence $\mathbb{B}_{1,\chi}$ is $\mathscr{P}$-integral. □

§7. Compatibility with the conjecture of Birch and Swinnerton-Dyer

In [7], Mazur gives a descent argument which shows that his congruence formulae are compatible with the conjecture of Birch and Swinnerton-Dyer. A generalization of this descent has been given by the first author [3]. We review the results here.

Let $f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}$ be a normalized weight two newform on $\Gamma_1(N)$, N prime, and let $K(f) = \mathbb{Q}(a_n | n = 1, 2, \ldots)$ be the field generated by the Fourier coefficients. Shimura [10] has shown how one can associate to f a simple abelian variety quotient $A_f/\mathbb{Q}$ of the Jacobian of $X_1(N)$. For a Dirichlet character χ the L-function of $A_f/\mathbb{Q}$ can be expressed as a product:

$$L(A_f, \chi, s) = \prod_{\sigma} L(f_\sigma, \chi, s)$$

where σ ranges through the imbeddings $\sigma : K(f) \hookrightarrow \mathbb{C}$, and f_σ is defined by

$$f_\sigma(z) = \sum_{n=1}^{\infty} a_n^\sigma e^{2\pi i n z}.$$

Now suppose χ is an odd quadratic character and let K_χ be the associated imaginary quadratic field. By a theorem of Shimura ([11], Theorem 1 (iii)),

$$L(f, \chi, 1) = 0 \Leftrightarrow L(f_\sigma, \chi, 1) = 0.$$

Hence the conjecture of Birch and Swinnerton-Dyer predicts:

$$L(f, \chi, 1) = 0 \Leftrightarrow rk(A(K_\chi)) = rk(A(\mathbb{Q})). \tag{7.1}$$

The following theorem is proved in [3].

THEOREM 7.2: *Let $P \supseteq I$ be a locally principal prime ideal in $\mathbb{T}$ of residual characteristic $p \nmid 2 \cdot \mathrm{ord}(\varepsilon)$. If $K_\chi = \mathbb{Q}(\sqrt{-3})$ assume $p \neq 3$. Let $\mathscr{P} \subseteq \mathbb{Z}[\varepsilon]$ be the prime associated to P by (5.3). Then the following implication holds:*

$$\mathbb{B}_{1,\chi} \cdot \mathbb{B}_{1,\chi\varepsilon} \not\equiv 0 \pmod{\mathscr{P}}$$
$$\Rightarrow rk(A_f(K_\chi)) = 0. \qquad \square$$

If this theorem is combined with Theorem 6.3, we obtain a weak form of one implication of (7.1).

In [7] Mazur shows that if f is a newform on $\Gamma_0(N)$, χ is an odd quadratic character, and $L(f, \chi, 1) = 0$ for trivial reasons (i.e. because of the sign of the functional equation), then $rk(A_f(K_\chi)) \geq 1$. In fact he explicitly constructs a point of infinite order, namely, the Birch-Heegner point. In our case, where f is a newform on $\Gamma_1(N)$ with nontrivial Nebentypus, the functional equation relates $L(f, \chi, 1)$ to $L(\bar{f}, \chi, 1)$ where

$$\bar{f}(z) = \sum_{n=1}^{\infty} \bar{a}_n e^{2\pi i n z}$$

is distinct from $f(z)$. Hence there are no "trivial" zeroes. The construction of Birch-Heegner points also does not generalize.

§8. An example: $X_1(13)$

The second author has used an algorithm due to Birch [1] and Manin [5] to compute the universal modular symbol ([5], [7]) for $\Gamma = \Gamma_1(13)$. The results of this calculation show that the curve $X = X_1(13)$ has genus two and that precisely two Nebentypus characters occur in the character decomposition

$$\mathscr{S}_2(\Gamma) = \bigoplus_{\varepsilon} \mathscr{S}_2(\Gamma, \varepsilon)$$

of the space of weight two cusp forms.

Let $\varepsilon : \mathbb{Z} \to \mathbb{C}$ be the Dirichlet character of conductor 13 satisfying $\varepsilon(7) = \omega = e^{2\pi i/6}$. Then

$$\dim_{\mathbb{C}}(\mathscr{S}_2(\Gamma, \varepsilon)) = \dim_{\mathbb{C}}(\mathscr{S}_2(\Gamma, \bar{\varepsilon})) = 1.$$

Let $f \in \mathscr{S}_2(\Gamma, \bar{\varepsilon})$ be the normalized element. Then f is a parabolic $\mathbb{T}$-eigenform with character $\bar{\varepsilon}$. The homomorphism h_f of (6.1) defines an isomorphism

$$h_f : \mathbb{T} \xrightarrow{\sim} \mathbb{Z}[\omega].$$

A direct calculation shows $\frac{1}{2}\mathbb{B}_{2,\varepsilon} = \dfrac{2\cdot(3 + 2\omega)}{(1 + 3\omega)}$. Let $\mathscr{P} \subseteq \mathbb{Z}[\omega]$ be the prime ideal of norm 19 generated by $3 + 2\omega$, and $P = h_f^{-1}(\mathscr{P}) \subseteq \mathbb{T}$. By (5.3) the Eisenstein ideal satisfies $I \subseteq P$. But $h_f(T_2) = (-1 - \omega), h_f(\langle 2 \rangle) = \omega$, so

$$3 + 2\omega = h_f(2 + \langle 2 \rangle - T_2) \in h_f(I).$$

Hence $P = I$, and

$$A \cong (\mathbb{Z}[\omega]/\mathscr{P}) \cong (\mathbb{Z}/19\mathbb{Z}).$$

The question in Remark 5.4 has an affirmative answer in this case.

Since $\mathbb{T}$ is a Dedekind domain, Proposition 6.2 shows that H_P^- is a free rank 1 $\mathbb{T}_P$-module. So Theorem 6.3 applies.

In the following pages we display the values

$$\Lambda_f(\chi) = \frac{\tau(\bar{\chi})L(f, \chi, 1)}{\Omega}$$

where $\Omega \in \mathbb{C}^*$ has been chosen so that

(1) $\Lambda_f(\chi) \in \mathbb{Z}[\omega]$;

(2) $\Lambda_f(\chi) \equiv 3 \cdot \chi(13) \cdot \bar{\varepsilon}(m_\chi) \cdot \mathbb{B}_{1,\chi} \cdot \mathbb{B}_{1,\varepsilon\chi} \pmod{\mathscr{P}}$.

The character χ ranges through the imaginary quadratic characters associated to the fields $\mathbb{Q}(\sqrt{-P_\chi})$ where $0 < p_\chi \leq 3001$ is prime. These numbers have been computed using an algorithm due to Birch [1] and Manin [5].

$X_1(13)$: Algebraic parts, $\Lambda_f(\chi)$, of the special value of the L-function twisted by odd quadratic characters, χ. $\Lambda_f(\chi) = a + b\omega$, $\omega = (1 + \sqrt{-3})/2$.

p_χ	a	b	p_χ	a	b	p_χ	a	b	p_χ	a	b
3	2	−2	5	0	−2	7	0	−2	11	6	−6
13	10	−10	17	8	−8	19	0	−14	23	2	−2
29	0	0	31	−4	0	37	−4	0	41	12	0
43	0	−22	47	4	0	53	0	10	59	0	−2
61	−4	4	67	38	−38	71	0	−2	73	0	−38
79	−8	0	83	28	0	89	6	0	97	54	−54
101	20	0	103	−4	0	107	14	−14	109	0	−56
113	8	−8	127	10	−10	131	−4	0	137	52	−52
139	0	−50	149	56	−56	151	−8	0	157	0	−98
163	0	−50	167	2	−2	173	2	−2	179	14	−14
181	0	−62	191	0	−10	193	−8	0	197	46	0
199	0	−6	211	22	−22	223	18	−18	227	0	−10
229	0	−168	233	0	−8	239	8	0	241	4	−4
251	0	−18	257	16	0	263	6	−6	269	34	−34
271	14	−14	277	20	−20	281	0	−2	283	54	−54
293	52	−52	307	48	0	311	0	0	313	0	−112
317	0	22	331	0	−74	337	0	−94	347	0	10
349	−82	0	353	80	0	359	0	0	367	14	−14
373	28	−28	379	58	−58	383	0	6	389	0	−46
397	158	−158	401	36	0	409	40	−40	419	46	−46
421	0	−134	431	6	−6	433	116	−116	439	2	−2
443	52	0	449	14	−14	457	20	0	461	32	−32
463	−8	0	467	16	0	479	14	−14	487	0	−2
491	70	−70	499	−52	0	503	0	−14	509	88	0
521	0	−58	523	22	−22	541	0	−94	547	8	0
557	76	0	563	0	2	569	14	0	571	−128	0
577	0	−26	587	18	−18	593	0	66	599	8	0
601	−40	0	607	0	−14	613	144	0	617	76	−76
619	−116	0	631	0	−26	641	12	−12	643	0	−122
647	2	−2	653	22	0	659	0	−54	661	16	0
673	−40	0	677	0	8	683	0	18	691	90	−90
701	0	−72	709	84	−84	719	0	2	727	−44	0
733	0	−206	739	150	−150	743	14	−14	751	−30	30
757	−62	0	761	42	−42	769	−122	0	773	38	−38
787	0	−106	797	58	−58	809	28	0	811	−96	0
821	50	0	823	0	−50	827	44	0	829	−40	0
839	0	−18	853	0	−150	857	0	−18	859	−124	0
863	8	0	877	88	−88	881	−84	0	883	84	0
887	18	−18	907	18	−18	911	−4	0	919	0	−38

p_χ	a	b	p_χ	a	b	p_χ	a	b	p_χ	a	b
929	44	-44	937	0	-178	941	0	-24	947	10	-10
953	82	-82	967	24	0	971	0	-26	977	92	0
983	4	0	991	2	-2	997	-24	24	1009	0	-248
1013	0	-18	1019	40	0	1021	148	-148	1031	0	-6
1033	384	-384	1039	-16	0	1049	84	-84	1051	42	-42
1061	0	-114	1063	38	-38	1069	-18	0	1087	-4	0
1091	-8	0	1093	0	-74	1097	0	16	1103	22	-22
1109	32	-32	1117	0	-224	1123	-20	0	1129	20	0
1151	0	-26	1153	96	-96	1163	0	-6	1171	-144	0
1181	98	0	1187	0	-62	1193	10	0	1201	0	-166
1213	100	-100	1217	0	118	1223	4	0	1229	76	-76
1231	0	-26	1237	-8	0	1249	0	-282	1259	158	-158
1277	56	0	1279	-88	0	1283	0	54	1289	-16	0
1291	0	-178	1297	12	0	1301	0	24	1303	6	-6
1307	0	30	1319	0	2	1321	0	-146	1327	4	0
1361	16	-16	1367	26	-26	1373	0	88	1381	64	0
1399	-28	0	1409	0	2	1423	0	-42	1427	22	-22
1429	0	-54	1433	72	0	1439	0	-14	1447	0	-42
1451	16	0	1453	-96	0	1459	34	-34	1471	42	-42
1481	0	-14	1483	40	0	1487	52	0	1489	10	-10
1493	152	0	1499	0	-10	1511	14	-14	1523	42	-42
1531	22	-22	1543	0	-38	1549	-116	0	1553	20	-20
1559	0	0	1567	0	-42	1571	78	-78	1579	0	-266
1583	22	-22	1597	118	0	1601	-30	0	1607	16	0
1609	-224	0	1613	0	-72	1619	0	22	1621	-126	126
1627	298	-298	1637	0	0	1657	356	-356	1663	-24	0
1667	62	-62	1669	0	-126	1693	32	0	1697	20	-20
1699	0	-218	1709	74	-74	1721	0	-90	1723	0	-262
1733	-8	8	1741	0	-110	1747	-32	0	1753	-52	0
1759	0	-6	1777	170	-170	1783	82	-82	1787	0	82
1789	0	-158	1801	504	-504	1811	0	-86	1823	22	-22
1831	-38	38	1847	28	0	1861	-76	0	1867	-164	0
1871	-12	0	1873	0	-206	1877	0	90	1879	0	-46
1889	84	-84	1901	-26	0	1907	0	-46	1913	188	0
1931	0	-150	1933	82	-82	1949	0	-210	1951	-68	0
1973	116	0	1979	58	-58	1987	542	-542	1993	108	-108
1997	0	2	1999	-34	34	2003	8	0	2011	0	-46
2017	-52	0	2027	24	0	2029	0	22	2039	50	-50
2053	0	-218	2063	0	2	2069	114	0	2081	0	-94
2083	138	-138	2087	0	-2	2089	104	-104	2099	0	-38
2111	4	0	2113	438	-438	2129	88	0	2131	-128	0
2137	0	-304	2141	60	-60	2143	118	-118	2153	0	8
2161	-320	0	2179	-228	0	2203	0	-6	2207	10	-10
2213	16	0	2221	-412	0	2237	0	26	2239	-30	30
2243	0	58	2251	-10	10	2267	240	0	2269	28	-28
2273	202	0	2281	-132	132	2287	-52	0	2293	0	32
2297	16	-16	2309	0	-62	2311	-18	18	2333	34	-34
2339	0	0	2341	0	-266	2347	0	18	2351	18	-18
2357	282	-282	2371	128	0	2377	418	0	2381	76	0
2383	0	2	2389	-336	0	2393	0	-34	2399	0	-34
2411	0	-138	2417	0	106	2423	4	0	2437	0	0
2441	0	0	2447	2	-2	2459	38	-38	2467	190	-190
2473	58	0	2477	38	-38	2503	0	-26	2521	0	-26
2531	0	-74	2539	0	-86	2543	44	0	2549	0	-14