JOSE M. BAYOD
J. MARTINEZ MAURICA

A characterization of the spherically complete normed spaces with a distinguished basis


<http://www.numdam.org/item?id=CM_1983__49_1_143_0>
A CHARACTERIZATION OF THE SPHERICALLY COMPLETE NORMED SPACES WITH A DISTINGUISHED BASIS

Jose M. Bayod and J. Martinez Maurica

The theory of normed spaces over a trivially valued field (or valued spaces) was developed mainly by P. Robert in his series of papers [3]. He introduced the concept of distinguished basis, also called orthogonal bases in the literature, and in order to deal with spaces that possess distinguished bases, he restricted himself to $V$-spaces ([3], p. 16), that is, complete valued spaces $E$ such that

$$
\|E\| = \{\|x\|: x \in E\} \subset \{0\} \cup \{\rho^n: n \in \mathbb{Z}\}
$$

for some real number $\rho > 1$. K.-W. Yang, [5], has given a different proof of the fact that $V$-spaces have a distinguished basis. All $V$-spaces are easily shown to be spherically complete.

In this note we give a characterization of all valued spaces which are spherically complete and have a distinguished basis. These spaces need not be $V$-spaces. Moreover, we answer a question of Robert ([3], p. 8), by giving examples of valued spaces without a distinguished basis.

For notations, we refer to [3] and [4].

THEOREM: Let $E$ be a complete valued space over a field $K$ (i.e., a non-archimedean Banach space over a field with the trivial valuation). Then, the following are equivalent:

(i) $E$ has a distinguished (or orthogonal) basis, and it is spherically complete.

(ii) Every strictly decreasing sequence in $\|E\|$ converges to zero.

PROOF: Assume (ii). Let $X \subset E$ be a maximal orthogonal subset of $E$ ([3], p. 9). It is very easy to prove that our hypothesis (ii) implies the
closed linear span of $X$, $[X]$, is spherically complete. Then by Ingleton’s Theorem ([4], Ex. 4.H; the proof also works when $K$ is trivially valued), if $[X] \not= E$, there is a linear projection $P: E \to [X]$ of norm one, and for any $z \in E \setminus [X]$, $z - Pz$ is orthogonal to $[X]$ and different from zero, contradicting the maximality of $X$.

Conversely, assume $E$ has a distinguished basis $X$ and is spherically complete, and that there is a sequence in $\|E\|$ strictly decreasing and bounded away from zero. Since for every nonzero element of $E$ there is some basic vector with the same norm, there must exist a sequence $(x_n)$ in $X$ with strictly decreasing norms but not convergent to zero.

Call $F$ the closed vector subspace $[x_n : n \in \mathbb{N}]$. Then $F$ is linearly isometric to the quotient of $E$ by the subspace generated by the other members of $X$, hence it must be spherically complete (Cf. [4], Th. 4.2). But it is not: consider the sequence of closed balls

$$B(x_1 + \ldots + x_n, \|x_n\|), \quad n \in \mathbb{N}.$$ 

**Remarks:**

1. For non-archimedean Banach spaces over a *non-trivially* valued field, the same is true: a proof can be found in [4], Th. 5.16. That proof also works in our setting, but it is much more elaborated than the one given above; our proof is also valid when the valuation is not trivial, with a minor modification: in that case one cannot be sure that the set of norm values of a basis is the same as $\|E\| \setminus \{0\}$, and one has to change $(x_n)$ into $(\lambda_n x_n)$ for suitable $\lambda_n \in K$.

2. It is not difficult to prove that a valued space is spherically complete and has a distinguished basis if and only if it is linearly isometric with a space $c_0(I : s)$ defined as the set

$$\{x : I \to K \mid \|x(i)\| s(i) \to 0 \text{ for the Frechet filter on } I\}$$

(where $I$ is any nonempty set) endowed with the norm

$$\|x\|_s = \max \{s(i) \mid x(i) \neq 0\}$$

where $s : I \to [0, +\infty)$ is a function whose range does not contain any strictly decreasing sequence with a positive limit.

Consequently, one can give examples of valued spaces with a distinguished basis, apart from $V$-spaces.

3. Now we can produce several examples of valued spaces without a distinguished basis:

   a. Over the real field: the fields $^o\mathbb{R}$ introduced by A. Robinson, regarded as valued spaces over $\mathbb{R}$ (trivially valued), are spherically complete (see [1]), and have $\|^o\mathbb{R}\| = [0, +\infty)$. 
(b) Over any field $K$: the field $E$ of formal power series with coefficients in $K$ and rational exponents, with the set of exponents relative to nonzero coefficients well-ordered is spherically complete ([2], p. 38), and has $\|E\|$ dense in $[0, +\infty)$.

REFERENCES


(Oblatum 6-1-1982)

Facultad de Ciencias
Av. de los Castros
Santander
Spain