

COMPOSITIO MATHEMATICA

GONZALO RIERA

The geometry of the period mapping on cyclic covers of P^1

Compositio Mathematica, tome 51, n° 1 (1984), p. 131-147

http://www.numdam.org/item?id=CM_1984__51_1_131_0

© Foundation Compositio Mathematica, 1984, tous droits réservés.

L'accès aux archives de la revue « Compositio Mathematica » (<http://www.compositio.nl>) implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/legal.php>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

THE GEOMETRY OF THE PERIOD MAPPING ON CYCLIC COVERS OF \mathbb{P}_1

Gonzalo Riera

Introduction

The tangent space to the period space for Riemann surfaces of genus g at a curve C is naturally isomorphic to the second symmetric product

$$S^{(2)}H^0(C; \Omega_C^1)^* \quad (0)$$

of the dual of the vector space of holomorphic differentials on C . If C is Galois, then its group of automorphisms acts on the vector space (0), and the representation theory of this situation was analyzed classically by Chevalley and Weil. Our aim in this and a future paper is to analyze the relationship between subspaces of (0) described representation-theoretically and the geometric properties of deformations of C in directions lying in these subspaces. The present paper deals with the case in which C is cyclic over \mathbb{P}_1 .

This work was completed under National Science Foundation grant INT 7927206 to the Universidad de Santiago de Chile, Santiago, Chile, and the University of Utah, U.S.A.

Algebraic differential on \mathbb{P}_1

A rational differential form ω on $\mathbb{C} - \{0\}$ is the pull-back of a form on \mathbb{P}_1 if it is homogeneous of degree 0 and satisfies

$$\langle \omega, \theta \rangle = 0$$

where $\theta = x_0 \partial / \partial x_0 + x_1 \partial / \partial x_1$ (see [6]).

Thus an algebraic one-form on \mathbb{P}_1 can be expressed as

$$p(x_0, x_1)\Omega/q(x_0, x_1)$$

where $\Omega = \langle \theta, dx_0 \wedge dx_1 \rangle = x_0 dx_1 - x_1 dx_0$ and p, q are homogeneous polynomials such that

$$\deg p = \deg q - 2$$

Algebraic differentials on a cyclic cover of \mathbb{P}_1

Let n, m be integers $n \geq 2, m \geq 1$ and consider a divisor

$$a_1 + a_2 + \dots + a_{nm} \tag{1}$$

of distinct non-zero complex numbers in $\mathbb{C} \cup \{\infty\} = P_1$. Also, denote by M the line bundle associated to that divisor. Since $H^1(\mathbb{P}_1, \mathcal{O}^*) = \mathbb{Z}$ and the Chern class $c_1(M) = nm$ there exists a line bundle L such that the following diagram commutes

$$\begin{array}{ccc} L & \xrightarrow{\lambda} & M \\ \downarrow \text{id} & & \downarrow \\ \mathbb{P}_1 & \rightarrow & \mathbb{P}_1 \end{array}$$

where λ raises a local section to the power n .

If s is a global section of M given by a homogeneous polynomial

$$F(x_0, x_1) = \prod_{i=1}^{nm} (x_0 - a_i x_1)$$

then $C = \lambda^{-1}(s(\mathbb{P}_1))$ is a cyclic covering whose ramification divisor lies over the given divisor (1).

The function $y = F^{1/n}$ is a homogeneous form of degree m on C so that in affine coordinates for \mathbb{P}_1 , the surface is the Riemann surface of the equation

$$y^n = (x - a_1) \dots (x - a_{nm}).$$

Let $\sigma: C \rightarrow C$ be a generator of the natural cyclic group of analytic automorphisms of the curve. The automorphism σ^* acts on differentials on C and thus any differential can be expressed as a linear combination of

$$\omega_k \quad k = 0, 1, \dots, n-1$$

where

$$\sigma^*(\omega_k) = \zeta^{-k} \omega_k \quad (\zeta = \exp(2\pi i/n)).$$

If we multiply this differential by the function

$$y^k / x_0^{mk}$$

we obtain a differential on \mathbb{P}_1 :

$$y^k \omega_k / x_0^{mk} = p\Omega/q.$$

Thus redefining p and q we have that any algebraic differential on C is a linear sum of

$$p\Omega/qy^k \tag{2}$$

with p, q relatively prime homogeneous polynomials such that

$$\deg p + 2 = \deg q + km.$$

A differential (2) will have no-poles on C if and only if $q = 1$ and we can write

$$H^{1,0}(C, \mathbb{C}) = \bigoplus_{k=1}^{n-1} H_k^{1,0}$$

where

$$H_k^{1,0} = \{ p\Omega/y^k; \deg p = km - 2 \} \tag{3}$$

for all $k = 1, 2, \dots, n-1$. Adding up the dimensions of these eigenspaces we obtain

$$\sum_{k=1}^{n-1} (km - 1) = m(n-1)/2 - (n-1) = g$$

where g is the genus of C .

Algebraic Hodge decomposition

Since we have just obtained an explicit expression for $H^{1,0}$ we have to characterize the quotient

$$H^1(C, \mathbb{C})/H^{1,0}(C, \mathbb{C}) = H^{0,1}(C, \mathbb{C}).$$

The answer is given in terms of meromorphic differentials on the curve that have poles on the branch points.

THEOREM: *The automorphism $\sigma : C \rightarrow C$ induces a decomposition in eigenspaces*

$$H^1/H^{1,0} = \bigoplus_{k=1}^{n-1} (H^1/H^{1,0})_k$$

where the k^{th} summand can be naturally identified with

$$\{\omega^k = p\Omega/y^{n+k}; \deg p = m(n+k) - 2\}/I_k \quad (4)$$

where I_k consists of those forms ω_k that satisfy

$$p \subset (\partial F/\partial x_0, \partial F/\partial x_1)I,$$

the homogeneous Jacobian ideal.

PROOF: Let B be the divisor on C which maps to the divisor (1) on P . The natural inclusion $C-B \rightarrow C$ induces in cohomology the exact sequence

$$0 \rightarrow H^1(C) \rightarrow H^1(C-B) \xrightarrow{R} \bigoplus_{p \in B} \mathbb{C}_p \xrightarrow{\Sigma} \mathbb{C} \rightarrow 0$$

where R applied to a differential form gives the residues at the branch points.

The cyclic group generated by σ acts on each term of this sequence and if we set

$$H_k^1(C) = \{w \in H^1(C); \sigma^*(w) = \zeta^{-k}w\}$$

($k = 0, \dots, n-1$) and similarly for $H_k^1(C-B)$ we obtain the exact sequences:

$$0 \rightarrow H_0^1(C) \rightarrow H_0^1(C-B) \rightarrow \bigoplus_{p \in B} \mathbb{C}_p \rightarrow \mathbb{C} \rightarrow 0$$

$$0 \rightarrow H_k^1(C) \rightarrow H_k^1(C-B) \rightarrow 0 \quad k = 1, \dots, n-1.$$

Since $H_0^1(C)$ is the space of forms invariant under σ , it is the space of forms on \mathbb{P}_1 with no singularities, thus

$$H_0^1(C) = 0$$

and

$$H^1(C) = \bigoplus_{k=1}^{n-1} H_k^1(C) \cong \bigoplus_{k=1}^{n-1} H_k^1(C-B).$$

Moreover since σ is an analytic map, the decomposition into eigenspaces is compatible with the Hodge decomposition, that is

$$H_k^{1,0} \subset H_k \quad \text{and}$$

$$H^1/H^{1,0}(C) \cong \bigoplus_{k=1}^{n-1} H_k^1(C-B)/H_k^{1,0}(C-B).$$

We will compute these last terms using the “algebraic de Rham theorem” by Grothendieck (cf. [5]).

For an affine variety S , $H^1(S) \cong H^1(A^*)$, where A^* is the complex of algebraic differentials.

The decomposition into eigenspaces gives

$$H_k^1(C - B) \cong H^1(A_k^*, d)$$

where the complex $A_k^0 \xrightarrow{d} A_k^1$ has an increasing filtration

$$\begin{aligned} A_k^0(l) &= \{p/y^{nl+k}\} & l = 0, 1, 2, \dots \\ A_k^1(l) &= \{q\Omega/y^{n(l+1)+k}\} & l = 1, 0, 1, \dots \end{aligned} \quad (5)$$

for homogeneous polynomials p and q of appropriate degrees. We can now write a Koszul resolution as in Clemens (cf. [4]).

Recall that

$$\theta = x_0 \partial / \partial x_0 + x_1 \partial / \partial x_1$$

and set

$$v = \mathbb{C} dx_0 \oplus \mathbb{C} dx_1.$$

For $l, r \in \mathbb{Z}$ let $P_{k,l}^r$ be the vector space of homogeneous forms of degree $mn(l+r) + mk - 1$ in x_0 and x_1 . We can then define natural epimorphisms

$$\alpha: P_{k,l}^0 v \rightarrow A_k^0(1) / A_k^0(l-1)$$

$$\omega \rightarrow \langle \theta, \omega \rangle / y^{nl+k}$$

and

$$\beta: P_{k,l}^1 \oplus \Lambda^2 v \rightarrow A_k^1(l) / A_k^1(l-1)$$

$$\omega \rightarrow \langle \theta, \omega \rangle / y^{n(l+1)+k}.$$

Next recall that

$$\langle \theta, dF \wedge \omega \rangle = mnF\omega - dF \wedge \langle \theta, \omega \rangle. \quad (6)$$

Now $\alpha(\omega) = 0$ if and only if F divides $\langle \theta, \omega \rangle$ that is

$$\langle \theta, \omega \rangle = F\gamma.$$

But then

$$\langle \theta, \omega - \gamma dF/nm \rangle = 0$$

so that

$$\omega - \gamma dF/nm = \langle \theta, q dx_0 \wedge dx_1 \rangle.$$

Thus

$$\ker \alpha = \{ p dF + q \Omega \}.$$

Also

$$\ker \beta = \{ p F d x_0 \wedge d x_1 \}.$$

Using (6) to insure commutativity, we can write the following diagram

$$\begin{array}{ccccccc}
 0 & \rightarrow & P_{k,l}^{-1} & \rightarrow & \ker \alpha & \rightarrow & \ker \beta & \rightarrow & P_{k,l}^1/I_{k,l} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 0 & \rightarrow & P_{k,l}^{-1} & \rightarrow & P_{k,l}^0 \oplus V & \rightarrow & P_{k,l}^1 \oplus \Lambda^2 V & \rightarrow & P_{k,l}^1/I_{k,l} & \rightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
 0 & \rightarrow & 0 & \rightarrow & A_k^0(l)/A_k^0(l-1) & \rightarrow & A_k^1(l)/A_k^1(l-1) & \rightarrow & 0 & & \\
 & & & & & & & & & & \\
 & & & & & & & & & & (7)
 \end{array}$$

where $I_{k,l} = I \cap P_{k,l}^1$.

The middle row is exact, it is just a Koszul resolution. The top row is exact except at $P_{k,l}^1/I_{k,l}$. To see this, exactness at $\ker \beta$ is just the identity (6) that, for two-forms ω ,

$$mnF\omega = dF \wedge \langle \theta, \omega \rangle.$$

Exactness at $\ker \alpha$ is just the fact that

$$dF \wedge \Omega = mnF dx_0 \wedge dx_1.$$

So (7) is a short exact sequence of complexes. Our remarks on each of the other two complexes then gives, via the long exact sequence in cohomology,

$$\begin{aligned}
 H^0(A_k^*(l)/A_k^*(l-1)) &= 0 \\
 H^1(A_k^*(l)/A_k^*(l-1)) &= P_{k,l}^1/I_{k,l}.
 \end{aligned} \tag{8}$$

The complexes $A_k^*(l)$ filter the complex A_k^* whose cohomology is

$H_k^*(C-B)$. The spectral sequence associated to this filtration has

$$E_1^{l,q} = H^{l+q}(A_k^*(l)/A_k^*(l-1))$$

so that

$$E_1^{l,q} = 0$$

unless $l+q=1$. So this spectral sequence degenerates at E_1 and we can compute the resulting filtration on $H_k^1(C-B)$ via (8), namely,

$$E_\infty^{-1,2} = H^1(A_k^*(-1)) = P_{mk-2}$$

$$E_\infty^{0,1} = H^1(A_k^*(0)/A_k^*(-1))$$

$$= P_{m(n+k)-2}/I_{m(n+k)-2}$$

etc.

To finish the proof of the theorem, notice that, referring to (3),

$$E_\infty^{-1,2} = H_k^{1,0}$$

and, by the exactness of the middle row of (7), we have that if $l \geq 1$,

$$\begin{aligned} \dim E_\infty^{l,1-l} &= [m(l+1)n+k] - 1 - 2[m(l+1)n+k] + mn \\ &\quad + [m(l+1)n+k] - 2mn + 1 = 0. \end{aligned}$$

We will compute the dimension of $E_\infty^{0,1}$ in order to motivate our next result. Namely $\dim E_\infty^{0,1} = [m(n+k)-1] - 2[m(n+k)-mn] = -[m(n+k)-2mn+1] = mn-k+1 = \dim H_{n-k}^{1,0}$.

This is as it should be since, under the cup-product pairing, if $w \in H_r^{1,0}$, $y \in H_s^1$, then

$$w \wedge y \in H_{r+s}^2 = 0 \quad \text{unless } r+s=n.$$

More precisely, we have the following result.

PROPOSITION: *Under the cup product $H^{1,0} \times H^{0,1} \rightarrow \mathbb{C}$ the space $H_k^{1,0}$ is orthogonal to $(H^1/H^{1,0})_i$ for $i \neq n-k$.*

For differential forms

$$\varphi = p\Omega/y^k \in H_k^{1,0}$$

$$\psi = q\Omega/y^{2n-k} \in (H^1/H^{1,0})_{n-k}$$

the following identity holds

$$\begin{aligned} (\varphi, \psi) &= \int \varphi \wedge \psi \\ &= (2\pi i/n - k)n^2 \sum_{j=1}^{nm} p(a_j)q(a_j)/ \\ &\quad \times (a_j - a_1)^2 \dots \wedge \dots (a_j - a_{nm})^2. \end{aligned} \quad (9)$$

PROOF: The first statement follows from the fact that, under the cup-product,

$$H_{k_1}^{1,0}(C) \otimes (H^1(C)/H^{1,0}(C))_{k_2}$$

goes into $H_{k_1+k_2}^{1,1}(C)$, which is zero for $k_1 + k_2 \neq n$.

For the second statement, we proceed as follows. Using affine coordinates for our curve C

$$y^n = F(x) = \prod_{j=1}^{nm} (x - a_j)$$

so that

$$ny^{n-1}dy = F'(x)dx.$$

Near $x = a_j$ we write

$$\begin{aligned} \varphi &= p(x)dx/y^k = p(x)y^{n-k-1}dx/y^{n-1} = np(x)dy/F'(x)y^{n-k-1} \\ &= \beta_j(x)y^{n-k-1}dy. \end{aligned}$$

Similarly $\psi = nq(x)dy/F'(x)y^{n-k+1} = \alpha_j(x)dy/y^{n-k+1}$.

For each $j = 1, \dots, nm$, let ρ_j be a C^∞ function on C supported in a neighborhood $|y| < \varepsilon$ of $x = a_j$ and identically equal to one the smaller neighborhood $|y| \leq \varepsilon/2$.

The form

$$\tilde{\psi} = d\left(\frac{1}{n-k} \sum_{j=1}^{nm} \rho_j \alpha_j / y^{n-k}\right) + \psi$$

is in the same cohomology class as ψ but is C^∞ in C , analytic outside the neighborhoods $|y| < \varepsilon$.

We the have

$$(\varphi, \psi) = (\varphi, \tilde{\psi}) = \sum_{j=1}^{nm} \int_{|y| \leq \varepsilon} \varphi \wedge \tilde{\psi}.$$

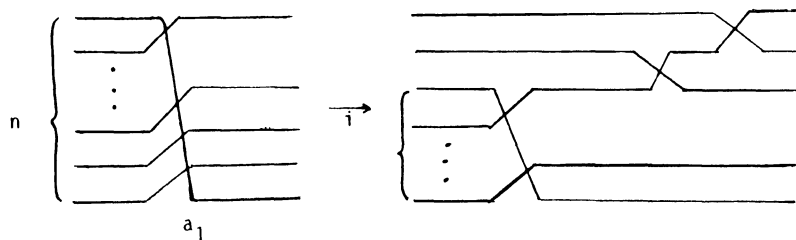


Figure 1

Since $\tilde{\psi}$ is analytic also in $|y| \leq \epsilon/2$ we get

$$(\varphi, \psi) = \sum_{j=1}^{nm} \int_{\epsilon/2 \leq |y| \leq \epsilon} \varphi \wedge \partial/\partial \bar{y} (1/n - k\rho_j \alpha_j / y^{n-k}) d\bar{y}$$

and by Stokes theorem

$$\begin{aligned} (\varphi, \psi) &= \frac{1}{n-k} \sum_{j=1}^{nm} \int_{|y|=\epsilon/2} \beta_j(x) y^{n-k-1} \alpha_j dy / y^{n-k} \\ &= \frac{2\pi i}{n-k} \sum_{j=1}^{nm} \beta_j(a_j) \alpha_j(a_j). \end{aligned}$$

proving the identity.

Deformations of a cyclic cover

Let τ_n^1 be the period space of curves of a given genus that can be expressed as an n to 1 branched cover of \mathbb{P}_1 . We will consider a neighborhood of the cyclic cover C given by an equation

$$y^n = (x - a_1) \dots (x - a_{nm}).$$

A deformation within τ_n^1 will split each branch point of order n into several branch points of lower order; a stratification of these deformations can be given in the following form:

At... a branch point, a_1 say, there is associated a cyclic permutation $(1, 2 \dots n)$ of the sheets of the Riemann surface. For each $i = 2, \dots, n$ we will say that a deformation is of type i if it splits a_1 into a branch point with associated permutation $(1, 2 \dots i)$ and $(n - i)$ simple branch points (see Fig. 1).

A locally irreducible, analytic curve in the period space will determine a non-zero tangent vector (which generates its tangent cone at C) which we consider as an element in

$$Sym^{(2)}(H^{1,0})^*.$$

We denote by v_i the tangent cone to deformations of type greater than or equal to i . Thus $v_{i+1} \subset v_i$, v_n is the set of all tangent vectors to deformations by cyclic covers and v_2 is the tangent space to τ_n^1 .

On the other hand, the cyclic group of order n generated by the automorphism $\sigma : C \rightarrow C$ acts on the space $Sym^{(2)}(H^{1,0})$. We say that a quadratic differential q_j is of type j , $1 \leq j \leq n$, if it is an eigenvector for the eigenvalue ζ^j , i.e.

$$\sigma^*(q_j) = \zeta^j q_j \quad (\zeta = \exp(2\pi i/n)).$$

An element q in $Sym^{(2)}(H^{1,0})$ is a linear sum of these eigenvectors, that is

$$q = \sum_{j=1}^n q_j,$$

and we will call q_j the component of type j of the quadratic differential q .

There is a close relationship between eigenvalues and splitting of branch points given by the following.

THEOREM: *The tangent cone v_i is always a (possibly non reduced) linear space.*

A quadratic differential is orthogonal to v_i if and only if its components of type $j = 2, \dots, n - i + 1, n$ vanish to orders

$$n, \dots, 2n - i - 1, 2n - 2 \text{ at branch points.}$$

In particular the cotangent space of cyclic deformations consists of those quadratic differentials whose invariant component vanishes at branch points to order $2n - 2$.

PROOF: We first fix the notation for a basis of $H^{1,0}(C)$.

$$\text{Let } e_\nu(\lambda) = x^\nu dx / y^\lambda$$

where $0 \leq \nu \leq \lambda m - 2$, $\lambda = 1, \dots, n - 1$. For fixed λ these differentials form a basis of $H_\lambda^{1,0}(C)$; $\sigma^*(e_\nu(\lambda)) = \zeta^{-\lambda} e_\nu(\lambda)$.

In terms of a local parameter centered at a_1 ,

$$(x - a_1) = z^n$$

we can write

$$e_\nu(\lambda) = p(z) dz, p(z) = C_{\lambda,\nu} z^{n-\lambda} (1 + \alpha z^n + \dots)$$

where $C_{\lambda, \nu} = na_1^\nu / [(a_1 - a_2) \dots (a_1 - a_{nm})]^{\lambda/\nu}$ and $\alpha \in C$. (There will of course be analogous expressions for a_2, \dots, a_{nm} .)

The vectors of the dual basis of $(H^{1,0})^*$ will be denoted by $e^\nu(\lambda)$.

To deform the curve C we will take a covering of C by coordinate charts whose changes of parameters depend on t , $|t| < \varepsilon$ (see [1]).

Here we take a local deformation of the parameter z centered at a_1 given by

$$\psi(z, t) = (x - a_1)$$

with $\psi(z, 0) = z^n$; the rest of the local parameters remain fixed. A differential of the first kind on C_t will be written in terms of z as

$$p(z, t)dz = p(z, t)dx/(\partial\psi/\partial z)$$

where $p(z, t)$ is an analytic function. Since

$$\frac{\partial\psi}{\partial z} \frac{dz}{dt} + \frac{\partial\psi}{\partial t} = 0$$

we compute the derivate of the differential at $t = 0$ to be

$$\left(\frac{\partial\psi}{\partial z}\right)^{-2} \left[p(z) \left(-\frac{\partial^2\psi}{\partial z\partial t} \frac{\partial\psi}{\partial z} + \frac{\partial^2\psi}{\partial z^2} \frac{\partial\psi}{\partial t} \right) - \frac{\partial p}{\partial z} \frac{\partial\psi}{\partial z} \frac{\partial\psi}{\partial t} \right] dz + \frac{\partial p}{\partial t} dz. \quad (10)$$

This is in general a differential of the 2nd kind with pole at a_1 , that is, an element in

$$H^1/H^{1,0}.$$

The linear map $\varphi \in \text{Hom}(H^{1,0}, H^1/H^{1,0}) \cong H^{1,0*} \otimes H^{1,0*}$ corresponding to the deformation associates to $p(z)dz$ the differential (10) (where we can disregard the analytic part $\partial p/\partial t dz$). In general, a deformation ψ depends on $n - 1$ parameters t_2, \dots, t_n given as

$$\psi(z, t_2, \dots, t_n) = z^n + t_2 z^{n-2} + \dots + t_n = (x - a_1).$$

(See [2] also.) We compute separately the linear mapping φ_i associated to

$$\psi_i(z, t_i) = z^n + t_i z^{n-i} \quad i = 2, \dots, n.$$

In this case (10) gives

$$(nz^i)^{-1} ((i-1)p(z) - z(p'(z)))dz$$

and replacing $p(z)$ by the corresponding expressions for the differentials $e_\nu(\lambda)$ we obtain

$$n^{-1} C_{\lambda, \nu} z^{n-(\lambda+i)-1} [(i+\lambda-n) + \beta z^n + \dots] dz$$

for some constant $\beta \in \mathbb{C}$. This differential has no poles for $i + \lambda \leq n$ and for $i + \lambda \geq n + 1$ we can compute the cup product

$$\begin{aligned} & (e_\mu(\lambda'), \varphi_i(e_\nu(\lambda))) \\ &= C_{\lambda,\nu} C_{\lambda',\mu} \int_{|z|=e} z^{2n-(\lambda+\lambda'+i)-1} (1 + \gamma z^n + \dots) dz. \end{aligned}$$

This integral vanishes unless $\lambda + \lambda' + i = 2n$ and we obtain

$$\varphi_i = \sum_{\nu,\mu} d_{\nu,\mu} \sum_{\lambda \geq n-i+1} e^\nu(\lambda) \otimes e^\mu(2n - (i + \lambda))$$

where

$$d_{\nu,\mu} = (2\pi\sqrt{-1}) a_1^{\nu+\mu} / [(a_1 - a_2) \dots (a_1 - a_{nm})]^{2-i/n}.$$

Thus $\sigma^*(\varphi_i) = \zeta^{-i} \varphi_i$ and it follows that a quadratic differential will be orthogonal to φ_i if and only if its component of type i vanishes at a_1 at the order $n + i - 2$. More precisely, let

$$q_i = \sum_{\nu,\mu} \alpha^{\nu,\mu} \left(\sum_{\lambda+\lambda'=n-i} e_\nu(\lambda) \otimes e_\mu(\lambda') + \sum_{\lambda+\lambda'=2n-i} e_\nu(\lambda) \otimes e_\mu(\lambda') \right)$$

be a differential of type i ; the first sum in parenthesis vanishes always to order $n + i - 2$. q_i is orthogonal to φ_i if

$$(q_i, \varphi_i) = (cte) \sum_{\nu,\mu} \alpha^{\nu,\mu} a_1^{\nu+\mu} = 0.$$

But this means that in terms of the local parameter z the expansion of the second sum has a first coefficient equal to zero, and thus q_i vanishes to order $n + i - 2$.

It remains to characterize the tangent cones v_i in terms of the parameters t_2, \dots, t_n . We have drawn in Fig. 2 the situation for $n = 4$.

The mapping, from the z -disk to a neighborhood of a_1 given by

$$\psi(z, t_2, \dots, t_n) = x - a_1$$

will be branched at the zeroes of the derivative with respect to z . Two of these branch points will coincide for the values of (t_2, \dots, t_n) that satisfy both equations

$$\partial\psi/\partial z = nz^{n-1} + (n-2)t_2z^{n-3} + \dots + t_{n-1} = 0$$

$$\partial^2\psi/\partial z^2 = n(n-1)z^{n-2} + \dots + 2t_{n-2} = 0.$$

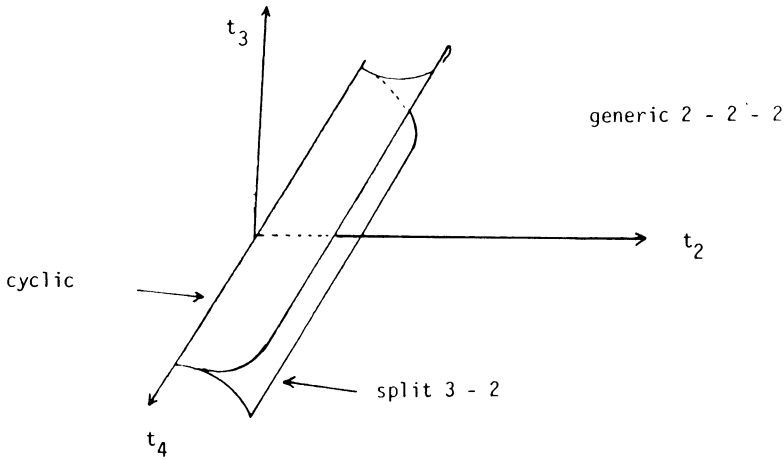


Figure 2

These equations in z will have a common root if their resultant

$$A = \begin{vmatrix} n & 0 & & \dots & & t_{n-1} & 0 & 0 \\ 0 & n & 0 & \dots & & t_{n-1} & 0 & 0 \\ & & & & & & & 0 \\ & & & & & & & t_{n-1} \\ n & (n-1) & 0 & \dots & & 2t_{n-2} & 0 & 0 \\ 0 & n & (n-1) & \dots & & 2t_{n-2} & 0 & 0 \\ & & & \ddots & & \dots & & \\ & & & & & & & 0 \\ 0 & & & \dots & n(n-1) \dots & & 2t_{n-2} & \end{vmatrix}$$

vanishes. This represents an hypersurface in (t_2, \dots, t_n) space corresponding to a triple branch point or more. More generally the rank of the matrix is smaller or equal to $2n - i - 1$ if and only if $\psi(z, t_2 \dots t_n)$ has a branch point of order i at least.

The hypersurface $A = 0$ is singular at 0 and its tangent cone is given by

$$\sum_{\alpha_2 + \dots + \alpha_n = n-2} \partial^{n-2} A / \partial t_2^{\alpha_2} \dots \partial t_n^{\alpha_n} t_2^{\alpha_2} \dots t_n^{\alpha_n} = 0$$

But from the last $n - 2$ columns in the matrix it is clear that all partial derivatives vanish at zero except

$$\partial^{n-2} A / \partial t_{n-1}^{n-2}(0)$$

and so the tangent cone is given by

$$t_{n-1}^{n-2} = 0, \quad \text{or} \quad t_{n-1} = 0.$$

This is then the tangent cone corresponding to v_3 , the tangent cone to deformations of type greater or equal to 3.

For a deformation to be of type greater than or equal to 4 it is necessary and sufficient that the discriminant of $\partial^2\psi/\partial z^2$, $\partial^3\psi/\partial z^3$ vanishes also. Denoting this discriminant by B , the locus in (t_2, \dots, t_n) space is given by $A = 0$ and $B = 0$. (Furthermore, the singular locus of the surface $A = 0$ is given by this intersection.)

The tangent cone at 0 is then the intersection of the tangent cones to both surfaces, that is

$$\{t_{n-1}^{n-2} = 0, t_{n-2}^{n-3} = 0\} \quad \text{or} \quad \{t_{n-1} = t_{n-2} = 0\},$$

and so on.

The tangent cone at 0 corresponding to v_i will be given by

$$\{t_{n-1} = \dots = t_{n-i+2} = 0\}$$

in general, and then the tangent cone to v_i will be the linear space generated by

$$\{\varphi_n, \varphi_2, \dots, \varphi_{n-i+1}\}.$$

A quadratic differential will be orthogonal to this subspace if and only if its components

$$\{q_n, q_2, \dots, q_{n-i+1}\}$$

vanish at branch points to orders $2n-2$, n , $n+1, \dots, 2n-i-1$ as required.

We observe also that in the generic case, v_2 , we can obtain for these cyclic covers the result proved in general by Donagi-Green concerning the orthogonal space to deformations (see [1]).

Finally, it is interesting to consider the cases $n = 2, 3$ since it is possible to write global variations for the original equation.

For $n = 2$, the cyclic cover is given as

$$C: y^2 = (x - a_1) \dots (x - a_{2m})$$

and a basis for $H^{1,0}(C)$ is

$$e_i = x^i/y dx \quad 0 \leq i \leq m-2.$$

The only possible deformation is given by the families of curves

$$C_t: y^2 = (x - a_1) \dots (x - a_k + t) \dots (x - a_{2m})$$

with a basis for $H^{1,0}(C_t)$

$$e_i(t) = x^i dx / ((x - a_1) \dots (x - a_k + t) \dots (x - a_{2m}))^{1/2}.$$

The derivative at $t = 0$ gives

$$(-1/2)x^i(x - a_1) \dots \hat{k} \dots (x - a_{2m}) dx / y^3$$

and in view of the proposition, its cup product with e_j is

$$\lambda_k^{i,j} = (2\pi\sqrt{-1}) a_k^{i+j} / (a_k - a_1) \dots (a_k - a_{2m}).$$

The corresponding tangent vector is then

$$\sum_{i,j} \lambda_k^{i,j} e^i \otimes e^j$$

and a quadratic differential is orthogonal to all deformations if it vanishes at the branch points a_1, \dots, a_{2m} .

For $n = 3$, the equation of C is

$$y^3 = (x - a_1) \dots (x - a_{3m})$$

with a basis of holomorphic differentials

$$e_i = x^i dx / y, \quad f_j = x^j dx / y^2.$$

A cyclic variation is similar to the preceding case the tangent vector being

$$\sum \lambda_k^{i,j} e^i \otimes f^j.$$

However, to write a global equation in the case of splitting of a branch point the following considerations are necessary:

We have to find an equation

$$f(x, y) = y^3 + p(x)y + q(x) = 0$$

where p, q are polynomials in x whose coefficients depend on a parameter t . The polynomials should be chosen so that the equation represents a

Riemann surface ramified of order 3 over the points $a_2 \dots a_{3m}$ and branched of order 2 over $a_1, a_1 - t$ and such that for $t = 0$ reduces itself to C .

The branch points are the common solutions of $f = 0$ and $\partial f / \partial y = 0$, and they are found among the solutions of the equation obtained by setting the discriminant equal to zero:

$$\Delta = -(4p^3 + 27q^2) = 0.$$

The branch points of order three have to satisfy also

$$\partial f / \partial y, \quad \partial^2 f / \partial y^2 = 0$$

and this implies that they are among the roots of

$$36p = 0.$$

We choose then

$$p(x) = t^{2/3}(x - a_2) \dots (x - a_{3m})$$

where the particular power of t not only appears there for convenience but also reflects the fact that the parameter t is not natural. A natural parameter will be taken to be $s = t^{2/3}$. With this value for p we may compute again the discriminant

$$-\Delta = (x - a_2)^2 \dots (x - a_{3m})^2 \Delta_2$$

for q has to vanish also at these $3m - 1$ points. Here we have written

$$\Delta_2 = 4t^2(x - a_2) \dots (x - a_{3m}) + 27h^2$$

where $q(x) = (x - a_2) \dots (x - a_{3m})h(x)$. The polynomial h must be chosen so as to satisfy the last condition concerning the branch points at $a_1, a_1 - t$. This forces the vanishing of Δ_2 there exactly to the first order, and we are led to an equation of the form

$$4t^2(x - a_2) \dots (x - a_{3m}) + 27h^2(x) = g^2(x)(x - a_1)(x - a_1 + t).$$

This is a sort of Pell's equation for the polynomials $h(x), g(x)$ and can be solved explicitly; to make a long story short, the solution appears as power series in t with polynomial coefficients, that is

$$h(x) = h_0(x - a_1) + h_1(x)t + \dots$$

$$g(x) = g_0 + g_1(x)t + \dots$$

for some constants h_0 and g_0 that can be computed directly from the equation.

Given now the formula for C_t , we have to write a basis for the analytic differentials:

$$\left(x^i y - l(x) dx/f_y, \quad x^i g(x) dx/f_y \right)$$

where l is some polynomial (see [3] for the details). We compute now the derivative with respect to $s = t^{2/3}$ and the answer does not depend on $l(x)$ or $g_1(x)$ but is of the form

$$x^i (x - a_2) \dots (x - a_{3m}) / y^4 dx,$$

in the second case say, and the tangent vector then is obtained as

$$\sum \lambda_i^{i,j} f^i \otimes f^j$$

as in the proof of the theorem.

References

- [1] E. ARBARELLO, M. CORNALBA, P. GRIFFITHS and J. HARRIS: *Special Divisors on Algebraic Curves*. Regional Algebraic Geometry Conference Athens, Georgia, May 1979.
- [2] E. ARBARELLO and M. CORNALBA: *Su Una Congettura di Petri* Preprint Gruppi per la Matematica del C.N.R., June 1980.
- [3] G.A. BLISS: *Algebraic Functions*. Dover Publications, Inc. New York, 1966.
- [4] H. CLEMENS: *Double Solids*. Preprint, University of Utah, May 1979.
- [5] P. GRIFFITHS and J. HARRIS: *Principles of Algebraic Geometry*. John Wiley & Sons Interscience Publication, 1978.
- [6] P. GRIFFITHS: On the periods of certain rational integrals. Part I. *Annals Math.* 90 (1969) 460–541.

(Oblatum 27-IV-1981 & 21-VII-1982)

Universidad de Santiago de Chile
 Casilla S659 Correo 2
 Santiago
 Chile