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## THE EXCEPTIONAL REPRESENTATIONS OF $GL_2$

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The purpose of this paper is to provide a characterization of the set of exceptional supercuspidal representations of  $GL_2(F)$  where  $F$  is a local field of residual characteristic  $p$  and, in particular, to provide a proof for Lemma 4.2.2 of [5].

In §1, we describe the construction of a set of supercuspidal representations of  $GL_2(F)$  by the method of Weil; supercuspidal representations which cannot be constructed in this way are said to be exceptional. In §2, we show that a “Weil representation” which belongs to a ramified quadratic extension of  $F$  may be constructed by induction from a one-dimensional representation of an open subgroup of  $GL_2(F)$  and we show that the inducing representation must satisfy a certain condition ((3.01)). In §3, we show that, conversely, any supercuspidal representation which is induced from a representation satisfying (3.01) is a Weil representation. In §4, we show that condition (3.01) is equivalent to that given in Lemma 4.2.2 of [5]. In what follows we denote the ring of integers in  $F$  by  $\mathfrak{o}_F$ , the maximal ideal of  $\mathfrak{o}_F$  by  $\mathfrak{p}_F$  and we set  $\mathfrak{q} = [\mathfrak{o}_F : \mathfrak{p}_F]$ . Other notation used here is explained in [5].

### Section 1

Let  $E/F$  be quadratic and separable, let  $\tau$  be the nontrivial  $F$ -automorphism of  $E$ , denote by  $N_{E/F}$  and  $\text{Tr}_{E/F}$  the norm and trace maps of  $E/F$  and let  $\omega_{E/F}$  be the nontrivial character of the multiplicative group,  $F^\times$ , of  $F$  which is trivial on  $N_{E/F}E^\times$ .

Let  $C_c^\infty(E)$  be the space of compactly supported, locally constant, complex-valued functions on  $E$ , let  $\psi$  be a nontrivial character of the additive group,  $F^+$ , of  $F$  and set  $\psi_{E/F} = \psi \circ \text{Tr}_{E/F}$ . Then there is a unique choice of Haar measure,  $\mu_\psi$ , on  $E^+$  for which *Fourier inversion* holds with respect to  $\psi_{E/F}$ ; that is, if we define the map  $f \mapsto \hat{f}$  on  $C_c^\infty(E)$  by  $\hat{f}(\beta) = \int_E f(\alpha) \psi_{E/F}(\alpha\beta) d\mu_\psi(\alpha)$  then we have  $\hat{\hat{f}}(x) = f(-x)$ .

Now it is a consequence of the work of Weil [7] on symplectic groups

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(see [2], p. 7) that there is a representation  $r$  of  $\mathrm{Sl}_2(F)$  on  $C_c^\infty(E)$  such that

$$r\left(\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}\right)f(\beta) = \omega_{E/F}(x)|x|_E^{1/2}f(x\beta) \quad (1.01)$$

$$r\left(\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}\right)f(\beta) = \psi(yN_{E/F}\beta)f(\beta) \quad (1.02)$$

$$r\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)f(\beta) = \gamma_{E/F}\hat{f}(\beta^\tau) \quad (1.03)$$

where  $\gamma_{E/F}$  is a complex number whose value may be found in Lemma 1.2 of [2].

In [2] it is shown that this representation commutes with left translations by elements  $\alpha$  of  $E$  for which  $N_{E/F}\alpha = 1$  so that  $C_c^\infty(E)$  may be decomposed into a sum of  $\mathrm{Sl}_2(F)$  invariant subspaces which are parametrized by characters of the subgroup  $\ker N_{E/F}$  of  $E^\times$ . It is then shown that the representations of  $\mathrm{Sl}_2(F)$  thus obtained are irreducible and that those representations which are parametrized by nontrivial characters of  $\ker N_{E/F}$  induce to supercuspidal representations of  $\mathrm{Gl}_2(F)$  whose irreducible constituents will be referred to here as *Weil representations* of  $\mathrm{Gl}_2(F)$  belonging to  $E/F$ .

Cartier has observed that the Weil representations belonging to  $E/F$  may also be obtained by first inducing the representation  $r$  to  $\mathrm{Gl}_2(F)$  and then decomposing the resulting representation under a certain natural action of  $E^\times$  and it is this approach, summarized in the following two lemmas, which we will use. Since this approach has been described in detail elsewhere [N] we will omit proofs.

**LEMMA 1.1:** *There is a unique representation  $\tilde{r}$  on the space  $C_c^\infty(F^\times \times E)$  for which*

$$\tilde{r}\left(\begin{bmatrix} x & 0 \\ 0 & x^{-1} \end{bmatrix}\right)f(z, \beta) = \omega_{E/F}(x)|x|_E^{1/2}f(z, x\beta) \quad (1.04)$$

$$\tilde{r}\left(\begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix}\right)f(z, \beta) = \psi(yzN_{E/F}\beta)f(z, \beta) \quad (1.05)$$

$$\tilde{r}\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)f(z, \beta) = \gamma_{E/F}\omega_{E/F}(z)|z|_E^{1/2}\hat{f}(z, z\beta^\tau) \quad (1.06)$$

$$\tilde{r}\left(\begin{bmatrix} w & 0 \\ 0 & 1 \end{bmatrix}\right)f(z, \beta) = f(zw, \beta) \quad (1.07)$$

where  $f \mapsto \hat{f}$  is the Fourier transform in the second variable.

LEMMA 1.2: Let  $\theta$  be a character of  $E^\times$  and let  $C_\theta$  be the subspace of functions  $f$  in  $C_c^\infty(F^\times \times E)$  for which  $f(xN_{E/F}\alpha, \beta\alpha^{-1}) = \theta(\alpha)|\alpha|_E^{1/2}f(x, \beta)$ ,  $\alpha$  in  $E^\times$ . Then  $C_\theta$  is stable under  $\bar{r}$  and if  $\theta$  is not of the form  $\chi \circ N_{E/F}$  then  $C_\theta$  is an irreducible supercuspidal  $\mathrm{Gl}_2(F)$  subspace of  $C_c^\infty(F^\times \times E)$ .

LEMMA 1.3: Denote by  $W_\psi(\theta)$  the representation of  $\mathrm{Gl}_2(F)$  on  $C_\theta$  obtained as above. Then  $W_\psi(\theta)$  is equivalent to the representation  $\pi(\theta)$  defined on page 144 of [2]. In particular,  $W_\psi(\theta) = \pi(\mathrm{Ind}_{W_E \uparrow W_F} \theta)$ ; that is,  $W_\psi(\theta)$  corresponds in the sense of Langlands to the representation  $\mathrm{Ind}_{W_E \uparrow W_F} \theta$  of the Weil group,  $W_F$ , of  $F$ .

PROOF: We recall that the representation  $\pi(\theta)$  is induced from a representation  $\pi(\theta, \psi)$  of the subgroup  $G_{E/F}$  of  $\mathrm{Gl}_2(F)$  consisting of elements  $g$  in  $\mathrm{Gl}_2(F)$  for which  $\det g$  lies in  $N_{E/F}F^\times$ .  $\pi(\theta, \psi)$  acts on the subspace  $\bar{C}_\theta$  of functions  $f$  in  $C_c^\infty(E)$  which satisfy  $f(\alpha\beta) = \theta^{-1}(\alpha)f(\beta)$  for  $\alpha$  in  $N_{E/F}$  and may be characterized by the following formulae ([2], p. 11):

$$\pi(\theta, \psi) \left( \begin{bmatrix} N_{E/F}\alpha & 0 \\ 0 & 1 \end{bmatrix} \right) f(\beta) = |\alpha|_E^{1/2} \theta(\alpha) f(\alpha\beta) \quad (1.08)$$

$$\pi(\theta, \psi)(g) = r(g) \quad \text{for } g \text{ in } \mathrm{Sl}_2(F). \quad (1.09)$$

(One should note that  $\bar{C}_\theta$  is invariant under  $r$ .)

By Frobenius reciprocity, it will be enough to show that  $\bar{C}_\theta$  is  $G_{E/F}$ -isomorphic to a subspace of  $C_\theta$ . In fact, one checks easily that if  $C_\theta^+$  is the subspace of  $C_\theta$  consisting of functions  $f(x, \beta)$  for which  $f(x, \beta) = 0$  when  $x$  is not a norm from  $E$  then  $C_\theta^+$  is the required subspace and that  $f \mapsto \bar{f}$  where  $\bar{f}(\beta) = f(1, \beta)$  is the required  $G_{E/F}$ -isomorphism from  $C_\theta^+$  to  $\bar{C}_\theta$ .

COROLLARY 1.4: The equivalence class of  $W_\psi(\theta)$  is independent of  $\psi$ . If  $\theta_1, \theta_2$  are characters of  $E^\times$  then  $W_\psi(\theta_1)$  is equivalent to  $W_\psi(\theta_2)$  if and only if either  $\theta_2 = \theta_1$  or  $\theta_2 = \theta_1^\tau$ .

We note that a Weil representation  $W$  may belong to more than one quadratic extension of  $F$ . If  $W$  belongs to the unramified quadratic extension of  $F$ , we say that  $W$  is an *unramified* Weil representation; otherwise we call  $W$  ramified. An irreducible supercuspidal representation of  $\mathrm{Gl}_2(F)$  which is not a Weil representation will be called *exceptional*.

## Section 2

The goal of this section is to describe a given Weil representation as an induced representation. To this end we need some preliminaries concern-

ing the construction of supercuspidal representations by induction from open subgroups. Further details and proofs are given in [5]. Let  $V$  be the standard plane over  $F$ ; i.e.,  $V = F \oplus F$ . Then by a *lattice flag* in  $V$  we mean a sequence  $L = \dots L_{-1}, L_0, L_1, \dots$  of free, rank two  $\mathcal{O}_F$ -sub-modules of  $V$  such that  $L_k \supset L_{k+1}$ ,  $P_F L_k = L_{k+2}$  and  $\dim_{\mathcal{O}_F/P} L_k/L_{k+1} = 1$ . There is a natural action of the ring,  $M_2(F)$ , of  $2 \times 2$  matrices over  $F$  on the set of lattice flags which is, in fact, transitive; if we call two lattice flags  $L^1$  and  $L^2$  equivalent when there exists an integer  $m$  such that  $L_k^2 = L_{k+m}^1$  for all  $k$  then  $M_2(F)$  acts transitively on the set of classes of flags as well.

Given a lattice flag  $L$ , we denote by  $\mathfrak{b}_m(L)$  the subset of elements  $g$  in  $M_2(F)$  for which  $gL_k \subset L_{k+m}$  for all  $k$ ; we set  $\mathfrak{b}(L) = \mathfrak{b}_0(L)$  and note that for  $k \geq 0$ ,  $\mathfrak{b}_k(L)$  is a principal two-sided ideal in  $\mathfrak{b}(L)$ .

We set  $B(L) = \mathfrak{b}^\times(L)$  and for  $k \geq 1$  set  $B_k(L) = 1 + \mathfrak{b}_k(L)$ . We note that for  $k \geq m/2 \geq 1$ , the map  $x \mapsto x - 1$  induces an isomorphism of abelian groups of  $B_k(L)/B_m(L)$  and  $\mathfrak{b}_k(L)/\mathfrak{b}_m(L)$ . We note also that the pairing of  $\mathfrak{b}_k(L)/\mathfrak{b}_m(L) \times \mathfrak{b}_{1-m}(L)/\mathfrak{b}_{1-k}(L)$  into  $F^+/P_F$  given by  $(x, y) \mapsto \text{tr } xy$  is nondegenerate. It follows that if  $\psi$  is a character of  $F^+$  of conductor  $P_F$  and if for  $b$  in  $\mathfrak{b}_{1-m}(L)$  we define the character  $\psi_b$  on  $B_k(L)$  by  $\psi_b(x) = \psi(\text{tr } b(x - 1))$  then  $b \mapsto \psi_b$  induces an isomorphism of  $\mathfrak{b}_{1-m}/\mathfrak{b}_{1-k}$  with the complex dual,  $\overline{B_k/B_m}$ , of  $B_k/B_m$  whenever  $k \geq m/2$ .

Let, now,  $\pi$  be an irreducible supercuspidal representation of  $\text{Gl}_2(F)$ . Call  $\pi$  *unramified* if it may be  $c$ -induced (see [3] for the precise definition) from the subgroup  $F^\times \cdot \text{Gl}_2(\mathcal{O}_F)$  and call  $\pi$  *ramified* otherwise. Then it is well known (see, e.g., [1]) that a Weil representation is unramified as a Weil representation if and only if it is unramified in the above sense.

On the other hand, [3], ramified supercuspidal representations may be characterized as representations which may be induced from the normalizer,  $K(L)$ , of some subgroup  $B(L)$  of  $\text{Gl}_2(F)$  (all such subgroups are, of course, conjugate).

To be precise, call an element  $b$  in  $M_2(F) \setminus \mathfrak{b}(L)$ -*generic* of level  $2k + 1$  if

1.  $F[x]/F$  is quadratic ramified;
2.  $F[x] \cap \mathfrak{b}(L) = \mathcal{O}_{F[x]}$ ;
3.  $\nu_{F[x]}(x) = 2k + 1$ .

It is easy to see that  $x$  lies in  $\mathfrak{b}_{2k+1}(L)$  and that, in fact, the set of  $\mathfrak{b}(L)$ -generic elements of level  $2k + 1$  is precisely  $\Pi_L^{2k+1} B(L)$  where  $\Pi_L$  is any generator of the ideal  $\mathfrak{b}_1(L)$  of  $\mathfrak{b}(L)$ .

**PROPOSITION 2.1:** 1. *With notation as above, let  $n$  be a positive integer and let  $b$  be a  $\mathfrak{b}(L)$ -generic element of level  $1 - 2n$ . Let  $\theta$  be a character of the subgroup  $T_b = (F[b])^\times$  of  $\text{Gl}_2(F)$  such that  $\theta(\beta) = \psi(\text{Tr}_{F[b]/F} b(\beta - 1))$  for  $\beta$  in  $U_{F[b]}^n$ . Then the complex-valued function  $\theta\psi_b$  on  $T_b B_n(L)$  defined*

by  $\theta\psi_b(\beta k) = \theta(\beta)\psi_b(k)$ ,  $\beta$  in  $T_b$ ,  $k$  in  $B_n(L)$  is in fact a well-defined character of  $T_b B_n(L)$  which induces an irreducible supercuspidal representation  $\pi(L; \psi_b, \theta)$  of  $\mathrm{Gl}_2(F)$ . We have  $\pi(L; \psi_b, \theta_1) \cong \pi(L; \psi_b, \theta_2)$  if and only if  $\theta_1 = \theta_2$ .

2. Given an irreducible ramified supercuspidal representation  $\pi$  of  $\mathrm{Gl}_2(F)$  and a lattice flag  $L$  there exist  $n, b, \theta$  as above and a character  $\chi$  of  $F^\times$  so that  $\pi \cong \pi(L; \psi_b, \theta) \otimes \chi \circ \det$ . If  $f(\chi) \leq n$  then  $\chi$  may be taken to be trivial.

PROOF: This is Proposition 3.1.1 of [5].

In order to describe a given Weil representation  $W$  as an induced representation it will be helpful to write  $W$  as  $W(\theta)$  where  $\theta$  enjoys certain properties. Specifically, if we denote by  $f(\theta)$  the exponent of the conductor of  $\theta$  and by  $d(E/F)$  the exponent of the different of the extension  $E/F$  then the existence of an appropriate character  $\theta$  is given by the following lemma.

LEMMA 2.2: Let  $W$  be a ramified Weil representation of  $\mathrm{Gl}_2(F)$ . Then there exists an extension  $E/F$ , a character  $\theta$  of  $E^\times$  such that  $f(\theta) \geq 2d(E/F) - 1$  and  $f(\theta) - d(E/F)$  is odd, and a character  $\chi$  of  $F^\times$  so that  $W$  is equivalent to the representation  $W(\theta) \otimes \chi \circ \det$ . If there exist  $E', \theta', \chi'$  with the above properties and if  $E' \neq E$  then  $p = 2$ ,  $f(\theta) = 2d(E/F) - 1 = 2d(E'/F) - 1 = f(\theta')$  and  $f(\omega_{E'/F} \cdot \omega_{E/F}^{-1}) = d(E/F)$ .

PROOF: This follows from Corollary 1.18 of [4] and the fact that  $W(\theta) = \pi(\mathrm{Ind}_{W_E \uparrow W_F} \theta)$ .

In what follows, we fix a ramified quadratic extension  $E/F$  and a character  $\theta$  of  $E^\times$  for which  $f(\theta) - d(E/F)$  is odd and  $f(\theta) \geq 2d(E/F) - 1$ ; we set  $n(\theta) = 1/2(f(\theta) + d(E/F) - 1)$ . In addition we fix a character  $\psi$  of  $F^+$  of conductor  $P_F$  which if  $p = 2$  has the additional property that  $\psi(x^2 + x) = 1$  for  $x$  in  $\mathcal{O}_F$ . We denote by  $b = b_\psi(\theta)$  an element of  $E$  for which  $\theta(\beta) = \psi(\mathrm{Tr}_{E/F} b(\beta - 1))$  for  $\beta$  in  $U_E^{[(f(\theta)+1)/2]}$  and by  $c_\psi = c_\psi(E/F)$  an element of  $F$  for which  $\omega_{E/F}(x) = \psi(c_\psi(x - 1))$  for  $x$  in  $U_F^{[k(E/F)+1/2]}$ .

Finally, we fix a lattice flag  $L^n$ ,  $n = n(\theta)$ , by setting  $L_0^n = P_F^{1-n} \oplus \mathcal{O}_F$ ;  $L_1^n = P_F^{1-n} \oplus P_F$ . We note that then

$$\mathfrak{b}_{2k}(L^n) = P_F^k \begin{bmatrix} \mathcal{O}_F & P_F^{1-n} \\ P_F^n & \mathcal{O}_F \end{bmatrix}; \quad \mathfrak{b}_{2k+1}(L^n) = P_F^k \begin{bmatrix} P_F & P_F^{1-n} \\ P_F^n & P_F \end{bmatrix}.$$

PROPOSITION 2.3: With notation as above, define the function  $f_0$  in the space  $C_\theta$  by  $f_0(x, \beta) = \theta^{-1}(\beta)|\beta|_E^{-1/2}$  if  $xN_{E/F}\beta$  lies in  $U_F^{[(n+1)/2]}$ ,  $f_0(x,$

$\beta) = 0$  otherwise. Then for  $k$  in  $B_n(L^n)$  we have that

$$W(\theta)(k)f_0 = \psi_{\bar{b}}(k)f_0$$

where

$$\bar{b} = \begin{bmatrix} 0 & -N_{E/F}b \\ 1 & \text{Tr}_{E/F}b + c_\psi \end{bmatrix}.$$

PROOF: It is a straightforward computation, using formulae (1.04), (1.05) and (1.07), that

$$W(\theta)(k)f_0 = \psi_{\bar{b}}(k)f_0$$

when  $k$  lies in  $B_n(L^n)$  and is upper triangular. Our result will thus follow if we show that

$$W(\theta) \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} f_0 = \psi(-yN_{E/F}b)f_0$$

when  $b$  lies in  $P_F^{n+[n/2]}$ . Since

$$\begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -y \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

it will suffice, by (1.05), (1.06), to show that if  $\hat{f}_0(z, z\beta^\tau) \neq 0$  then  $\psi(-yzN_{E/F}\beta) = \psi(-yN_{E/F}b)$ ; that is, to show that the support of the function  $\hat{f}_0(z, z\beta^\tau)$  is contained in the set of  $(z, \beta)$  for which  $zN_{E/F}(\beta b^{-1})$  lies in  $U_F^{[(n+1)/2]}$ .

Now we have that

$$\hat{f}_0(z, z\beta^\tau) = \int_Y \theta^{-1}(\alpha) |\alpha|_E^{-1/2} \psi_{E/F}(\alpha z\beta^\tau) d\mu_\psi(\alpha)$$

where  $Y$  is the set of  $\alpha$  for which  $N_{E/F}\alpha$  lies in  $z^{-1}U^{[(n+1)/2]}$ . Since  $f(\theta) = 2n - d(E/F) + 1 \geq d(E/F)$  we have that  $N_{E/F}(U_E^{[(f(\theta)+1)/2]}) \subset U_F^{[(n+1)/2]}$  and thus that  $\hat{f}_0(z, z\beta^\tau)$  is a nonzero multiple of

$$\begin{aligned} & \int_{P_E^{[(f(\theta)+1)/2]}} \int_Y \theta^{-1}(\alpha(1+\gamma)) |\alpha|^{-1/2} \\ & \quad \times \psi_{E/F}(\alpha(1+\gamma)z\beta^\tau) d\mu_\psi(\alpha) d\mu_\psi(\gamma) \\ & = \int_Y \theta^{-1}(\alpha) |\alpha|_E^{-1/2} \psi_{E/F}(\alpha z\beta^\tau) \\ & \quad \times \int_{P_E^{[(f(\theta)+1)/2]}} \psi_{E/F}((\alpha z\beta^\tau - b)\gamma) d\mu_\psi(\gamma) d\mu_\psi(\alpha) \\ & = 0 \end{aligned}$$

unless  $\alpha z \beta^\tau - b$  lies in  $P_E^{2-d(E/F)-[(f(\theta)+1)/2]}$ , that is, unless  $\alpha z \beta^\tau b^{-1}$  lies in  $U_E^{[(f(\theta)/2)]}$ . (Here, one uses the fact that  $\nu_E(b) = 1 - 2n$  so that  $2 - d(E/F) - [(f(\theta) + 1)/2] - \nu_E(b) = f(\theta) - [(f(\theta) + 1)/2] = [f(\theta)/2]$ .) Finally, since  $zN_{E/F}\alpha$  lies in  $U_F^{[(n+1)/2]}$  and since, in general,  $N_{E/F}U_E^r \subset U_F^s$  where  $s = \min([(r + d(E/F))/2], r)$  one checks that  $\hat{f}_0(z, z\beta^\tau) = 0$  unless  $zN_{E/F}(\beta b^{-1})$  lies in  $U_F^{[(n+1)/2]}$ .

**COROLLARY 2.4:** *With notation as above, there exists a character  $\bar{\theta}$  of  $T_{\bar{b}}$  such that  $W(\theta)$  is equivalent with  $\pi(L^n; \psi_{\bar{b}}, \bar{\theta})$ .*

**PROOF:** We note first that  $\bar{b}$  is  $\mathfrak{b}(L^n)$ -generic of level  $1 - 2n$  since  $\nu_F(\mathrm{Tr}_{E/F}b + c_\psi) \geq \min(1 - n, 1 - d(E/F)) = 1 - n$ . Next, since  $\psi_{\bar{b}}$  is stable under  $T_{\bar{b}}B_n(L^n)$ , the span under  $T_{\bar{b}}B_n(L^n)$  of  $f_0$  decomposes into a sum of the form  $\oplus \langle f_{\bar{\theta}_j} \rangle$  where  $\bar{\theta}_j$  is a character of  $T_{\bar{b}}$  of the form described in Proposition 2.1 and where  $W(\theta)(h)f_{\bar{\theta}_j} = \theta_j \psi_{\bar{b}}(h)f_{\bar{\theta}_j}$  for  $h$  in  $T_{\bar{b}}B_n(L^n)$ . Finally, since distinct characters  $\theta_j \psi_{\bar{b}}$  induce distinct irreducible supercuspidal representations of  $\mathrm{Gl}_2(F)$ , we see that the span under  $T_{\bar{b}}B_n(L^n)$  of  $f_0$  is one-dimensional, that we may set  $\bar{\theta} = \bar{\theta}_1$  whence  $f_{\bar{\theta}_1} = f_0$ , and  $W(\theta)$  is equivalent to  $\pi(L^n; \psi_{\bar{b}}, \bar{\theta})$ .

### Section 3

In this section we fix, once and for all, an integer  $n \geq 1$  and a  $\mathfrak{b}(L^n)$ -generic element,  $\bar{b}$ , of level  $1 - 2n$ . Our goal is to determine whether some or all of the representations  $\pi(L^n; \psi_{\bar{b}}, \theta)$  are Weil representations. From Proposition 2.3, it is clear that in order that some representation  $\pi(L^n; \psi_{\bar{b}}, \theta)$  be Weil it is necessary that there exist a ramified quadratic extension  $E/F$  with  $3d(E/F) \leq 2(n+1)$  and an element  $b$  in  $E$  with  $\nu_E(b) = 1 - 2n$  such that

$$\begin{aligned} i. \quad \mathrm{tr} \bar{b} &\equiv \mathrm{Tr}_{E/F}b + c_\psi(E/F) \pmod{P_F^{-(n-1)/2}} \\ ii. \quad (\det \bar{b})/N_{E/F} &\equiv 1 \pmod{P_F^{[(n+1)/2]}}. \end{aligned} \tag{3.01}$$

We will say that such an element  $\bar{b}$  is *Weil-generic*. Our main result in this section is

**PROPOSITION 3.1:** *The representation  $\pi(L^n; \psi_{\bar{b}}, \theta)$  is Weil if and only if  $\bar{b}$  is Weil-generic.*

We will need several lemmas.

**LEMMA 3.2:** *Suppose that the pair  $(E, b)$  satisfies condition (3.01). Let  $E_1/F$  be ramified quadratic and suppose for some  $b_1$  in  $E_1$  we have*



$\mathrm{Tr}_{E_1/F} b_1 \equiv \mathrm{Tr}_{E/F} b \pmod{P_F^{-(n-1)/2}}$ ,  $N_{E_1/F} b_1 / N_{E/F} b \equiv 1 \pmod{P_F^{[(n+1)/2]}}$ . Then the pair  $(E_1, b_1)$  satisfies condition (3.01).

PROOF: We must show that  $c_\psi(E/F) \equiv c_\psi(E_1/F) \pmod{P_F^{-(n-1)/2}}$ . To begin with, we note that since  $2(n+1) \geq 3d(E/F)$  it follows that  $-[(n-1)/2] > \frac{1}{2}d(E/F) - n$ . In addition, we have that

$$d(E/F) = \min\left(2\left(\nu_F(\mathrm{Tr}_{E/F} b) + n\right), 2\nu_F(2) + 1\right),$$

$$d(E_1/F) = \min\left(2\left(\nu_F(\mathrm{Tr}_{E_1/F} b_1) + n\right), 2\nu_F(2) + 1\right).$$

One may then deduce from the congruence  $\mathrm{Tr}_{E_1/F} b_1 \equiv \mathrm{Tr}_{E/F} b \pmod{P_F^{-(n-1)/2}}$  that  $d(E_1/F) = d(E/F)$ .

Now since  $-[(n-1)/2] \leq 1 - [(d(E/F) + 1)/2]$ , the congruence  $c_\psi(E/F) \equiv c_\psi(E_1/F) \pmod{P_F^{-(n-1)/2}}$  is equivalent to the statement that the restrictions of  $\omega_{E/F}$  and  $\omega_{E_1/F}$  to  $U_F^{[(n+1)/2]}$  coincide. However  $\omega_{E/F}|_{U_F^{[(n+1)/2]}}$  is determined by the data  $f(\omega_{E/F}) = d(E/F)$ ,  $\omega_{E/F}^2 = 1$ ,  $\omega_{E/F}(1 + x \mathrm{Tr}_{E/F} b + x^2 N_{E/F} b) = 1$  for  $x$  with  $2\nu_F(x) \geq 2n - 1 + [(n+1)/2]$ . Since

$$\begin{aligned} & \frac{1}{2}(2n-1) + [(n+1)/2] + \nu_F\left(\mathrm{Tr}_{E_1/F} b_1 - \nu_F(\mathrm{Tr}_{E/F} b)\right) \\ & \geq d(E/F); \end{aligned}$$

$$2n - 1 + [(n+1)/2] + \nu_F\left(N_{E_1/F} b_1 - N_{E/F} b\right) \geq d(E/F),$$

we see that  $\omega_{E_1/F}$  satisfies the above data, whence our result.

Let  $E/F$  be quadratic ramified with  $3d(E/F) \leq 2(n+1)$  and let  $b$  be an element of  $E$  with  $\nu_E(b) = 1 - 2n$ . Denote by  $W(E; b)$  the set of representations  $W(\theta)$  where  $\theta$  is a character of  $E^\times$  such that  $\theta(\beta) = \psi(\mathrm{Tr}_{E/F} b(\beta - 1))$  for  $\beta$  in  $U_F^{[(2n-d(E/F)+2)/2]}$  and  $\theta(\tilde{\omega}_F)\omega_{E/F}(\pi_F) = 1$  for some fixed prime element  $\tilde{\omega}_F$  of  $F$ .

LEMMA 3.3: Let  $m = \lfloor \frac{1}{2}(2n - d(E/F) + 2) \rfloor$ . Then  $W(E; b)$  consists of  $(q-1)q^{m-1}$  distinct representations if  $3d(E/F) < 2(n+1)$  and  $\frac{1}{2}(q-1)q^{m-1}$  distinct representations if  $3d(E/F) = 2(n+1)$ .

PROOF: This follows from the fact that  $[U_E : U_E^m] = (q-1)q^{m-1}$  together with Corollary 1.4 and the fact that  $b \equiv b^\tau \pmod{P_E^{2-d(E/F)-m}}$  if and only if  $-2n + d(E/F) \geq 2 - d(E/F) - m$ , that is, if and only if  $3d(E/F) \geq 2(n+1)$ .

LEMMA 3.4: Let  $S$  be the subgroup of  $F \times F^\times$  consisting of pairs  $(x, y)$  with  $x$  in  $\mathrm{Tr}_{E/F} P_E^{1-n-[(d(E/F)/2]}$  and  $y$  in  $N_{E/F} U_E^{n-[(d(E/F)/2]}$ . Suppose  $E_1, E_2$  are

ramified quadratic extensions of  $F$ ,  $b_i$  lies in  $E_i$  and  $\text{Tr}_{E_i/F} b_i = \text{Tr}_{E/F} b$  (mod  $P_F^{-(n-1)/2}$ );  $N_{E_i/F} b_i / N_{E/F} b \equiv 1 \pmod{P_F^{(n+1)/2}}$ . Suppose further that  $(\text{Tr}_{E_1/F} b_1, N_{E_1/F} b_1) \not\equiv (\text{Tr}_{E_2/F} b_2, N_{E_2/F} b_2) \pmod{S}$ . Then  $W(E_1, b_1)$  and  $W(E_2, b_2)$  are disjoint sets.

PROOF: It was shown in Lemma 3.2 that  $d(E_1/F) = d(E_2/F) \geq \frac{2}{3}(n+1)$  and that if  $d(E_i/F) = \frac{2}{3}(n+1)$  then  $f(\omega_{E_i/F} \omega_{E_2/F}^{-1}) < d(E_1/F)$ . It follows by Lemma 2.2 that  $W(E_1, b_1)$  and  $W(E_2, b_2)$  are disjoint unless  $E_1 = E_2$ .

Suppose now that  $E_1 = E_2$ , that  $W(\theta_1)$  lies in  $W(E_i, b_i)$  and that  $W(\theta_1)$  is equivalent with  $W(\theta_2)$ . By Corollary 1.4, there exists an element  $\nu$  in the galois group of  $E_1/F$  such that

$$b_1 \equiv b_2^\nu \pmod{P_E^{1-n-d(E/F)/2}}$$

which contradicts our hypothesis.

LEMMA 3.5:  $[P_E^{-(n-1)/2} \times U_F^{[(n-1)/2]}; S] = q^{\lfloor 1/2(d(E/F)-1) \rfloor}$  if  $2(n+1) > 3d(E/F)$ ;  $[P_E^{-(n-1)/2} \times U_F^{[(n+1)/2]}; S] = 2q^{\lfloor 1/2(d(E/F)-1) \rfloor}$  if  $2(n+1) = 3d(E/F)$ .

PROOF: Straightforward.

PROOF OF PROPOSITION 3.1: Suppose that  $\bar{b}$  is Weil-generic. Then  $\bar{b}$  is  $K(L^n)$  conjugate to

$$\begin{bmatrix} 0 & -\det \bar{b} \\ 1 & \text{tr } \bar{b} \end{bmatrix}$$

and we have thus produced, by Lemmas 3.3, 3.4, 3.5,  $(q-1)q^{n-1}$  distinct irreducible Weil summands of  $\text{Ind}_{B_n(L^n) \uparrow_{G_{12}(F)}} \psi_{\bar{b}}$  each having central character which is trivial at  $\tilde{\omega}_F$ . On the other hand, the total number of such summands is

$$[U_{F[\bar{b}]} B_n(L^n) : B_n(L^n)] = [U_{f[\bar{b}]} : U_{f[\bar{b}]}] = (q-1)q^{n-1}.$$

Since given any representation  $\pi(L^n; \psi_{\bar{b}}, \theta)$  we may find a character  $\chi$  of  $F^\times$  such that  $f(\chi) = 0$  and  $\pi(L^n; \psi_{\bar{b}}, \theta) \otimes \chi \circ \det$  has a central character trivial on  $\tilde{\omega}_F$  we have shown that all representations  $\pi(L^n; \psi_{\bar{b}}, \theta)$  are Weil representations.

## Section 4

The purpose of this section is to prove the following proposition which gives a simple characterization of the property of being Weil-generic.

PROPOSITION 4.1: Fix  $n \geq 1$  and let  $L^n$  be the lattice flag described in §3. Let  $\bar{b}$  be  $\mathfrak{b}(L^n)$ -generic of level  $1 - 2n$  and set  $\bar{E} = F(\bar{b})$ . Then the following are equivalent.

1.  $\bar{b}$  is Weil-generic.
2. Either  $2(n+1) > 3d(\bar{E}/F)$  or the polynomial  $X^3 - (\text{tr } \bar{b})X^2 + \det \bar{b}$  has a root in  $F$ .
3. There exists a ramified quadratic extension  $E/F$  with  $3d(E/F) \leq 2(n+1)$  and an element  $b$  in  $E$  with  $N_{E/F}b = \det \bar{b}$  and  $\text{Tr}_{E/F}b + c_\psi(E/F) \equiv \text{tr } \bar{b} \pmod{P_F^{[d(E/F)/2]+1-n}}$ .

PROOF:  $1 \Rightarrow 2$ . Suppose that  $\bar{b}$  is Weil-generic and that  $2(n+1) \leq 3d(\bar{E}/F)$ . Pick  $b$  in  $E$  satisfying (3.01). We show first that  $3d(E/F) = 2(n+1)$ . Suppose that  $d(E/F)$  is odd. Then since, by assumption,  $3d(E/F) \leq 2(n+1)$  we must have  $3d(E/F) < 2(n+1)$ . Now (see Lemma 3.2),  $d(E/F) = \min(2(\nu_F(\text{Tr}_{E/F}b) + n), 2\nu_F(2) + 1)$  so that  $2\nu_F(2) + 1 = d(E/F) < 2(n+1)/3$  and also  $\nu_F(\text{Tr}_{E/F}b) \geq \nu_F(2) + 1 - n$ . By (3.01) -  $i$ ,

$$\nu_F(\text{tr } \bar{b}) \geq \min(\nu_F(2) + 1 - n, -2\nu_F(2), -[(n-1)/2]).$$

However, from  $2\nu_F(2) + 1 < 2(n+1)/3$  we obtain that  $\nu_F(2) + 1 - n \leq -2\nu_F(2)$  while  $\nu_F(2) + 1 - n \leq -[(n-1)/2]$  since  $n \geq d(E/F) = 2\nu_F(2) + 1$ . Thus  $\nu_F(\text{tr } \bar{b}) \geq \nu_F(2) + 1 - n$  whence  $d(\bar{E}/F) = 2\nu_F(2) + 1 = d(E/F)$ . Therefore  $3d(\bar{E}/F) < 2(n+1)$  which is false.

Now suppose that  $d(E/F)$  is even so that  $d(E/F) = 2(\nu_F(\text{Tr}_{E/F}b) + n) \leq 2\nu_F(2)$ . Then if  $3d(E/F) < 2(n+1)$ , we would have  $\nu_F(\text{Tr}_{E/F}b) = \frac{1}{2}d(E/F) - n < 1 - d(E/F) = \nu_F(c_\psi(E/F))$ . Since  $\frac{1}{2}d(E/F) - n \leq -[(n-1)/2]$  it would follow that  $\nu_F(\text{Tr}_{\bar{E}/F}\bar{b}) = \frac{1}{2}d(E/F) - n$  whence  $d(\bar{E}/F) = d(E/F) < \frac{2}{3}(n+1)$ . Thus we have shown that  $3d(E/F) = 2(n+1)$  and we note that  $d(E/F)$  is even.

Now by definition,

$$\begin{aligned} 1 &= \psi\left(c_\psi(E/F)(N_{E/F}(1+xb) - 1)\right) \\ &= \psi\left(c_\psi(E/F)(x \text{Tr}_{E/F}b + x^2 N_{E/F}b)\right) \end{aligned}$$

for  $x$  in  $P_F^{d/2+n-1}$ . Setting  $x = y \text{Tr}_{E/F}b / N_{E/F}b$  and noting that  $\nu_F(\text{Tr}_{E/F}b) = \frac{1}{2}d(E/F) - n$  while  $\nu_F(N_{E/F}b) = 1 - 2n$  we see that

$$\psi\left(\left(c_\psi(E/F)(\text{Tr}_{E/F}b)^2 / N_{E/F}b\right)(y + y^2)\right) = 1$$

for  $y$  in  $\mathcal{O}_F$ . Since  $\psi$  has been picked so that  $\psi(y + y^2) = 1$  for  $y$  in  $\mathcal{O}_F$  ( $p=2$  here since  $d(E/F)$  is even) we see that  $c_\psi(E/F)(\text{Tr}_{E/F}b)^2 /$

$N_{E/F}b \equiv 1 \pmod{P_F}$ . By (3.01) it follows that  $X = \mathrm{Tr}_{E/F}b$  satisfies the congruence  $X^3 - \mathrm{tr} \bar{b} X^2 + \det \bar{b} \equiv 0 \pmod{P_F^{1-2n}}$ . A Hensel's lemma argument now shows that the polynomial  $X^3 - \mathrm{tr} \bar{b} X^2 + \det \bar{b}$  has a root in  $F$ .

$2 \Rightarrow 3$ . If  $2(n+1) > 3d(\bar{E}/F)$  then  $v_F(c_\psi(\bar{E}/F)) = 1 - d(\bar{E}/F) \geq [d(\bar{E}/F)/2] + 1 - n$  and so we may take  $E = \bar{E}$ ,  $b = \bar{b}$ .

Now suppose that  $2(n+1) \leq 3d(\bar{E}/F)$  and let  $s$  be a root in  $F$  of the polynomial  $X^3 - (\mathrm{tr} \bar{b})X^2 + \det \bar{b}$ . Then since  $v_F(\mathrm{tr} \bar{b}) \geq [(d(\bar{E}/F) + 1)/2] - n$  while  $v_F(\det \bar{b}) = 1 - 2n$ , a standard argument shows that  $v_F(s) = \frac{1}{3}(1 - 2n) \leq v_F(2) - n$ . It follows that the polynomial  $X^2 - sX + \det \bar{b}$  is irreducible over  $F$  and that if  $E/F$  is a splitting field then  $3d(E/F) = 2(n+1)$ . Let  $b$  be a root in  $E$  of the polynomial  $X^2 - sX + \det \bar{b}$ . Then since  $d(E/F)$  is even we obtain, as above, that  $c_\psi(E/F)(\mathrm{Tr}_{E/F}b)^2/N_{E/F}b \equiv 1 \pmod{P_F}$  whence  $c_\psi(E/F) \equiv N_{E/F}b/(\mathrm{Tr}_{E/F}b)^2 \pmod{P_F^{1+(1/3)(1-2n)}}$ . Finally,  $N_{E/F}b = \det \bar{b}$  while  $\mathrm{Tr}_{E/F}b$  satisfies  $X^3 - \mathrm{Tr} \bar{b} X^2 + \det \bar{b} = 0$  so that  $N_{E/F}b/(\mathrm{Tr}_{E/F}b)^2 = \mathrm{tr} \bar{b} - \mathrm{Tr}_{E/F}b$ . Combining this last equation with the congruence preceding it and noting that  $1 + \frac{1}{3}(1 - 2n) = [d(E/F)/2] + 1 - n$  yields our result.

$3 \Rightarrow 1$ . Set  $d = d(E/F)$  and suppose by induction that for  $1 \leq j \leq k$  we have picked quadratic extensions  $E_j/F$  and elements  $b_j$  in  $E_j$  such that  $d(E_j/F) = d$ ,  $N_{E_j/F}b_j = \det b$  and  $\mathrm{Tr}_{E_j/F}b_j + c_\psi(E_j/F) \equiv \mathrm{tr} \bar{b} \pmod{P_F^{[d/2]-n+j}}$ . Set  $\bar{s} = \mathrm{tr} \bar{b}$ ,  $s_k = \mathrm{Tr}_{E_k/F}b_k$ ,  $\Delta = \det b$ , let  $a$  be an element of  $P_F^{[d/2]-n+k}$  and set  $s_a = s_k + a$ . Let  $E_a$  be a splitting field of  $X^2 - s_a X + \Delta$  over  $F$  and pick a root,  $b_a$ , of this polynomial in  $E_a$ .

Now since  $v_F(s_a - s_k) \geq [d/2] - n + k$  and since  $d(E_k/F) = \min(2(v_F(s_k) + n), 2v_F(2) + 1)$  while  $d(E_a/F) = \min(2(v_F(s_a) + n), 2v_F(2) + 1)$  it follows that  $d(E_a/F) = d(E_k/F) = d$ . Since  $v_F(s_a - s_k) \geq [d/2] - n + k$  while  $N_{E_k/F}b_k = N_{E_a/F}b_a$  it follows that  $U_F^l \cap N_{E_k/F}E_k^x = U_F^l \cap N_{E_a/F}E_a^x$  where  $l = \max 2[(d+1)/2] - 2k, [(d+1)/2]$  and thus that  $c_\psi(E_a/F) \equiv c_\psi(E_k/F) \pmod{P_F^{1-l}}$ .

Since  $1 + 2k - 2[(d+1)/2] \geq [d/2] - n + k + 1$  when  $k \geq 1$  while  $1 - [(d+1)/2] \geq [d/2] - n + k + 1$  when  $k \leq n - d$  we see that if we set  $a = \bar{s} - c_\psi(E_k/F) - s_k$ ,  $b_{k+1} = b_a$ ,  $E_{k+1} = E_a$  then the pair  $(b_{k+1}, E_{k+1})$  satisfies our inductive hypothesis whenever  $k \leq n - d$ . Finally since  $-[(n-1)/2] + n - [(d)/2] \leq n - d + 1$  we see that we may find  $(b_k, E_k)$  as above for  $k = [(n-1)/2] + n - [(d)/2]$ . The pair  $(b_k, E_k)$  then satisfies (3.01) whence  $\bar{b}$  is Weil-generic.

We may now state our main result.

**THEOREM 4.2:** *Let  $\pi$  be an irreducible ramified supercuspidal representation of  $\mathrm{Gl}_2(F)$  and let  $L$  be a lattice flag. Pick  $n, b, \theta$  as in Proposition 2.1 so that  $\pi \cong \pi(L; \psi_b, \theta) \otimes \chi \circ \det$  and set  $E = F(b)$ . Then  $\pi$  is an exceptional representation of  $\mathrm{Gl}_2(F)$  if and only if  $2(n+1) \leq 3d(E/F)$  and the polynomial  $X^3 - (\mathrm{tr} b)X^2 + \det b$  is irreducible over  $F$ .*

PROOF: Propositions 3.1, 4.1.

We note, in conclusion, that we obtain as a consequence

COROLLARY 4.3:  $GL_2(F)$  has no exceptional representations unless  $p = 2$ .

PROOF: If  $p \neq 2$ , then we have  $d(E/F) = 1$  for all quadratic ramified extensions  $E/F$ . Since  $n \geq 1$  in all cases we have that  $2(n + 1) > 3d(E/F)$ .

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