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The dimensions of complemented hilbertian subspaces of uniformly convex Banach lattices


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THE DIMENSIONS OF COMPLEMENTED HILBERTIAN
SUBSPACES OF UNIFORMLY CONVEX BANACH LATTICES

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For $X$ a given Banach space Dvoretzky's Theorem [1] implies that every finite dimensional $E \subset X$ contains a Hilbertian subspace $F$. In this paper we are interested in spaces $X$ for which the $F$'s can always be chosen to be uniformly complemented in $X$, and especially in obtaining estimates for $\dim F$ in terms of $\dim E$. It is clearly necessary to suppose that $X$ doesn't contain $\ell^p_n$'s uniformly. For Banach lattices $X$ Johnson and Tzafriri [8] have shown that the last condition is also sufficient. The novelty of the results presented here is the estimates for $\dim F$ in terms of $\dim E$, which are quite sharp. The main technique used in the proofs is the version of Dvoretzky's Theorem proven by Figiel, Lindenstrauss and Milman in [2]; for properly chosen ellipsoids the Levy means involved there can be estimated using the properties of p-summing operators defined on the space $X$.

This paper was submitted in another place in 1978, and so has been delayed in appearing. Since that time the results presented here have been considerably strengthened: Figiel and Tomczak-Jaegermann [21] extend these results to uniformly convex and $k$-convex spaces; Benyamini and Gordon [20] consider random factorizations of maps more general than the identity on $\ell^p_2$: Pisier's theorem [22] a space not containing $\ell^p_n$'s must be $k$-convex shows all the results mentioned carry over to B-convex spaces.

The notation and terminology used here is for the most part standard. We only recall the definitions used in the statements of theorems. A Banach lattice $L$ is $q$-concave if there is a constant $A > 0$ with

$$\left( \sum_{i \leq n} \|x_i\|^q \right)^{1/q} \leq A \left( \sum_{i \leq n} |x_i|^q \right)^{1/q}$$

for all $x_1, x_2, \ldots, x_n \in L$. Similarly $L$ is $p$-convex if there is a constant

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$B > 0$ with
\[
\left\| \left[ \sum_{i=1}^{n} |x_i|^p \right]^{1/p} \right\| \leq B \left( \sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}
\]
for all $x_1, x_2, \ldots, x_n \in L$. In each case we write $K_q(L)$ and $K_p(L)$ for the best constants $A$ and $B$ appearing in the inequalities. The basic facts about $q$-concave and $p$-convex lattices may be found in [9], [13] and [12].

For $2 \leq s < \infty$ and $E$ a given space we take the $s$-cotype constant of $E$, $\alpha_s(E)$, to be the smallest $\alpha > 0$ such that
\[
\left[ \sum_{i=1}^{m} \|x_i\|^s \right]^{1/s} \leq \alpha \left[ \sum_{i=1}^{m} r_i(t) x_i^s \right]^{1/s}
\]
for any $x_1, x_2, \ldots, x_n \in E$. Here $r_1, r_2, \ldots, r_n, \ldots$ are the Rademacher functions on $[0, 1]$. It is obvious that $\alpha_s(P^N) = 1$, $2 \leq s < \infty$.

The Banach-Mazur distance between isomorphic spaces $E$ and $F$ is
\[
d(E, F) = \inf \{ \|u\| \|u^{-1}\| : u : E \to F \text{ an isomorphism} \}.
\]

Let $X$ be a fixed space and $\lambda \geq 1$. For $E \subset X$ a finite dimensional subspace we define $c_\lambda(E)$ to be the maximum of the dimensions of those $F \subset E$ for which
\begin{enumerate}
  \item[(i)] $d(F, \ell_2^{\dim F}) \leq 2$, and
  \item[(ii)] there is a projection of $X$ onto $F$ of norm at most $\lambda$.
\end{enumerate}
In the terminology of Pelczynski and Rosenthal [16] $X$ is called locally $\pi$-Euclidean if there is a constant $\lambda \geq 1$ and a function $f$ on the natural numbers such that $c_\lambda(E) \geq n$ whenever $\dim E \geq f(n)$.

Finally for $1 < p < \infty$, $p'$ is the conjugate of $p$ ($1/p + 1/p' = 1$).

**Theorem 1:** Let $X$ be a space which is a subspace of quotient of a $p$-convex and $q$-concave Banach lattice $L$, $1 < p \leq 2 \leq q < \infty$. There is a $\lambda \geq 1$ so that, for $E \subset X$ any $n$ dimensional subspace and $s \in [2, q]$,
\[
c_\lambda(E) \geq \lambda^{-1} \min \{ n^{2/p'}, \alpha_s(E)^{-2} n^{2/s} \}.
\]

Before giving the proof we point out some instances of the theorem.

(a) The hypothesis of the theorem implies that $X$ is cotype $q$ [13], and so for some constant $a > 0$ and $\alpha = \min(2/p', 2/q)$,
\[
c_\lambda(E) \geq a (\dim E)^\alpha
\]
whenever $X \subset E$ is finite dimensional. In particular $X$ is locally $\pi$-Euclidean. This result is also stated in [8], though no estimate for $c_\lambda(E)$ is given.
(b) The lattice \( L = L_p(\mu) \) is both \( p \)-convex and \( p \)-concave, \( 1 < p < \infty \), and consequently (\( * \)) holds with \( \alpha = \min(2/p', 2/p) \). For \( 2 \leq p < \infty \) this is well-known, and follows from the results of [2] and [14]. Taking \( E = \ell_p^n \) and using the results of [2] shows that this lower bound for \( c_\lambda(E) \) is best possible for \( L \) an \( L_p \)-space.

(c) In case \( 2 \leq s \leq q \) and \( d(E, \ell_s^n) \leq n^{1/s - 1/p'} \)

\[ \alpha_s(E)^{-2} n^{2/s} \geq d(E, \ell_s^n)^{-2} n^{2/s} \geq n^{2/p'}, \]

and hence (\( * \)) is true with \( \alpha = 2/p' \). For \( E \) a 2-isomorph of \( \ell_s^n, 2 \leq s \leq \min(p', q) \), this gives a lower bound for \( c_\lambda(E) \) depending only on the convexity of \( L \). For Hilbertian subspaces of \( L_p(\mu) \)-spaces \( 1 < p < 2 \) this lower estimate cannot be improved; by [2] \( \ell_p^n, 1 < p < 2 \), contains a Hilbert subspace of dimension \( c_1 n \), but no complemented Hilbert subspaces of dimension greater than \( c_2 n^{2/p'} \). For \( L \) a \( p \)-convex lattice \( (1 < p \leq 2) \) with some non-trivial concavity, every \( n \) dimensional Hilbert subspace is \( c_3 n^{1/p - 1/2} \)-complemented [11].

Below \( X, L, p \) and \( q \) have the same meaning as in the statement of Theorem 1. The proof is preceded by three short lemmas, the first mentioned by Pisier in [18].

The lattice structure enters into the proof only through Lemma 1.

**Lemma 1:** If \( u : X \to G \) is \( q' \)-integral then \( u' \) is \( p' \)-summing and

\[ \pi_{p'}(u') \leq K_p(L)K_q(L)i_q(u). \]

**Proof:** It is enough to show that for \( u : L \to G \) \( q' \)-summing,

\[ i_{p'}(u') \leq K_p(L)K_q(L)i_q(u). \]

By Proposition 3.1 of [11] \( u' \) maps the closed unit ball of \( G' \) into an order bounded set and

\[ \| \sup_{\|x\| \leq 1} |u'(x')| \| \leq K_q(L)\pi_q(u). \]

Since \( L' \) is \( p' \)-concave with \( K_p'(L') = K_p(L) \) [9], the same proposition gives

\[ i_{p'}(u') \leq K_p(L)\| \sup_{\|x\| \leq 1} |u'(x')| \|. \]

**Lemma 2:** If \( 2 \leq s < \infty \), \( H \) is an \( n \) dimensional Hilbert space and \( u : H \to G \) any map, there is a subspace \( A \subset H \) with \( \dim A \geq n/2 \) and

\[ \|u|A\| \leq (2/n)^{1/s} \pi_{(2,s)}(u). \]
PROOF: If the conclusion fails inductively choose \( m \) vectors \( x_1, x_2, \ldots, x_m \) \( \in H \) to satisfy

\[
\|x_i\| = 1, \quad \|u(x_i)\| > (2/n)^{1/s} \pi_{(2,s)}(u)
\]

and

\[
x_k \in [x_1, x_2, \ldots, x_{k-1}]^\perp, \quad 1 \leq k \leq m.
\]

It is clearly possible to choose \( m = \lceil n/2 + 1 \rceil \geq n/2 \) such vectors. But since the \( x_i \)'s are orthonormal

\[
\pi_{(2,s)}(u) \geq \left[ \sum_{i=1}^m \|u(x_i)\|^s \right]^{1/s} > m^{1/s}(2/n)^{1/s} \pi_{(2,s)}(u),
\]

a contradiction.

For \( G \) a finite dimensional space and \( \| \|_2 \) a Hilbertian norm on \( G \), \( G_2 \) denotes \( G \) under \( \| \|_2 \).

**Lemma 3:** Let \( E \subset X \) be any \( n \) dimensional subspace. There is a Hilbertian norm \( \| \|_2 \) on \( E \) and an operator \( v: X \to E_2 \) such that, if \( u: E_2 \to X \) is the formal inclusion, then \( vu = 1_E \) and \( \pi_q(u) = i'_q(v) = n^{1/2} \).

**Proof:** By Theorem 1.1 of [10] there is an isomorphism \( w: L_2^n \to E \) so that \( \pi_q(w) = 1 \) and \( i'_q(w^{-1}) = n \). Define \( \| \|_2 \) on \( E \) by \( \|x\|_2 = n^{-1/2} \|w^{-1}(x)\| \). Clearly \( \pi_q(u) = i'_q(u^{-1}) = n^{1/2} \). For \( v \) take any map \( v: X \to E_2 \) with \( v|E = u^{-1} \) and \( i'_q(v) = i'_q(u^{-1}) \) (such an extension exists by the defining factorization of \( q' \)-integral maps).

**Proof of Theorem 1:** Given \( E \subset X \) of dimension \( n \), choose \( \| \|_2, u \) and \( v \) as in Lemma 3. We claim there is a constant \( a > 0 \) (depending only on \( q \) and \( L \)) and a subspace \( B \subset E \) with \( \dim B \geq n/4 \) such that, if \( u_1 = u \upharpoonright B_2 \) and \( v_1 \) is \( v \) followed by the orthogonal projection of \( E_2 \) onto \( B_2 \), then

\[
v_1u_1 = 1_B,
\]

\[
\pi_q(u_1) \leq n^{1/2},
\]

\[
\pi_q'(v_1) \leq an^{1/2},
\]

\[
\|u_1\| \leq a\alpha_q(E)n^{1/2-1/s},
\]

and

\[
\|v_1\| \leq an^{1/p-1/2}.
\]
Trivially, $\pi_{(2,q)}(u) \leq \pi_q(u)$. For $2 \leq s < q$ the proof of a result of Maurey ([15], Proposition 74, p. 90) implies

$$\pi_{(2,s)}(u) \leq c(q) \alpha_s(E) \pi_q(u) \leq c(q) \alpha_s(E) n^{1/2},$$

where $c(q)$ is Khintchin’s constant. By Lemma 2 there is an $A \subset E$ with $\dim A \geq n/2$ and

$$||u||_{A_2} \leq (2/n)^{1/4} \pi_{(2,s)}(u) \leq c_2 \alpha_s(E) n^{1/2 - 1/2}.$$

By Lemma 1

$$\pi_{(2,p')} (v') \leq \pi_{p'}(v') \leq c_{L,p'} (v) = C_L n^{1/2}$$

and thus, again by Lemma 2, there is a $B \subset E$ with $\dim B \geq (\dim A)/2 \geq n/4$ and

$$||v'||_{B_2} \leq (4/n)^{1/p'} \pi_{(2,p')} (v') \leq c_4 n^{1/p' - 1/2}.$$ Properties (1), (2) and (3) follow immediately from the corresponding properties of $u$ and $v$ (and Lemma 1).

The remainder of the proof now follows using the results and techniques of [2]. Let $S \subset B_2$ be the unit sphere $||x|| = 1$ and $dm$ be the normalized, rotational invariant measure on $S$. Recall that the Levy mean of a continuous real valued function $f$ on $S$ is the number $M_f$ such that

$$m \{ x \in S : f(x) \geq M_f \} = m \{ x \in S : f(x) \leq M_f \}.$$ Let $M$ be the Levy mean of $x \mapsto ||u_1(x)|| = ||x||$ on $S$ and $M^#$ be the Levy mean of $x \mapsto ||v'_1(x)||$ on $S$ (of course $B'_2 = B_2$ naturally). Equality (1) implies that for $x \in S$,

$$1 = \langle u_1(x), v'_1(x) \rangle \leq ||u_1(x)|| ||v'_1(x)||,$$

and consequently

$$1 \leq MM^#.$$ (6)

We now claim that there is a constant $b > 0$, depending only on $p$, $q$ and $L$, such that

$$M \leq b \quad \text{and} \quad M^# \leq b.$$ (7)

To prove the first let $a(q)$ be the constant satisfying

$$||z||_2 = a(q) \left[ \int |(x,z)|^q m(dx) \right]^{1/q}, \quad z \in B_2;$$
\(a(q)\) is the \(q\)-summing norm of the identity on \(B_2\) and \(n/4 \leq \text{dim } B\), so \(a(q) \geq c_5 n^{1/2}\) for some constant \(c_5\) depending only on \(q\) (cf. [4]). By Pietsch’s integral representation theorem [17] there is a probability measure \(\mu\) on \(S\) with

\[
\|u_1(x)\| \leq \pi_q(u_1) \left[ \int \|x, z\|^q \mu(\mathrm{d}z) \right]^{1/q}, \quad x \in B_2.
\]

Thus

\[
M^q \leq 2 \int \|u_1(x)\|^q m(\mathrm{d}x)
\]

\[
\leq 2\pi_q(u_1)^q \int \|x, z\|^q m(\mathrm{d}x) \mu(\mathrm{d}z)
\]

\[
= 2\pi_q(u_1)^q a(q)^{-q}
\]

\[
\leq 2n^{q/2}c_5^{-q}n^{-q/2},
\]

the last by (2). The inequality \(M^\# \leq b\) follows similarly from (3).

By Theorem 2.6 of [2] (and the remarks following) there is an absolute constant \(c > 0\) and an \(F \subset E\) with

\[
\|\|2\text{-equivalent to } M\|\|_2 \text{ on } F,
\]

the norm \(x \mapsto \|v_1(x)\|\) 2-equivalent to \(M^\#\|\|_2 \text{ on } F\), and

\[
\dim F \geq cn \min\{\|u_1\|^{-1}M, \|v_1\|^{-1}M^\#\}^2.
\]

By (6) and (7) \(M\) and \(M^\#\) are at least \(b^{-1}\) so, using (4) and (5),

\[
\dim F \geq c_6 \min\{\alpha_s(E)^{-2}n^{2/s}, n^{2/p'}\}.
\]

Finally, let \(w: B_2 \to F_2\) be the orthogonal projection. Since \(\|v'(x)\| \leq 2b\|x\|_2\) for \(x \in F\), the projection \(wv_1\) has norm at most \(2b\) as an operator from \(X\) into \(F_2\). But \(\|y\| \leq 2b\|y\|_2\) for \(y \in F\), so \(\|wv_1\| \leq 4b^2\) as an operator from \(X\) into \(F\). This concludes the proof. \(\Box\)

A review of the proof of Theorem 1 shows that, once the Hilbert norm \(\|\|_2\) and the operators \(u, v\) have been chosen, the key inequalities are the upper estimates for \(M\) and \(M^\#\) given in terms of \(\pi_q(u)\) and \(\pi_{p'}(v')\). Such estimates are available in several other instances.

Given \(1 \leq p \leq \infty\) a space \(X\) contains \(\mathcal{L}_p^n\)'s uniformly if there is a sequence \((E_n)_{n \geq 1}\) of finite dimensional subspaces of \(X\) with \(\sup_n d(E_n, \mathcal{L}_p^n) < \infty\). If in addition there are projections \(u_n: X \to E_n\) with \(\sup_n \|u_n\| < \infty\), then \(X\) contains uniformly complemented \(\mathcal{L}_p^n\)'s.
THEOREM 2: Let $L$ be a Banach lattice not containing $\ell_\infty^n$'s uniformly, and let $X \subset L$. Then either

(a) $X$ contains uniformly complemented $\ell_1^n$'s,

or

(b) $X$ is locally $\pi$-Euclidean.

In the second case there are positive constants $\lambda$ and $\alpha$ such that

$$c_\lambda(E) \geq \lambda^{-1}(\dim E)\alpha$$

for all finite dimensional $E \subset X$.

THEOREM 3: There is an absolute constant $c > 0$ with the following property. If $E$ is an $n$ dimensional space with a monotone symmetric basis, there is an $m$ dimensional $F \subset E$ and a projection $w: E \rightarrow F$ with

(a) $d(F, \ell_2^m) \leq 2$,

(b) $m \geq c^{-1}d(E, \ell_2^n)^{-2}n$,

and

(c) $\|w\| \leq c \log n$.

Again the first part of Theorem 2 is stated without proof in [8]. The conclusion of Theorem 3 is of interest only in case $d(E, \ell_2^n)^{-2}n$ is substantially larger than $(\log n)^2$; by John’s Theorem [7] $d(E, \ell_2^n) \leq n^{1/2}$ for every $n$ dimensional space, and every $m$ dimensional $F \subset E$ is at least $M^{1/2}$-complemented (of [3]).

PROOF OF THEOREM 2: Assume that $X'$ doesn't contain $\ell_\infty^n$'s uniformly. The arguments of Pisier in [18] show that there is a constant $c > 0$ and indices $p$ and $q$, $1 < p \leq 2 \leq q < \infty$, so that $\pi_{p'}(v') \leq c\ell_q'(v)$ for every $q'$-integral map on $X$. Once this is established as a substitute for Lemma 1, the proof can proceed exactly as before.

PROOF OF THEOREM 3: Let $(e_i)_{i \leq n}$ be a monotone symmetric basis for $E$, set

$$\|x\|_2 = \left[\sum_{i \leq n} |x_i|^2\right]^{1/2} \quad \text{for} \quad x = \sum_{i \leq n} x_i e_i$$

and write $u: E_2 \rightarrow E$, $v: E \rightarrow E_2$ for the formal identities. Every map $g$ of
the form \( g(e_i) = \varepsilon_i e_{\pi(i)} \), with \(|\varepsilon_i| = 1\) for each \( i \) and \( \pi \) a permutation of \( \{1, 2, \ldots, n\} \), is an isometry of both \( E \) and \( E_2 \); further the only maps \( E_2 \to E \) which commute with all such \( g \) are scalar multiples of \( u \). By an averaging argument (cf. [5], Lemma 5.2) \( a(u)\alpha^*(v) = n \) for every Banach ideal norm \( \alpha \). Thus we may assume, normalizing \( \|\cdot\|_2 \) if necessary, that

\[
i_\infty(u) = n^{1/2} \quad \text{and} \quad \pi_1(v) = n^{1/2}.
\]

Let \( a \) and \( b \) be the best constants satisfying

\[
a^{-1}\|x\|_2 \leq \|x\| \leq b\|x\|_2, \quad x \in E.
\]

Another averaging argument shows

\[
ab = d(E, \mathcal{L}_2^n).
\]

\( M \) and \( M^* \) are defined as in the proof of Theorem 1. By that proof, for any \( q \geq 2 \),

\[
M \leq 2^{1/q} \pi_q(\mathcal{L}_2^n)^{-1} \pi_q(u) \quad \text{and} \quad M^* \leq 2^{1/q} \pi_q(\mathcal{L}_2^n)^{-1} \pi_q(v'),
\]

where \( \pi_q(\mathcal{L}_2^n) \) denotes the \( q \)-summing norm of the identity on \( \mathcal{L}_2^n \). Using the expression given in [4] for \( \pi_q(\mathcal{L}_2^n) \) and Stirling’s formula there is an absolute constant \( a > 0 \) such that \( \pi_q(\mathcal{L}_2^n)^{-1} \leq a(q/n)^{1/2} \) for all \( q \geq 2 \).

Any map \( w \) into an \( n \) dimensional space satisfies \( \pi_q(w) \leq n^{1/q}i_\infty(w) \) (cf. [11], Corollary 1.7). Consequently, combining inequalities yields

\[
M \leq 2^{1/q} a(q/n)^{1/2} \pi_q(u)
\]

\[
\leq 2an^{1/q}q^{1/2}n^{-1/2}i_\infty(u)
\]

\[
= 2an^{1/q}q^{1/2}
\]

and similarly

\[
M^* \leq 2an^{1/q}q^{1/2}i_\infty(v')n^{-1/2}
\]

\[
= 2an^{1/q}q^{1/2} \gamma_1(v)n^{-1/2}
\]

\[
\leq 2an^{1/q}q^{1/2} \pi_1(v)n^{-1/2}
\]

\[
= 2an^{1/q}q^{1/2},
\]

the last inequality by [6], Lemma 3.3, since \( E \) has a monotone uncondi-
tional basis. Now taking \( q = \log n \),

\[
MM^* \leq c_1 \log n
\]

for some absolute constant \( c_1 \). Trivially, \( M \geq a^{-1} \) and \( M^* \geq b^{-1} \). Using the method of Figiel-Lindenstrauss-Milman as in the proof of Theorem 1 produces an \( m \) dimensional \( F \subset E \), which is 2-isomorphic to \( \ell_2^m \), \( c_2 \log n \) complemented in \( E \) with

\[
m \geq c_2 n \min\{b^{-1}M, a^{-1}M^*\}^2 \geq c_2 n d(E, \ell_2^n)^{-2}.
\]

There are a number of natural questions about complemented Hilbert subspaces. Let \( X \) be a space with some Rademacher type. Is there a constant \( \lambda > 1 \), depending on \( X \), with

\[
c_\lambda(E) \geq \lambda^{-1}d(E, \ell_2^n)^{-2}
\]

for all \( n \) dimensional \( E \subset X \)? For \( X \) a \( p \)-convex and \( q \)-concave lattice it is known \([11]\) that \( d(E, \ell_2^n) \leq n^{1/p-q/q} \) for \( E \subset X \) having dimension \( n \). For such lattices Theorem 1 gives an apparently stronger result, although in this case it is likely that the correct distance estimate is

\[
d(E, \ell_2^n) \leq c_L \max\{n^{1/p-1/2}, n^{1/2-1/q}\}.
\]

The lattice structure enters into the proofs of our results only through the inequality

\[
\sigma_s(v') \leq c_i(v)
\]

for operators on \( X \). For \( X \) the Schatten \( p \)-trace class of operators on \( \ell_2 \), Pisier \([18]\) has shown that \( (\#) \) fails for every non-trivial pair \( 1 < r \leq s < \infty \). We know of no non-trivial lower estimates for \( c_i(E) \) if \( E \subset C_p \), \( 1 < p < 2 \), although sharp upper estimates for \( d(E, \ell_2^n) \) are available \([19]\). Finally, we know of no space \( X \) on which \( (\#) \) is true which is not a subspace of a quotient of a Banach lattice having some Rademacher type.

References


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