H. Kisilevsky

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SOME NON-SEMI-SIMPLE IWASAWA MODULES

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The purpose of this note is to show that the semisimplicity result of [2] may fail to be true in the cases not covered by the theorem in section 2. We base the examples on the idea of J.F. Jaulent [4] although our method in §1 is somewhat different. Theorem 1 of this note gives an alternate proof of Theorem 1 of [2] and Theorem 9 of [4]. We follow the notation in [2].

Let $k/\mathbb{Q}$ be a totally complex abelian extension, and denote by $\Delta = \text{Gal}(k/\mathbb{Q})$. Let $J \in \Delta$ be the automorphism given by complex conjugation (under some fixed embedding of an algebraic closure of $k$ into the complex numbers). Fix a prime $p$, such that $\delta_{p-1} = 1$ for all $\delta \in \Delta$. Let $\hat{\Delta} = \text{Hom}(\Delta, \mu_{p^{-1}}) = \text{Hom}(\Delta, \mathbb{Z}_{p}^*)$ and denote by $V$ the set of characters $\chi$ of $\Delta$ which are either odd or trivial, i.e. $V = \{\chi \in \hat{\Delta} | \chi(J) = -1 \text{ or } \chi = \chi_0\}$.

For each $\chi \in V$, there exists a (unique) $\mathbb{Z}_p$-extension (see [1]) $K_\chi/k$, $\text{Gal}(K_\chi/k) = \Gamma_\chi$ such that $K_\chi/\mathbb{Q}$ is normal and $\text{Gal}(K_\chi/\mathbb{Q}) \cong \Gamma_\chi \cdot \Delta$ a semidirect product with

$$\delta \sigma \delta^{-1} = \sigma^{\chi(\delta)} \text{ for all } \sigma \in \Gamma_\chi, \delta \in \Delta.$$ 

Let $L/K_\chi$ be the maximal abelian unramified $p$-extension of $K_\chi$ so that $\text{Gal}(L/K_\chi) = X \cong \lim_{\rightarrow} A_n$ (where $A_n$ is the $p$-primary subgroup of the ideal class group of $k_n \subseteq K_\chi$, and the limit is taken as usual with respect to the norm maps).

Then, as usual, $X$ is a noetherian torsion $\Delta$-module, so we have

$$X/TX \simeq TX \simeq TX_0 \simeq X_0/TX_0$$

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where $TX = \{x \in X | T x = 0\}$ and $X_0 = \{x \in X | T^k x = 0 \text{ some } k \geq 1\}$ and "\sim" here denotes pseudo-isomorphism.

Since $\Gamma_x$ acts trivially on $X/TX$ and on $\tau X$ we have a natural action of $\Delta$ on these groups and following [4], we study their $\Delta$-decompositions.

If $M$ is a $\Delta$-module which is also a (pro) $p$-group then for $\phi \in \Delta$ write

$$M_\phi = \{m \in M | \delta(m) = \phi(\delta) \cdot m \text{ for } \delta \in \Delta\}$$

and call this the $\phi$-component of $M$.

Now $X_0 \sim \Lambda^{a_1} + \ldots + \Lambda^{a_r}$ for integers $a_1, \ldots, a_r \geq 1$. We say $X_0$ is semi-simple $\iff a_1 = a_2 = \ldots = a_r = 1$ and in this case it is clear that

$$\tau X \sim X_0 \sim X/TX \text{ as } \Delta\text{-modules.}$$

In §1 we compute the $\Delta$-decomposition of $X/TX$ and in §2 we obtain some information of the $\Lambda$-decomposition of $\tau X$.

\section*{§1. The $\Delta$-structure of $X/TX$}

Let $L_0$ be the subfield of $L$ fixed by $TX$ so that $L_0$ is the largest subfield of $L$ abelian over $k$, and $\text{Gal}(L_0/k) \simeq X/TX$. Then $L_0$ is normal over $\mathbb{Q}$, and $L_0 \supseteq \Pi K_\phi$ (compositum taken over certain $\phi$ to be determined) where $[L_0 : \Pi K_\phi] < \infty$. (In fact the Galois group $\text{Gal}(L_0/\Pi K_\phi)$ is the torsion subgroup of $X/TX$ and has certain interest c.f. [3].)

To determine the characters $\phi$ for which $K_\phi \subseteq L_0$, we note $K_\phi \subseteq L_0 \iff K_\phi K_x/K_x$ is unramified at all primes over $p$ and this is a condition which we shall determine locally.

Let $p$ be a prime of $k$ dividing $p$, and let $F = F_0$ be the completion of $k$ at $p$. Let $F_\phi$ be the union of the completions of the finite layers of $K_\phi$ with respect to some consistent choice of primes over $p$. (In the notation of [2], $p = p_i$ some $i$, $F = F_{0,i}$, and $F_\phi = \bigcup_{n \geq 1} F_{n,i}$) Then $F_\phi/F$ is a $\mathbb{Z}_p$-extension, infinitely ramified, such that $F_\phi/\mathbb{Q}_p$ is Galois, and $\text{Gal}(F_\phi/\mathbb{Q}_p) \simeq \mathbb{Z}_p \cdot D$ a semi-direct product where $D \subseteq \Delta$ is the decomposition group of $p$ in $\text{Gal}(k/\mathbb{Q})$ and

$$\delta \sigma \delta^{-1} = \sigma^{\phi(\delta)}$$

for all $\sigma \in \mathbb{Z}_p = \text{Gal}(F_\phi/F)$ and $\delta \in \Lambda$, and $\phi' = \phi|D$.

We state the following two lemmas whose proofs we omit:
**Lemma 1:** If $M$ is the compositum of all $\mathbb{Z}_p$-extensions of $F$, and $G = \text{Gal}(M/F)$, then $G \simeq \mathbb{Z}^{[D]_p+1}$ and we have the $D$-decomposition of $G$ for all $\phi' \in D$,

$$
G_{\phi'} \simeq \mathbb{Z}_p \quad \text{if} \quad \phi' \neq \chi_0
$$

$$
\simeq \mathbb{Z}_p + \mathbb{Z}_p \quad \text{if} \quad \phi' = \chi_0.
$$

**Lemma 2:** $F^p_{\phi}F_X/F_X$ is unramified if and only if either (a) $F^p_{\phi} = F_X$ or (b) $F^\nr \subseteq F^p_{\phi}F_X$, where $F^\nr$ is the unique non-ramified $\mathbb{Z}_p$-extension of $F$ and is equal to $F \cdot \mathbb{Q}^\nr_p$, the compositum of $F$ with the non-ramified $\mathbb{Z}_p$-extension of $\mathbb{Q}_p$.

**Theorem 1:** $K_{\phi}K_X/K_X$ is unramified if and only if

$$
\phi \in V \quad \text{and} \quad \phi|D = \chi|D.
$$

**Proof:** Suppose $K_{\phi}K_X/K_X$ is unramified so that for each $p$ over $p$, we have $F^p_{\phi}F_X/F_X$ is unramified. Hence by Lemma 2, either (a) $F^p_{\phi} = F_X$ so that $\phi|D = \chi|D$ or (b) $F^\nr \subseteq F^p_{\phi}F_X$. In this case, (b), we must have $\text{Gal}(F^p_{\phi}F_X/F_X)_{\chi_0}$ is non-trivial since $\text{Gal}(F^\nr/F)_{\chi_0} \simeq \mathbb{Z}_p$. Since both $F^p_{\phi}/F$ and $F_X/F$ are infinitely ramified it follows that only the $\chi_0$ component of $\text{Gal}(F^p_{\phi}F_X/F)$ is non-zero and so $\phi|D = \chi_0|D = \chi|D$. Hence in either case, $\phi \in V$ and $\phi|D = \chi|D$.

Conversely, suppose $\phi \in V$, and $\phi|D = \chi|D$. If $\chi|D \neq \chi_0|D$ then $F^p_{\phi} = F_X$ by Lemma 1, and so $K_{\phi}K_X/K_X$ is unramified at primes over $p$.

If $\phi|D = \chi|D = \chi_0|D$ then again by Lemma 1 either $F^p_{\phi} = F_X$; or $F^\nr \subseteq F^p_{\phi}F_X$, so again $K_{\phi}K_X/K_X$ is unramified at primes over $p$. Since $K_{\phi}/k$ in unramified outside of primes over $p$, the conclusion of the theorem follows.

**Corollary:** $(X/TX)_\phi \sim \mathbb{Z}_p$ for $\phi \in V$, $\phi|D = \chi|D$, $\phi \neq \chi$, and

$$
\sim 0 \quad \text{otherwise}.
$$

**Remark:** This corollary furnishes another proof of Theorem 1 in [2] and Theorem 9 of [4].

We also note if for any $\phi$ we have $F^p_{\phi} = F_X$, then it follows that $K_{\phi}K_X \subseteq L$ in the notation of [2] and for each $\phi$, $(X'/TX')_\phi$ has non-zero $\mathbb{Z}_p$-rank. This gives many examples of $\mathbb{Z}_p$-extensions where $X'/TX'$ and $TX'$ are infinite.
§2. $\Delta$-structure of $\tau X$

Let $\Gamma = \Gamma_x = \text{Gal}(K_x/k)$, and so $\tau X = \lim A_n$. Since the limit is taken with respect to the norm maps $N_{m,n}$ and since $\delta N_{m,n} = N_{m,n}\delta$ for all $\delta \in \Delta$, it follows that

$$(\tau X)_{\phi} = \lim (A_n^\phi)_{\phi} \text{ for } \phi \in \hat{\Delta}.$$ 

We consider the usual exact sequences

$$1 \to P_n \to I_n \to C_n \to 1$$
$$1 \to E_n \to k_n^* \to P_n \to 1$$

where $I_n$, $C_n$, $P_n$, $E_n$ are the ideal group, class group, group of principal ideals, and unit group of the $n$th layer $k_n$ of $K_x$ respectively.

We obtain the exact sequence

$$1 \to P_n^\Gamma \to I_n^\Gamma \to C_n^\Gamma \xrightarrow{f} NP_n/P_n^\gamma \sim E_0 \cap Nk_n^*/NE_n \to 1$$

where the map $f$ is given below. Choose a fixed generator $\gamma$ of $\Gamma_x$. Then for $x \in C_n^\Gamma$, $\gamma x = x$ and so $\frac{\gamma A}{A} = (\alpha) \in P_n$ for an ideal $A \in x$, define $f(x) = (\alpha) \mod P_n^\gamma$. This is a group homomorphism which is not a $\Delta$-map, (c.f. [4]), but satisfies

$$f: (A_n^\phi)_{\phi} \to (NP_n/P_n^{\gamma-1})_{\phi x}.$$ 

Also the isomorphism is given by:

$$NP_n/P_n^{\gamma-1} \sim E_0 \cap N(k_n^*)/NE_n$$
$$\frac{\gamma A}{A} = (\alpha) \mod P_n^{\gamma-1} \to N(\alpha) \mod N(E_n)$$

where $N$ denotes the norm map $N_{n,0}$ from $k_n$ to $k = k_0$. Hence we obtain the exact sequence

$$1 \to \frac{P_n \cap I_0}{I_0} \to \frac{P_n^\Gamma}{P_0} \to \frac{I_n^\Gamma}{I_0} \to \frac{C_n^\Gamma}{j(C_0)} \to \frac{E_0 \cap N(k_n^*)}{N(E_n)} \to 1$$

(*)

where $j(C_0) \subseteq C_n^\Gamma$ is the subgroup generated by the ideals of $k = k_0$. We shall compute the $\phi$-components of the groups $E_0 \cap N(k_n^*)/E_0^\phi$ and $I_n^\Gamma/I_0$. Since the groups on either side of $C_n^\Gamma/j(C_0)$ are (at worst) quotients
of these, this will describe the set of φ-components of $A_n^T \sim C_n^T/j(C_0)$ which are possibly non-trivial. (As in [2], we use the notation $A_n \sim B_n$ for sequences of groups $\{A_n\}$ and $\{B_n\}$ to mean there are homomorphisms $\phi_n: A_n \to B_n$ whose kernels and cokernels have orders bounded independently of $n$.)

For each prime $p$ of $k$ dividing $p$, let $p = A(p)^* \in I_n$ where $e_p \sim p^n$ is the ramification index of $p$ for $k_n/k$. Since $A$ permutes the primes of $k$ over $(p)$ transitively it follows that

$$I_n^T/I_0 \simeq \bigoplus_{p \mid (p)} \langle A(p) \rangle / \langle p \rangle \simeq \mathbf{Z}/p^n\mathbf{Z}[A/D]$$

where $\langle A(p) \rangle$, $\langle p \rangle$ are the multiplicative subgroups of $I_n^T$ generated by $A(p)$ and $p$ respectively.

Hence it follows that $(I_n^T/I_0)_\phi \simeq \mathbf{Z}/p^n\mathbf{Z}$ if $\phi|D = \chi_0|D$

$\sim 0$ otherwise.

On the other hand by [2, Lemma 1] we have

$$(E_0 \cap N(k_n^*)/E_0p^n)_{\phi_1} \simeq \mathbf{Z}/p^n\mathbf{Z}$$

if $\phi_1 \neq \chi_0$ and $\phi_1|D \neq \chi|D$

$\sim 0$ otherwise.

Since $(A_n)_\phi \to (E_0 \cap N(k_n^*)/NE_n)_{\phi X}$, the possible φ-components of $A_n^T$ which have non-trivial image in this group are among those φ,

$$\phi(J) = \chi(J), \phi \neq \chi^{-1}$$

and $\phi|D \neq \chi_0|D$.

Hence the non-trivial φ-components of $A_n^T$ are among

$$\{\phi|\phi|D = \chi_0|D\} \cup \{\phi|\phi(J) = \chi(J), \phi \neq \chi^{-1}, \phi|D \neq \chi_0|D\}.$$

This provides no restriction in the case that $D \subseteq \ker \chi$ when in fact $X_0$ is semisimple [2].

If $\chi|D \neq \chi_0|D$, then we see that the $\chi^{-1}$ component of $A_n^T$ and that of $\tau X$ must be pseudo-null.

§3. Examples

We now describe a set of characters $\chi$ so that for the $\mathbf{Z}_p$-extensions $K_\chi/k$ the groups $X/\tau X$ and $\tau X$ have different $\Delta$-decompositions. This implies that the corresponding $X_0$ is not semi-simple.

By Corollary of §1, we see that $(X/\tau X)_{\chi^{-1}} \simeq \mathbf{Z}_p$ if $\chi^{-1} \neq \chi$ and $\chi^{-1}|D = \chi|D$, i.e. if $\chi^2 \neq \chi_0$ and $\chi^2|D = \chi_0|D$. On the other hand §2 implies that $(\tau X)_{\chi^{-1}} \sim 0$ if $\chi|D \neq \chi_0|D$ so we have:
For any character $\chi$, such that $\chi^2 \neq \chi_0$, $\chi|D \neq \chi_0|D$ and $\chi^2|D = \chi_0|D$
we have $(TX)_{X^{-1}} \sim 0$ and $(X/TX)_{X^{-1}} \sim \mathbb{Z}_p$.

The examples of Jaulent [4] are of this type.

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Department of Mathematics
Concordia University
Sir George Williams Campus
1455 De Maisonneuve Blvd. West
Montreal
Quebec H3G 1M8
Canada