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0. Introduction

Factorizable groups of homeomorphisms first arose when Hirsch considered $\mathrm{Diff}_0^r(X)$, the group of C^r diffeomorphisms of a closed differentiable manifold X, C^r isotopic to the identity [14]. It turns out that the identity components in the groups of automorphisms of most geometric structures on closed manifolds are factorizable (cf. [3,7,16,1,2,5]). Epstein [4] showed generally that if G is a factorizable group of homeomorphisms of a paracompact space X which acts transitively on a basis of open neighborhoods of X, then $[G, G]$, the commutator subgroup of G, is simple. This has proved to be a most fruitful beginning in the understanding of various groups of diffeomorphisms, leading to the deep results of Thurston [15] and Mather [11,12], namely $\mathrm{Diff}_0^r(X)$ is simple for X a connected closed manifold and $1 \leqslant r \leqslant +\infty$, $r \neq \dim X + 1$. In this paper, we give a more complete characterization of the factorizable groups. Our results assert that if G is a factorizable group of homeomorphisms of X without any fixed point, then $[G, G]$, the commutator subgroup of G, is perfect, i.e., is its own commutator subgroup. In fact, $[G, G]$ is the least subgroup in some sense even if it is not simple (cf. Theorem 1.8). If G has a fixed point, both G and $[G, G]$ can be not perfect. As a corollary of this, we obtain a somewhat improved form of Epstein's simplicity theorem [4].

There are many groups of homeomorphisms that satisfy our assumptions without satisfying Epstein's assumptions. Some examples are the group of compactly supported diffeomorphisms preserving the leaves of a foliation and isotopic to the identity through such diffeomorphisms; the group of compactly supported diffeomorphisms preserving the fibers of a fiber bundle isotopic to the identity through such diffeomorphisms; and similarly for diffeomorphisms preserving a submanifold of dimension greater than 0, or strata preserving diffeomorphisms of a stratafication without any 0-dimensional strata. Therefore, these groups have perfect commutator subgroups. However, we will not consider in this paper the interesting problem: are these groups perfect themselves? My original motivation in examining the commutator subgroups came from the following theorem [8]: the commutator subgroups classify the manifolds up to diffeomorphisms preserving the appropriate structures, e.g., each isomorphism $\tau: \mathrm{Diff}^r_{c0}(X) \approx \mathrm{Diff}^r_{c0}(Y)$ determines a C^r diffeomorphism $h: X \to Y$, and $\mathrm{Diff}^r_{c0}(X)$ is its own commutator subgroup for $r \neq \dim Y + 1$. In fact, $\tau(f) = hfh^{-1}$ for each $f \in \mathrm{Diff}^r_{c0}(X)$. Lastly, the results of this paper have generalizations to groups of noncompactly supported homeomorphisms [9].

1. The theorem

DEFINITION: The *support* of a homeomorphism h of X, denoted by *supp* h, is the closure of $\{x \in X \mid x \neq hx\}$.

DEFINITION: A group of homeomorphisms G of a space X is *factorizable* if given $g \in G$, then for each covering $\mathfrak{U}$ of X by open sets, there exists elements $g_1, \ldots, g_n \in G$ and $U_1, \ldots, U_n \in \mathfrak{U}$ such that $g = g_n \cdots g_1$ and $\operatorname{supp} g_i \subset U_i$ for each $i = 1, \ldots, n$.

THEOREM 1.1: *Suppose X is a paracompact space and G is a factorizable group of homeomorphisms of X. If G has no fixed point then $[G, G]$, the commutator subgroup of G, is perfect.*

The proof consists of several propositions and lemmas. Suppose N is a subgroup of G, we write $N \lhd G$ if N is a normal subgroup of G. Let $\mathfrak{B}_N = \{gU \mid nU \cap U = \emptyset$ for some $n \in N$, $U \subset X$ open, $g \in G\}$, and $\overline{\mathfrak{B}}_N = \{gU \mid nU \cap U = \emptyset$ for some $n \in N$, $U \subset X$ open, $g \in [G, G]\}$. Write $G(U)$ for $\{g \in G \mid \operatorname{supp} g \subset U\}$. Let $\Pi_N = \prod_{V \in \mathfrak{B}_N} [G(V), G(V)]$, the subgroup of G generated by $[G(V), G(V)]$, and similarly

$$\overline{\Pi}_N = \prod_{V \in \overline{\mathfrak{B}}_N} [[G(V), G(V)], [G(V), G(V)]].$$

The following proposition shows what happens if we assume no extra property on G. Write $[a, b]$ for $a^{-1}b^{-1}ab$.

PROPOSITION 1.2: *If $N \lhd G$, then $\Pi_N \lhd N$. If N is normalized by $[G, G]$, i.e., $gNg^{-1} \subset N$ for each $g \in [G, G]$, then $\overline{\Pi}_N \subset N$.*

PROOF: Since both statements are alike, we proved only the first. It suffices to show that $[G(U), G(U)] \subset N$ for any U satisfying $U \cap nU = \emptyset$ for some $n \in N$. In fact, if $g, h \in G(U)$, then $\operatorname{supp} n^{-1}g^{-1}n \cap \operatorname{supp} h \subset n^{-1}U \cap U = \emptyset$, i.e., $[n^{-1}gn, h] = 1$. Therefore, using the formula $[ab, c] = b^{-1}[a, c]b[b, c]$, the equality $[n^{-1}gn, h] = 1$, and $[N, G] \subset N$, we have

$$[g, h] = [n^{-1}g^{-1}ng, h] = [[n, g], h] \in N.$$

If G is factorizable, then

LEMMA 1.3: *If G is a factorizable group of homeomorphisms of the space X, and $\mathcal{U}$ is a G-invariant covering of X by open sets, (i.e., $G\mathcal{U} \subset \mathcal{U}$,) then $[G, G] = \prod_{V \in \mathcal{U}} [G(V), G(V)]$.*

PROOF: By definition, if G is factorizable, then $G = \prod_{V \in \mathcal{U}} G(V)$ for any covering $\mathcal{U}$ of X. The lemma now follows from an induction using the formula $[ab, c] = b^{-1}[a, c]b[b, c]$.

The following Corollary will lead to the proof of Theorem 1.1.

COROLLARY 1.4: *Suppose G is a factorizable group of homeomorphisms. If $N \subset G$ has no fixed point, then $\Pi_N = [G, G]$. If in addition $N \lhd G$, then $[G, G] \lhd N$.*

PROOF: This is because $\mathcal{B}_N$ is a G-invariant covering of X, since N moves each point of X. The second statement now follows from 1.2 and 1.3.

By taking $N = [[G, G], [G, G]]$ in Corollary 1.4, and since $[[G, G], [G, G]]$ has no fixed point as we shall prove later in 1.7, one sees that $[G, G] \subset [[G, G], [G, G]]$, i.e., $[G, G]$ is perfect. This shows that Theorem 1.1 is true. But we will prove that $[G, G]$ is the "least" such group in 1.7 and 1.8, a harder fact that yields the Epstein's simplicity theorem.

The next lemma will eliminate some technicalities for commutators of homeomorphisms on paracompact spaces.

LEMMA 1.5: *Suppose X is a paracompact space and $\mathcal{V}$ is a cover of X. Then we can find a refinement $\mathcal{U}$ of $\mathcal{V}$ such that any two intersecting members of $\mathcal{U}$ lie in a single member of $\mathcal{V}$.*

PROOF: Since X is paracompact, we may assume that $\mathcal{V}$ is locally finite. Thus, a point $x \in X$ has an open neighborhood W_x which intersects only finitely many members of $\mathcal{V}$, namely, $V_x^1, \ldots,$ and V_x^n. The refinement $\mathcal{U}$ asked for in the lemma will come from shrinking the neighborhoods W_x, and this in turn comes from shrinking the neighborhoods $V_x^1, \ldots, V_x^n$.

Since a paracompact space is normal, there is a refinement $\mathcal{Q} = \{Q^i \mid i \in I\}$ of $\mathcal{V} = \{V^i \mid i \in I\}$ such that $\overline{Q}^i \subset V^i$ for all $i \in I$. Moreover, normality furnishes a function f_i: $X \to [0, 1]$ for each $i \in I$ such that $f_i|_{\overline{Q}^i} = 1$ and $f_i|_{X-V^i} = 0$. Let $f_1, \ldots, f_n$ be the functions associated with the neighborhoods $V_x^1, \ldots, V_x^n$. Let $V_x^j(r) = f_j^{-1}(r, 1]$ for $j = 1, \ldots, n$ and $r \in (0, 1)$. The neighborhoods $V_x^1(r), \ldots, V_x^n(r)$ will yield the appropriate shrinking of W_x.

Suppose $r \neq f_j(x)$ for each $j = 1, \ldots, n$. Then x lies either in the interior of $V_x^i(r)$ or is in the complement of $\overline{V}_x^i(r)$. Arrange the indices so that $V_x^1(r), \dots, V_x^m(r)$ contain x, while $V_x^{m+1}(r), \dots, V_x^n(r)$ do not. Thus, $V_x = \cap_{i=1}^m V_x^i(r) - \cup_{i=m+1}^n \bar{V}_x^i(r)$ is an open set containing x. Consider $\mathcal{U} = \{U_x \mid U_x = W_x \cap V_x, x \in X\}$, we claim this has the property described by the lemma.

In fact, let U_x and U_y be two intersecting members of U. By the foregoing construction, we have $U_x \subset V_x = \cap_{i=1}^m V_x^i(r) - \cup_{i=m+1}^n \bar{V}_x^i(r)$ and $U_y \subset V_y \subset \cap_{i=1}^p V_y^i(t)$ for some p and t. It suffices to prove the claim for $t \geqslant r$. Since $U_x \cap U_y \neq \emptyset$, so $U_x \cap V_y^i(t) \neq \emptyset$, hence $U_x \cap V_y^i \neq \emptyset$. Now, $U_x = W_x \cap V_x$ implies $W_x \cap V_y^i \neq \emptyset$. But W_x intersects exactly n members in $\mathcal{V}$, namely, $V_x^1, \dots, V_x^n$, which shows that $V_y^i = V_x^j$ for some $j \in \{1, \dots, n\}$. Therefore, $U_x \cap V_x^j(t) = U_x \cap V_y^i(t) \neq \emptyset$. This together with the inclusions $V_x^j(t) \subset V_x^j(r)$ and $U_x \subset \cap_{i=1}^m V_x^i(r) - \cup_{i=m+1}^n \bar{V}_x^i(r)$ leads to the conclusion $U_x \subset V_x^j(r) = V_y^i(r) \subset V_y^i$. Consequently, both U_x and U_y lie in V_y^i, a member of $\mathcal{V}$. This completes the proof.

We will alter $\mathcal{B}_N$ and $\overline{\mathcal{B}}_N$ slightly, the altered sets will be useful as G-invariant coverings of X. Let $\mathcal{C}_N$ be $\{U \mid nU \cap U = \emptyset$, some $n \in N$, U open$\}$, so that $G\mathcal{C}_N = \mathcal{B}_N$ and $[G, G]\mathcal{C}_N = \overline{\mathcal{B}}_N$. If X is paracompact, we can find a refinement $\mathcal{D}$ of $\mathcal{C}_N$ by 1.5 so that the union of any two intersecting members of $\mathcal{D}$ lies in a member of $\mathcal{C}_N$. Alter $\mathcal{B}_N$ and $\overline{\mathcal{B}}_N$ by letting $\mathcal{B}_N = G\mathcal{D}$ and $\overline{\mathcal{B}}_N = [G, G]\mathcal{D}$. This does not change 1.2 through 1.4.

Lemma 1.6: *Suppose X is a paracompact space and $N \lhd G$ is a group of homeomorphisms of X without fixed point. If G is factorizable, then $\mathcal{B}_N = \overline{\mathcal{B}}_N$. In particular, $\overline{\Pi}_N \lhd G$.*

Proof: Suppose $V \in \mathcal{B}_N$, by the factorizability of G, there are elements $g_1, \dots, g_n \in G$ and $D_1, \dots, D_n \in \mathcal{D}$ (a cover since N has no fixed point) such that $V = g_n \dots g_1 U$ for some $U \in \mathcal{D}$, and $\operatorname{supp} g_i \subset D_i$ for each $i = 1, \dots, n$. By the definition of $\mathcal{D}$ either $\operatorname{supp} g_i \cup U \subset W_i \in \mathcal{C}_N$ or else $U \subset X - \operatorname{supp} g_i$. In the first case $n_i W_i \cap W_i = \emptyset$ for some $n_i \in N$; this means $n_i U \subset X - \operatorname{supp} g_i$.

Let $U_i = g_i \dots g_1 U$, we claim there is some $h_i \in \mathcal{G}$ such that $h_i U_{i-1} \subset X - \operatorname{supp} g_i$. In fact, simply let $h_i = n_i(g_{i-1} \cdots g_1)^{-1}$ or $(g_{i-1} \cdots g_1)^{-1}$. Consequently, $[g_1^{-1}, h_1]|_U = g_1 h_1^{-1} g_1^{-1} h_1|_U = g_1 h_1^{-1} h_1|_U = g_1|_U$, so that $[g_n^{-1}, h_n] \dots [g_i^{-1}, h_i] \dots [g_1^{-1}, h_1]U = g_n \dots g_1 U = V$. Hence, $\mathcal{B}_N = \overline{\mathcal{B}}_N$. Finally, since $\overline{\mathcal{B}}_N = \mathcal{B}_N$ is G-invariant and $gG(V)g^{-1} = G(gV)$, $\overline{\Pi}_N$ is a normal subgroup of G.

Since $\overline{\Pi}_N$ is a normal subgroup, we can apply 1.2 to this subgroup. Theorem 1.1 now follows from the following.

Theorem 1.7: *Suppose X is a paracompact space and G is a factorizable group of homeomorphisms of X. If N is a subgroup of G normalized by $[G, G]$ and without fixed point, then $[G, G] \subset N$.*

Proof: We may define $\mathcal{B}_{(\overline{\Pi}_N)}$ and $\Pi_{(\overline{\Pi}_N)}$. If N moves each point of X, so does $\overline{\Pi}_N$. Assuming this, we see $\mathfrak{B}_{(\overline{\Pi}_N)}$ is a G-invariant covering for X. Thus, $[G, G] = \Pi_{(\overline{\Pi}_N)}$ by 1.4. Moreover, since by 1.6, $\overline{\Pi}_N \lhd G$, and N is normalized by $[G, G]$, 1.2 yields

$$[G, G] = \Pi_{(\overline{\Pi}_N)} \subset \overline{\Pi}_N \subset N.$$

It remains to show that $\overline{\Pi}_N$ moves each point of X. The following will suffice: at each point x of X, there is a neighborhood $V \in \overline{\mathfrak{B}}_N$ such that $[[G(V), G(V)], [G(V), G(V)]]x \neq x$. Let us define V. Since N moves each point, there is a neighborhood V of x satisfying $nV \cap V = \emptyset$ for some $n \in N$. As an application of the factorizations in G, observe that by letting V be the only open neighborhood containing x in a covering of X, we see that there exists some $h_V \in G(V)$ such that $h_V x \neq x$. Thus, for some neighborhood W of x in V, $h_V W \cap W = \emptyset$. Again, there is some homeomorphism $h_W \in G(W)$ and an open neighborhood U of x in W such that $h_W U \cap U = \emptyset$, yielding some $h_U \in G(U)$ satisfying $h_U x \neq x$. Clearly,

$$\begin{aligned}[][[h_V, h_W], [h_V, h_U]]x &= [h_V^{-1} h_W^{-1} h_V h_W, h_V^{-1} h_U^{-1} h_V h_U]x \\ &= [h_W, h_U]x = h_U x \neq x.\end{aligned}$$

This completes the proof.

Proof of 1.1: Clearly $[[G, G], [G, G]]$ is a normal subgroup of G. Moreover, $[[G, G], [G, G]]$ moves each point of X as we saw in the last part of the proof of 1.7. Thus, 1.7 completes the proof.

We actually proved a stronger theorem in 1.7. This is similar to Epstein's theorem that under additional hypotheses, $[G, G]$ is the least subgroup of G normalized by $[G, G]$.

Theorem 1.8: *Suppose X is a paracompact space and G is a factorizable group of homeomorphisms of X. If G has no fixed point, then $[G, G]$ is the least subgroup of G, normalized by $[G, G]$, which acts without any fixed point.*

Theorem 1.8 is the main result of this paper and represents much improvement from Theorem 1.1. In fact, we saw in the remarks after Corollary 1.4 how 1.1 easily follows from the very simple 1.2 and 1.3. As a corollary of 1.8, we obtain a slightly improved form of Epstein's theorem [4].

Theorem 1.9: *Suppose X is a paracompact space with a basis of open neighborhoods $\mathfrak{B}$. If G is a group of homeomorphisms of X that satisfies the following two axioms*:

Axiom 1: *G acts transitively inclusively on $\mathfrak{B}$, i.e., given U, $V \in \mathfrak{B}$, there is some $g \in G$ such that $gU \subset V$;*
Axiom 2: *G is factorizable for each covering $\mathfrak{U}$ of X, where $\mathfrak{U} \subset \mathfrak{B}$. Then $[G, G]$, the commutator subgroup of G, is simple.*

Proof: It suffices to show that each subgroup N of G normalized by $[G, G]$ has no fixed point provided that $N \neq 1$. Theorem 1.8 then completes the proof. To prove the above, we claim that for each $U \in \mathfrak{B}$, $GU = [G, G]U$. This follows from the proof of 1.6 because Axiom 1 implies the following statement in 1.6 is valid: there is some $h_i \in G$ such that $h_i U_{i-1} \subset X - \text{supp } g_i$, in the notations of the proof of 1.6, so the claim is true. Since $N \neq 1$, there is a neighborhood $V \in \mathfrak{B}$ such that $nV \cap V = \emptyset$ for some $n \in N$. By Axiom 1 and the equality $GU = [G, G]U$, each point $x \in X$ has a basis neighborhood $W \in \mathfrak{B}$ such that $W \subset gV$ for some $g \in [G, G]$. Therefore, $gng^{-1}(W) \cap W \subset gng^{-1}(gV) \cap (gV) = g(nV \cap V) = \emptyset$, or $Nx \neq x$ since $gng^{-1} \in N$.

Remark 1.10: The above theorem is better than Epstein's theorem in several ways. Firstly, we do not require as Epstein chose to: $G\mathfrak{B} \subset \mathfrak{B}$ (his Axiom 1). Secondly, we do not require $G\mathfrak{B} = \mathfrak{B}$ (his Axiom 2). In fact, one does not know if $G\mathfrak{B} = \mathfrak{B}$ for X a topological 4-manifold and $\mathfrak{B}$ the collection of 4-balls, depending on the status of the annulus conjecture. Lastly, our factorization axiom is simpler than Epstein's, which has an additional inequality attached, we quote [4]: "*Let $g \in G$, $U \in \mathfrak{B}$ and $\mathfrak{U} \subset \mathfrak{B}$ be a covering of X. Then there exists an integer n, elements $g_1, \ldots, g_n \in G$ and $V_1, \ldots, V_n \in \mathfrak{B}$ such that $g = g_n g_{n-1} \cdots g_1$, supp $g_i \subset V_i$ and supp $g_i \cup (g_{i-1} \cdots g_1 U) \neq X$ for $1 \leqslant i \leqslant n$.*"

2. The examples

We will first mention the known (and well-known) examples and then the new examples. The first seven are examples for Epstein's theorem [4] (cf. also Theorem 1.9). All the manifolds in this section will be either closed or open, and connected.

Example 2.1: Let X^n be a differentiable manifold, $\mathfrak{B}$ the collection of C^r embedded n-balls, and $G = \text{Diff}^r_{c0}(X)$, the group of C^r diffeomorphisms of X compactly C^r isotopic to the identity (i.e., the isotopy has compact support). The axioms of 1.8 are satisfied by the triple $(G, X, \mathfrak{B})$ [4]. Moreover, $\text{Diff}^r_{c0}(X)$ is perfect by the result of Thurston [15] for $r = \infty$ and Mather [11,12] for $r \neq n+1$. Therefore, $\text{Diff}^r_{c0}(X)$ is simple for $r \neq n+1$.

Example 2.2: Let X^n be a piecewise linear manifold, $\mathfrak{B}$ the collection of PL n-balls in X, and $G = PL_{c0}(X)$ defined just as in 2.1. Again, Theorem

1.9 (cf. [7] for factorizations in G) implies $PL_{c0}(X)$ has a simple commutator subgroup. However, it is still unknown whether $PL_{c0}(X)$ is perfect, except when $X = \mathbb{R}$ or S^1 [4].

EXAMPLE 2.3: Just as in 2.1 and 2.2, by 1.9 $\mathrm{Top}_{c0}(X)$, the group of homeomorphisms of a topological manifold X compactly isotopic to the identity, has a simple commutator subgroup. In fact, $\mathrm{Top}_{c0}(X)$ is factorizable via the isotopy extension theorem [3], and an old result of FIsher [6] asserts that $\mathrm{Top}_{c0}(X)$ is simple.

EXAMPLE 2.4: Let X^n be a smooth manifold of dimension greater then 2 equipped with a volume form ω, i.e., $\omega = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_n$ locally. Consider the group $\mathrm{Diff}_{c0}(X, \omega)$, of compactly supported volume preserving diffeomorphisms of X isotopic to the identity through such diffeomorphisms. One can define the Calabi-Thurston-Weinstein homomorphism V: $\mathrm{Diff}_{c0}(X, \omega) \to H_c^{n-1}(X, \mathbb{R})/\Gamma$ [16]. Then Thurston [16] showed that $\ker V = [\mathrm{Diff}_{c0}(X, \omega), \mathrm{Diff}_{c0}(X, \omega)]$ is a perfect factorizable group if X is closed. Therefore, by Theorem 1.8, the commutator subgroup of $\mathrm{Diff}_{c0}(X, \omega)$ is simple. $\mathrm{Diff}_{c0}(\mathbb{R}^n, \omega)$ is also known to be simple if $n > 2$, and not simple if $n = 2$.

EXAMPLE 2.5: Let X^{2n} be a smooth manifold of dimensional greater than 2 equipped with a symplectic form ω, i.e., $\omega = \mathrm{d}x_1 \wedge \mathrm{d}x_2 + \ldots + \mathrm{d}x_{2n-1} \wedge \mathrm{d}x_{2n}$ locally. Consider the group $\mathrm{Diff}_{c0}(X, \omega)$, of compactly supported diffeomorphisms preserving ω and isotopic to the identity through such diffeomorphisms. One may again define the Calabi-Thurston-Weinstein homomorphism S: $\mathrm{Diff}_{c0}(X, \omega) \to H_c^1(X, \mathbb{R})/\Gamma$ [1]. Then Banyaga [1] showed for X closed, $\mathrm{Ker}\, S = [\mathrm{Diff}_{c0}(X, \omega), \mathrm{Diff}_{c0}(X, \omega)]$ is a perfect factorizable group, so by 1.8, this commutator subgroup is a simple group. If X is open, then $\mathrm{Ker}\,(S)$ is not always simple. Instead, the kernel of the homomorphism R: $\mathrm{Ker}\,(S) \to \mathbb{R}/\Lambda$, defined in [1], is simple.

EXAMPLE 2.6: Let X^{2n+1} be a smooth manifold, $n \geqslant 1$, equipped with a contact form ω, i.e., $\omega = x_1\mathrm{d}x_2 + \ldots + x_{2n-1}\mathrm{d}x_{2n} + \mathrm{d}x_{2n+1}$ locally. The group $\mathrm{Diff}_{c0}(X, \omega)$, of compactly supported contact diffeomorphisms of X preserving ω up to a positive function and isotopic to the identity through such diffeomorphisms, is factorizable because of a contact isotopy extension theorem [10,2]. Since Axiom 1 of 1.9 is easily seen to be true, $\mathrm{Diff}_{c0}(X, \omega)$ has a simple commutator subgroup [2]. It is not known if $\mathrm{Diff}_{c0}(X, \omega)$ is simple.

EXAMPLE 2.7: Let X be a compact manifold, m is a probability measure without atom whose support is the whose manifold. The group of homeomorphisms of X preserving m is locally contractible [5]. Thus, its identity component is factorizable. Moreover, if $\dim X \geqslant 3$, the commutator subgroup of the identity component is simple (by 1.9), and its abelianization is isomorphic to a quotient of $H_1(X, \mathbb{R})$ by a discrete subgroup.

• The next six examples satisfy the hypotheses of Theorems 1.1 and 1.8, but not Epstein's axioms (as in 1.9). The conclusions of 1.8 are omitted here.

EXAMPLE 2.8: Let X be a differentiable manifold with a foliation F and $\mathrm{Diff}^r_{c0}(X, F)$, the group of leaf preserving C^r diffeomorphisms of X and compactly C^r isotopic to the identity through such diffeomorphisms. Then clearly $\mathrm{Diff}^r_{c0}(X, F)$ has no fixed point. It is also factorizable (Lemma 2.8), so the perfectness of $[\mathrm{Diff}^r_{c0}(X, F), \mathrm{Diff}^r_{c0}(X, F)]$ follows.

LEMMA 2.8: *Given a covering* $\mathcal{U}$ *of* X *by open sets and* $f \in \mathrm{Diff}^r_{c0}(X, F)$ *in a small neighborhood* N *of the identity, then for* N *sufficiently small, there are* $f_1, \dots, f_n \in \mathrm{Diff}^r_{c0}(X, F)$ *and* $U_1, \dots, U_n \in \mathcal{U}$ *such that* $f = f_n \cdots f_1$, *supp* $f_i \subset U_i$ *for each* $i = 1, \dots, n$, *and* $f_i \in N_1$ *for each* $i = 1, \dots, n$, *where* N_1 *is a preassigned neighborhood of the identity in* $\mathrm{Diff}^r_{c0}(X, F)$.

PROOF: The proof is entirely analogous to the smooth case without a foliation (cf. [14, 3.1]). For the sake of completeness, we include it here. Let $H: X \times I \to X$ be a leaf preserving C^r isotopy * such that $H_1 = f$ and $H_0 =$ identity. Let $U_1, \dots, U_n \in \mathcal{U}$ be a cover of the compact set $\cup_{t \in I}$ supp H_t, and $\{\phi_i \mid i = 1, \dots, n\}$ a partition of unity subordinate to the cover $\{U_i \mid i = 1, \dots, n\}$. Define h_i: $X \to X \times I$ by $h_i(x) = (x, \phi_i(x))$, let $g_i = H(h_i + \dots + h_1)$ for $i = 1, \dots, n$, and $g_0 =$ identity. Then for f sufficiently small, H can be very near the projection π_X: $X \times I \to X$, so that $g_i \in \mathrm{Diff}^r_{c0}(X, F)$. Let $f_i = g_i g_{i-1}^{-1}$ for each $i = 1, \dots, n$, then supp $f_i \subset U_i$. Therefore, $f_n \cdots f_1 = (g_n g_{n-1}^{-1}) \cdots (g_2 g_1^{-1})(g_1 g_0^{-1}) = g_n = H(h_n + \dots + h_1) = H_1 = f$. Moreover, by assuming f small enough, f_i can be made as small as possible.

EXAMPLE 2.9: Let $\pi: X \to B$ be a C^r fiber bundle and $\mathrm{Diff}^r_{c0}(X, \pi)$, the group of fiber preserving C^r diffeomorphisms compactly fiber preserving C^r isotopic to the identity. This group acts without a fixed point. Moreover, Lemma 2.8 adapts trivially to this case and shows $\mathrm{Diff}^r_{c0}(X, \pi)$ is factorizable. Therefore, Theorem 1.1 concludes $[\mathrm{Diff}^r_{c0}(X, \pi), \mathrm{Diff}^r_{c0}(X, \pi)]$ is perfect.

EXAMPLE 2.10: As a generalization of 2.9, we can consider a C^r map between differentiable manifolds f: $X \to Y$ and $\mathrm{Diff}^r_{c0}(X, f)$, the group of C^r diffeomorphisms that preserve inverses of points and compact C^r

* We assume H is actually C^r, which is equivalent to the non-C^r definition of an isotopy as a path in $\mathrm{Diff}^r_{c0}(X, F)$ with respect to the fine C^r topology.

isotopic to the identity through such diffeomorphisms. Lemma 2.8 again adapts trivially to this case and shows $\mathrm{Diff}^r_{c0}(X, f)$ is factorizable. If $G = \mathrm{Diff}^r_{c0}(X, f)$ acts without any fixed point, then $[G, G]$ is perfect by 1.1. We note that if $f^{-1}y$ is a single point for some $y \in y$, then clearly G has a fixed point $f^{-1}y$.

EXAMPLE 2.11: Let X be a C^r manifold and Y a closed submanifold, $\dim Y > 0$. Consider $\mathrm{Diff}^r_{c0}(X, Y)$ the group of compactly supported C^r diffeomorphisms preserving Y setwise and C^r isotopic to the identity through such diffeomorphisms. Again, as in 2.8 and 2.9, Lemma 2.8 adapts to show $\mathrm{Diff}^r_{c0}(X, Y)$ is factorizable, so 1.1 shows $\mathrm{Diff}^r_{c0}(X, Y)$ has a perfect commutator subgroup. Note that if Y is 0-dimensional or immersed, $\mathrm{Diff}^r_{c0}(X, Y)$ may fix a point.

EXAMPLE 2.12: Let X be a C^r manifold and $\mathbb{S}$ a C^r stratification with closed strata of dimension greater than zero for X. Consider $\mathrm{Diff}^r_{c0}(X, \mathbb{S})$ the group of compactly supported strata preserving C^r diffeomorphisms isotopic to the identity through such diffeomorphisms. Again, as in 2.11, Lemma 2.8 and Theorem 1.1 apply to yield the perfectness of $[\mathrm{Diff}^r_{c0}(X, \mathbb{S}), \mathrm{Diff}^r_{c0}(X, \mathbb{S})]$. Note that if S has a 0-dimensional strata, then $\mathrm{Diff}^r_{c0}(X, \mathbb{S})$ may fix a point.

EXAMPLE 2.13: The above examples may be combined. For example, consider the groups of diffeomorphisms preserving both a foliation and a differential form, or both a fiber bundle and a differential form, etc. Then an isotopy extension theorem, where the appropriate structures are preserved, will prove the identity components to be factorizable. The rest follows immediately.

EXAMPLE 2.14: There is a simple example where a group leaves one point fixed and thereby both the group and its commutator subgroup fail to be perfect. Consider $\mathrm{Diff}^r_{c0}(X^n, x) = \{h \in \mathrm{Diff}^r_{c0}(X^n) \mid x = hx\}$, where $\mathrm{Diff}^r_{c0}(X^n)$ was defined in 2.1 and $x \in X$. Taking the differential at x gives a homomorphism d: $\mathrm{Diff}^r_{c0}(X^n, x) \to GL_n(\mathbb{R})$. This is actually onto by a classical result of Palais [13]. Hence, the fact that $GL_n(\mathbb{R})$ and $[GL_n(\mathbb{R}), GL_n(\mathbb{R})]$ are nonperfect for n large implies the same for $\mathrm{Diff}^r_{c0}(X^n, x)$.

Lastly, we remark that for manifolds, a factorizable group is always compactly supported. Therefore, a group of noncompactly supported homeomorphisms calls for different techniques [9].

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