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SOME NON-SEMI-SIMPLE IWASAWA MODULES

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The purpose of this note is to show that the semisimplicity result of [2] may fail to be true in the cases not covered by the theorem in section 2. We base the examples on the idea of J.F. Jaulent [4] although our method in §1 is somewhat different. Theorem 1 of this note gives an alternate proof of Theorem 1 of [2] and Theorem 9 of [4]. We follow the notation in [2].

Let $k/\mathbf{Q}$ be a totally complex abelian extension, and denote by $\Delta = \mathrm{Gal}(k/\mathbf{Q})$. Let $J \in \Delta$ be the automorphism given by complex conjugation (under some fixed embedding of an algebraic closure of k into the complex numbers). Fix a prime p, such that $\delta^{p-1} = 1$ for all $\delta \in \Delta$. Let $\hat{\Delta} = \mathrm{Hom}(\Delta, \mu_{p-1}) = \mathrm{Hom}(\Delta, \mathbf{Z}_p^*)$ and denote by V the set of characters χ of Δ which are either odd or trivial, i.e. $V = \{\chi \in \hat{\Delta} \mid \chi(J) = -1$ or $\chi = \chi_0\}$.

For each $\chi \in V$, there exists a (unique) $\mathbf{Z}_p$-extension (see [1]) K_χ/k, $\mathrm{Gal}(K_\chi/k) = \Gamma_\chi$ such that $K_\chi/\mathbf{Q}$ is normal and $\mathrm{Gal}(K_\chi/\mathbf{Q}) \cong \Gamma_\chi \cdot \Delta$ a semidirect product with

$$\delta\sigma\delta^{-1} = \sigma^{\chi(\delta)} \text{ for all } \sigma \in \Gamma_\chi, \delta \in \Delta.$$

Let L/K_χ be the maximal abelian unramified p-extension of K_χ so that $\mathrm{Gal}(L/K_\chi) = X \simeq \varprojlim A_n$ (where A_n is the p-primary subgroup of the ideal class group of $k_n \subseteq K_\chi$, and the limit is taken as usual with respect to the norm maps).

Then, as usual, X is a noetherian torsion Λ-module, so we have

$$X/TX \sim {}_TX \sim {}_TX_0 \sim X_0/TX_0$$

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where ${}_TX = \{x \in X \mid Tx = 0\}$ and $X_0 = \{x \in X \mid T^k x = 0 \text{ some } k \geq 1\}$ and "$\sim$" here denotes pseudo-isomorphism.

Since Γ_χ acts trivially on X/TX and on ${}_TX$ we have a *natural* action of Δ on these groups and following [4], we study their Δ-decompositions.

If M is a Δ-module which is also a (pro) p-group then for $\phi \in \hat{\Delta}$ write

$$M_\phi = \{m \in M \mid \delta(m) = \phi(\delta) \cdot m \ \text{ for } \ \delta \in \Delta\}$$

and call this the ϕ-component of M.

Now $X_0 \sim \Lambda/T^{a_1} + \ldots + \Lambda/T^{a_r}$ for integers $a_1, \ldots, a_r \geq 1$. We say X_0 is semi-simple $\Leftrightarrow a_1 = a_2, \ldots = a_r = 1$ and in this case it is clear that

$${}_TX \sim X_0 \sim X/TX \ \textit{as } \Delta\textit{-modules.}$$

In §1 we compute the Δ-decomposition of X/TX and in §2 we obtain some information of the Δ-decomposition of ${}_TX$.

§1. The Δ-structure of X/TX

Let L_0 be the subfield of L fixed by TX so that L_0 is the largest subfield of L abelian over k, and $\mathrm{Gal}(L_0/k) \simeq X/TX$. Then L_0 is normal over $\mathbf{Q}$, and $L_0 \supseteq \Pi K_\phi$ (compositum taken over certain ϕ to be determined) where $[L_0 : \Pi K_\phi] < \infty$. (In fact the Galois group $\mathrm{Gal}(L_0/\Pi K_\phi)$ is the torsion subgroup of X/TX and has certain interest c.f. [3].) To determine the characters ϕ for which $K_\phi \subseteq L_0$, we note $K_\phi \subseteq L_0 \Leftrightarrow K_\phi K_\chi/K_\chi$ is unramified at all primes over p and this is a condition which we shall determine locally.

Let $\mathfrak{p}$ be a prime of k dividing p, and let $F = F_0$ be the completion of k at $\mathfrak{p}$. Let F_ϕ be the union of the completions of the finite layers of K_ϕ with respect to some consistent choice of primes over $\mathfrak{p}$. (In the notation of [2], $\mathfrak{p} = \mathfrak{p}_i$ some i, $F = F_{0,i}$, and $F_\phi = \bigcup_{n \geq 1} F_{n,i}$.) Then F_ϕ/F is a $\mathbf{Z}_p$-extension, infinitely ramified, such that $F_\phi/\mathbf{Q}_p$ is Galois, and $\mathrm{Gal}(F_\phi/\mathbf{Q}_p) \simeq \mathbf{Z}_p \cdot D$ a semi-direct product where $D \subseteq \Delta$ is the decomposition group of p in $\mathrm{Gal}(k/\mathbf{Q})$ and

$$\delta\sigma\delta^{-1} = \sigma^{\phi'(\delta)}$$

for all $\sigma \in \mathbf{Z}_p = \mathrm{Gal}(F_\phi/F)$ and $\delta \in V$, and $\phi' = \phi \mid D$.

We state the following two lemmas whose proofs we omit:

LEMMA 1: *If M is the compositum of all $\mathbf{Z}_p$-extensions of F, and $G = \mathrm{Gal}(M/F)$, then $G \simeq Z_p^{|D|+1}$ and we have the D-decomposition of G for all $\phi' \in \hat{D}$,*

$$G_{\phi'} \simeq \mathbf{Z}_p \quad \text{if} \quad \phi' \neq \chi_0'$$
$$\simeq \mathbf{Z}_p + \mathbf{Z}_p \quad \text{if} \quad \phi' = \chi_0'.$$

LEMMA 2: *$F_\phi F_\chi/F_\chi$ is unramified if and only if either (a) $F_\phi = F_\chi$ or (b) $F^{nr} \subseteq F_\phi F_\chi$, where F^{nr} is the unique non-ramified $\mathbf{Z}_p$-extension of F and is equal to $F \cdot \mathbf{Q}_p^{nr}$, the compositum of F with the non-ramified $\mathbf{Z}_p$-extension of $\mathbf{Q}_p$.*

THEOREM 1: *$K_\phi K_\chi/K_\chi$ is unramified if and only if*

$$\phi \in V \text{ and } \phi | D = \chi | D.$$

PROOF: Suppose $K_\phi K_\chi/K_\chi$ is unramified so that for each $\mathfrak{p}$ over p, we have $F_\phi F_\chi/F_\chi$ is unramified. Hence by Lemma 2, either (a) $F_\phi = F_\chi$ so that $\phi | D = \chi | D$ or (b) $F^{nr} \subseteq F_\phi F_\chi$. In this case, (b), we must have $\mathrm{Gal}(F_\phi F_\chi/F)_{\chi_0'}$ is non-trivial since $\mathrm{Gal}(F^{nr}/F)_{\chi_0'} \cong \mathbf{Z}_p$. Since both F_ϕ/F and F_χ/F are infinitely ramified it follows that only the χ_0' component of $\mathrm{Gal}(F_\phi F_\chi/F)$ is non-zero and so $\phi | D = \chi_0 | D = \chi | D$. Hence in either case, $\phi \in V$ and $\phi | \mathrm{D} = \chi | \mathrm{D}$.

Conversely, suppose $\phi \in V$, and $\phi | D = \chi | D$. If $\chi | D \neq \chi_0 | D$ then $F_\phi = F_\chi$ by Lemma 1, and so $K_\phi K_\chi/K_\chi$ is unramified at primes over $\mathfrak{p}$.

If $\phi | D = \chi | D = \chi_0 | D$ then again by Lemma 1 either $F_\phi = F_\chi$; or $F^{nr} \subseteq F_\phi F_\chi$, so again $K_\phi K_\chi/K_\chi$ is unramified at primes over $\mathfrak{p}$. Since K_ϕ/k in unramified outside of primes over p, the conclusion of the theorem follows.

COROLLARY: $(X/TX)_\phi \sim \mathbf{Z}_p$ *for* $\phi \in V$, $\phi | D = \chi | D$, $\phi \neq \chi$, *and*
~ 0 *otherwise.*

REMARK: This corollary furnishes another proof of Theorem 1 in [2] and Theorem 9 of [4].

We also note if for any ϕ we have $F_\phi = F_\chi$, then it follows that $K_\phi K_\chi \subseteq L'$ in the notation of [2] and for each ϕ, $(X'/TX')_\phi$ has non-zero $\mathbf{Z}_p$-rank. This gives many examples of $\mathbf{Z}_p$-extensions where X'/TX' and ${}_TX'$ are infinite.

§2. Δ-structure of ${}_TX$

Let $\Gamma = \Gamma_\chi = \mathrm{Gal}(K_\chi/k)$, and so ${}_TX = \varprojlim A_n^\Gamma$. Since the limit is taken with respect to the norm maps $N_{m,n}$ and since $\delta N_{m,n} = N_{m,n}\delta$ for all $\delta \in \Delta$, it follows that

$$({}_TX)_\phi = \varprojlim (A_n^\Gamma)_\phi \quad \text{for} \quad \phi \in \hat{\Delta}.$$

We consider the usual exact sequences

$$1 \to P_n \to I_n \to C_n \to 1$$
$$1 \to E_n \to k_n^* \to P_n \to 1$$

where I_n, C_n, P_n, E_n are the ideal group, class group, group of principal ideals, and unit group of the n^{th} layer k_n of K_χ respectively.

We obtain the exact sequence

$$1 \to P_n^\Gamma \to I_n^\Gamma \to C_n^\Gamma \xrightarrow{f} NP_n/P_n^{\gamma-1} \simeq E_0 \cap Nk_n^*/NE_n \to 1$$

where the map f is given below. Choose a fixed generator γ of Γ_χ. Then for $x \in C_n^\Gamma$, $\gamma x = x$ and so $\dfrac{\gamma A}{A} = (\alpha) \in P_n$ for an ideal $A \in x$, define $f(x) = (\alpha) \bmod P_n^{\gamma-1}$. This is a group homomorphism which is not a Δ-map, (c.f. [4]), but satisfies

$$f : (A_n^\Gamma)_\phi \to ({}_NP_n/P_n^{\gamma-1})_{\phi\chi}.$$

Also the isomorphism is given by:

$${}_NP_n/P_n^{\gamma-1} \xrightarrow{\sim} E_0 \cap N(k_n^*)/N(E_n)$$
$$(\alpha) \bmod P_n^{\gamma-1} \to N(\alpha) \bmod N(E_n)$$

where N denotes the norm map $N_{n,0}$ from k_n to $k = k_0$. Hence we obtain the exact sequence

$$1 \to \frac{P_n \cap I_0}{I_0} \to P_n^\Gamma/P_0 \to I_n^\Gamma/I_0 \to C_n^\Gamma/j(C_0) \to \frac{E_0 \cap N(k_n^*)}{N(E_n)} \to 1 \qquad (*)$$

where $j(C_0) \subseteq C_n^\Gamma$ is the subgroup generated by the ideals of $k = k_0$. We shall compute the ϕ-components of the groups $E_0 \cap N(k_n^*)/E_0^{p^n}$ and I_n^Γ/I_0. Since the groups on either side of $C_n^\Gamma/j(C_0)$ are (at worst) quotients of these, this will describe the set of ϕ-components of $A_n^\Gamma \sim C_n^\Gamma/j(C_0)$ which are possibly non-trivial. (As in [2], we use the notation $A_n \sim B_n$ for sequences of groups $\{A_n\}$ and $\{B_n\}$ to mean there are homomorphisms $\phi_n : A_n \to B_n$ whose kernels and cokernels have orders bounded independently of n.)

For each prime $\mathfrak{p}$ of k dividing p, let $\mathfrak{p} = A(\mathfrak{p})^{e_\mathfrak{p}}$ in I_n where $e_\mathfrak{p} \sim p^n$ is the ramification index of $\mathfrak{p}$ for k_n/k. Since Δ permutes the primes of k over (p) transitively it follows that

$$I_n^\Gamma/I_0 \simeq \bigoplus_{\mathfrak{p}|(p)} \langle A(\mathfrak{p})\rangle/\langle \mathfrak{p}\rangle \sim \mathbf{Z}/p^n\mathbf{z}[\Delta/D]$$

where $\langle A(\mathfrak{p})\rangle$, $\langle \mathfrak{p}\rangle$ are the multiplicative subgroups of I_n^Γ generated by $A(\mathfrak{p})$ and $\mathfrak{p}$ respectively.

Hence it follows that $(I_n^\Gamma/I_0)_\phi \sim \mathbf{Z}/p^n\mathbf{Z}$ if $\phi | D = \chi_0 | D$
~ 0 otherwise.

On the other hand by [2, Lemma 1] we have

$$(E_0 \cap N(k_n^*)/E_0 p^n)_{\phi_1} \sim \mathbf{Z}/p^n\mathbf{Z} \text{ if } \phi_1 \text{ even, } \phi_1 \neq \chi_0 \text{ and } \phi_1 | D \neq \chi | D$$
$$\sim 0 \text{ otherwise.}$$

Since $(A_n)_\phi^\Gamma \to (E_0 \cap N(k_n^*)/NE_n)_{\phi\chi}$, the possible ϕ-components of A_n^Γ which have non-trivial image in this group are among those ϕ,

$$\phi(J) = \chi(J),\ \phi \neq \chi^{-1} \text{ and } \phi | D \neq \chi_0 | D.$$

Hence the non-trivial ϕ-components of A_n^Γ are among

$$\{\phi | \phi | D = \chi_0 | D\} \cup \{\phi | \phi(J) = \chi(J),\ \phi \neq \chi^{-1},\ \phi | D \neq \chi_0 | D\}.$$

This provides no restriction in the case that $D \subseteq \ker \chi$ when in fact X_0 is semisimple [2].

If $\chi | D \neq \chi_0 | D$, then we see that the χ^{-1} component of A_n^Γ and that of ${}_TX$ must be pseudo-null.

§3. Examples

We now describe a set of characters χ so that for the $\mathbf{Z}_p$-extensions K_χ/k the groups X/TX and ${}_TX$ have *different* Δ-decompositions. This implies that the corresponding X_0 is *not* semi-simple.

By Corollary of §1, we see that $(X/TX)_{\chi^{-1}} \sim \mathbf{Z}_p$ if $\chi^{-1} \neq \chi$ and $\chi^{-1} | D = \chi | D$, i.e. if $\chi^2 \neq \chi_0$ and $\chi^2 | D = \chi_0 | D$. On the other hand §2 implies that $({}_TX)_{\chi^{-1}} \sim 0$ if $\chi | D \neq \chi_0 | D$ so we have:

For any character χ, such that $\chi^2 \neq \chi_0$, $\chi|D \neq \chi_0|D$ and $\chi^2|D = \chi_0|D$ we have $({}_TX)_{\chi^{-1}} \sim 0$ and $(X/{}_TX)_{\chi^{-1}} \sim \mathbf{Z}_p$.

The examples of Jaulent [4] are of this type.

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