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WEAK NORMALITY AND LIPSCHITZ SATURATION FOR ORDINARY SINGULARITIES

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1. Introduction

If X is an algebraic variety defined over an algebraically closed field k, then associated to X are two homeomorphic varieties, the weak normalization X^* of X, defined by Andreotti and Bombieri [2], and the Lipschitz saturation $\tilde{X}$ of X, defined by Pham-Teissier [12], and Lipman [8] (see Section 2 for the definitions). Since X^* is maximal among varieties birationally homeomorphic to X it follows that if X is weakly normal, i.e. $X^* = X$, then also $X^* = \tilde{X} = X$. The main result of this paper shows that if X is obtained from a nonsingular projective variety by means of a linear projection from a center in general position, then $X^* = \tilde{X}$. An old conjecture of Andreotti-Bombieri states that in this situation X is in fact weakly normal, from which our result would immediately follow. We will briefly review the status of this conjecture and the relationship of the present paper to the Andreotti-Bombieri conjecture.

Assume that dim $X = n$ and that X is obtained by generic linear projection of a nonsingular Y into $\mathbb{P}^{n+r}(k)$. Then if $k = \mathbb{C}$ the conjecture is known to be true when $r = 1$ and when $r \geqslant (n-1)/2$ [1,3]. If char $k = 0$ then the conjecture is known to be true for $r = 1$ [4].

The technique used in proving the weak normality theorem for $r = 1$ depended heavily upon being able to compute the depth of the structure sheaf of X and this seems highly unlikely to be possible if the codimension of X is at least 2. The other case which has been proved, $r \geqslant (n-1)/2$, relies upon explicit canonical forms for the singularities which arise. Each such singularity is checked directly to be weakly normal. Again, this is not a technique which can be exploited to obtain the general conjecture.

The result to be proved in the present paper is valid for all r and for arbitrary characteristic. Of course, it already follows from the weak normality conjecture in the range of dimensions and codimensions where that conjecture has been proved, but outside that range, it is a new result. The motivation for studying the relationship between weak normality and Lipschitz saturation (at least when $k = \mathbb{C}$) is provided by a result of Spallek [14,15]. The result of Spallek states that if Y is a (reduced) complex space and if g is a weakly holomorphic function on Y which is m-times continuously differentiable across the singular set of Y, then if m is sufficiently large g is automatically holomorphic. For $k = \mathbb{C}$ our result states that any continuously holomorphic function g on a variety X with only ordinary singularities is automatically Lipschitz. But at present we are unable to prove any automatic differentiability of g. If this were possible then we could apply Spallek's theorem to conclude that X is weakly normal.

The way one compares the weak normalization and Lipschitz saturation is via the double point schemes studied in Kleiman [7] and Roberts [13]. The relevant definitions and results are recalled in Sections 2 and 3 and the proof of the main theorem occupies most of Sections 4 and 5. Theorem 5.1 is the main result of the paper.

2. Preliminaries

All rings are assumed to be commutative with identity; k will denote a fixed algebraically closed field. Suppose $A \subseteq B$ are two k-algebras such that B is integral over A. The inclusion of k in A induces a natural map $B \otimes_k B \to B \otimes_A B$. Let I be the kernel of this map and define two rings $A_B{}^*$ and $\tilde{A}_B$ as follows.

2.1. Definition:

$$\text{(a) } A_B{}^* = \{ b \in B \colon b \otimes_k 1 - 1 \otimes_k b \in \mathrm{Rad}(I) \}$$

$$\text{(b) } \tilde{A}_B = \{ b \in B \colon b \otimes_k 1 - 1 \otimes_k b \in \bar{I} \}.$$

In (a) Rad(I) denotes the nilradical of I and in (b) $\bar{I}$ refers to the integral closure of the ideal I in the algebra $B \otimes_k B$. This is defined by

$$\bar{I} = \{ d \in B \otimes_k B \colon d^n + a_1 d^{n-1} + \dots + a_n = 0 \text{ for some } n \text{ and } a_i \in I^i \}.$$

Since $\bar{I} \subseteq \mathrm{Rad}(I)$ there is an inclusion of k-algebras:

2.2. Lemma: $A \subseteq \tilde{A}_B \subseteq A_B{}^* \subseteq B$.

The ring $\tilde{A}_B$ is called the *Lipschitz saturation* of A in B while $A_B{}^*$ is called the *weak normalization* of A in B. If B is the normalization of A, then $\tilde{A}_B$ and $A_B{}^*$ are denoted by $\tilde{A}$ and A^* respectively and are called the Lipschitz saturation and weak normalization of A respectively. The concept of Lipschitz saturation was introduced by Pham-Teissier [12] (see also Lipman [8]), while the concept of weak normalization was introduced in a different form by Andreotti-Bombieri [2]. The paper of Manaresi [9] contains the verification that the definition of weak normality we have given above agrees with the original definition.

These concepts will now be applied to schemes over k. By algebraic variety over k is meant a reduced separated scheme of finite type over k. Let $(X, \mathcal{O}_X)$ be an algebraic variety over k, let $(X^\#, \mathcal{O}_{X^\#})$ be the normalization of X and let $\pi\colon X^\# \to X$ be the normalization map. There is a map of schemes $X^\# \times_X X^\# \to X^\# \times_k X^\#$ corresponding to the map of sheaves $\mathcal{O}_{X^\#} \otimes_k \mathcal{O}_{X^\#} \to \mathcal{O}_{X^\#} \otimes_{\mathcal{O}_X} \mathcal{O}_{X^\#}$. Let I be the kernel of this sheaf map, and define the sheaves $(\tilde{\mathcal{O}}_X)$ and $(\mathcal{O}_X)^*$ locally by means of Definition 2.1. Each of these sheaves is representable as the structure sheaf of a variety over k and from Lemma 2.2 one obtains the following sequence of maps of k-varieties.

2.3. LEMMA: $X^\# \to X^* \xrightarrow{g} \tilde{X} \xrightarrow{f} X.$

Furthermore, $\mathcal{O}_{X^*} = (\mathcal{O}_X)^*$ and $\mathcal{O}_{\tilde{X}} = (\tilde{\mathcal{O}}_X)$ and the morphisms f and g are homeomorphisms. The variety $\tilde{X}$ is the *Lipschitz saturation* of X and X^* is the *weak normalization* of X. X is said to be *Lipschitz saturated* if $\tilde{X} = X$ and X is said to be *weakly normal* if $X = X^*$. From lemma 2.3 one obtains

2.4. LEMMA: *If X is weakly normal, then X is Lipschitz saturated.*

The converse of lemma 2.4 is false. For example, if char $k \neq 2, 3$ then $X = \operatorname{Spec} k[t^2, t^3]$ is Lipschitz saturated, but $X^* = \operatorname{Spec} k[t]$.

The main result of this paper is that if X is obtained by generic projection of a nonsingular projective variety defined over k, then $X^* = \tilde{X}$. Of course, this would be automatic if X were weakly normal, but, as indicated in the introduction, this stronger result can be proved only in some special cases at present.

3. Subtransversality

Let $f\colon X \to Y$ be a morphism of nonsingular algebraic varieties defined over k. Let $\pi\colon (X \times X)' \to X \times X$ be the morphism obtained by blowing up the diagonal Δ_X in $X \times X$ and let $E = \pi^{-1}(\Delta_X)$ be the exceptional divisor. Let $J \subseteq \mathcal{O}_{(X \times X)'}$ be the sheaf of ideals defining the subscheme $((f \times f) \circ \pi)^{-1}(\Delta_Y) \subseteq (X \times X)'$ and let $I = \mathcal{O}_{(X \times X)'}[-E]$ be the invertible sheaf of ideals defining the exceptional locus E.

3.1. DEFINITION: (a) The *double point scheme* $Z(f)$ is the subscheme of $(X \times X)'$ defined by the sheaf of ideals $I^{-1}J$.

(b) The morphism f is said to be *weakly subtransversal* if $Z(f) = \emptyset$ or if

$Z(f)$ is reduced of dimension $2(\dim(X)) - \dim(Y)$ and if $\dim(Z(f) \cap E) = 2(\dim(X)) - \dim(Y) - 1$, (or $Z(f) \cap E = \emptyset$).

(c) The morphism f is said to be *strongly subtransversal* if $Z(f) = \emptyset$ or if $Z(f)$ is smooth of dimension $2(\dim(X)) - \dim(Y)$ and if $Z(f)$ is transversal to E.

3.2. Lemma: *If f is a weakly subtransversal morphism, then J is a reduced sheaf of ideals.*

Proof: Let $m = \dim(Y)$. Then J is defined locally by m functions $F_1, \dots, F_m$ and E is defined locally by a single function u. Let $g_j = F_j/u$. These are well defined functions since $J \subseteq I$. Let x be a closed point of $Z(f) \cap E$ and let A be the local ring of $(X \times X)'$ at x. The weak subtransversality assumption implies that the ideal $(g_1, \dots, g_m, u)$ is an unmixed ideal of A of height $m + 1$, and since A is Cohen-Macaulay it follows that $g_1, \dots, g_m$, u is an A-sequence. This implies that $(g_1, \dots, g_m) \cap (u) = (g_1 u, \dots, g_m u)$ and since both ideals on the left are radical ideals, one has that $J = (g_1 u, \dots, g_m u)$ is a radical ideal in A.

Consider an imbedding $X \subseteq \mathbb{P}^N$ and a morphism f_L: $X \to \mathbb{P}^m$ induced by projection from a linear subspace L of $\mathbb{P}^N$ with codim $(L) = m + 1$ and $X \cap L = \emptyset$.

3.3. Theorem: *Let $X \subseteq \mathbb{P}^N$ be nonsingular. Then, if the linear subspace L is chosen in general position, the associated projection f_L: $X \to \mathbb{P}^m$ is weakly subtransversal. (If char $k > 0$ it may be necessary to reimbed X in a larger projective space before applying the projection. We refer the reader to Robert's paper* [13] *for the precise conditions concerning this reimbedding.)*

Proof: Theorem (0.2) of Robert's paper [13] shows that $Z(f_L)$ is smooth of dimension $2(\dim(X)) - m$ if L is chosen in general position. It remains only to check that $\dim(Z(f_L) \cap E) = 2(\dim(X)) - m - 1$ (or $Z(f_L) \cap E = \emptyset$) for L chosen in general position. But this is a consequence of Kleiman's transversality theorem [6] as follows. If H is the Grassman variety that parametrizes lines in $\mathbb{P}^N$, the secant map $(X \times X) \backslash \Delta_X \to H$ extends to a morphism φ: $(X \times X)' \to H$. If L is a fixed linear subspace of $\mathbb{P}^N$ of codimension $m + 1$, then the schubert subvariety of H defined by $S = \{\lambda \in H: \lambda \cap L \neq \emptyset$ and $\lambda \not\subseteq L\}$ is smooth and $Z(f_L) = \varphi^{-1}(S)$ ([13] page 70). Now let $G = \mathrm{Aut}\,(\mathbb{P}^N)$ and apply Kleiman's theorem 2(i) [6] to the pair of maps $S \hookrightarrow H$ and $E \hookrightarrow (X \times X)' \xrightarrow{\varphi} H$ to conclude the proof.

3.4. Remarks: (a) Theorem 3.3 is true with the stronger conclusion that f_L is strongly subtransversal if char $k \neq 2$. This is proved by the techniques of Robert's paper. Since we shall not need this stronger result we do not present the details; however, note that if char $k = 0$ then part (ii) of Kleiman's Theorem 2 [6] applies to the proof of Theorem 3.3 presented above to conclude the strong subtransversality of f_L.

(b) In the case $k = \mathbb{C}$, Theorem 3.3, with the stronger conclusion of strong subtransversality, also follows from Theorem 19.3 of Andreotti-Holm [3] and from Mather's theory of generic linear projections [10,11].

4. Cohomology of blowing-up

Let $(A, \mathfrak{m})$ be a regular local ring of dimension r, and let $X = \operatorname{Spec} A$; π: $X' \to X$ will denote the blow-up of X with center $\mathfrak{m}$. Thus $X' = \operatorname{Proj}(\oplus_{n=0}^{\infty} \mathfrak{m}^n)$. Let $E = \pi^{-1}(\mathfrak{m})$ be the exceptional divisor. With these notations there is the following result.

4.1. Proposition: *$H^q(X', \mathcal{O}[sE]) = 0$ for $q > 0$ and $s < r$.*

Proof. Let $F = \mathcal{O}[sE]$ and note that $F = \mathcal{O}_{X'}(-s)$. By the formal function theorem

$$H^q(X', F)^{\wedge}_{\mathfrak{m}} = \lim_{\leftarrow} H^q(X'_n, F_n)$$

where $X'_n = X' \times_X \operatorname{Spec}(A/\mathfrak{m}^n)$. Now X'_n is just the ringed space $(E, \mathcal{O}_{X'}/I^n)$ where $I = \pi^{-1}(\mathfrak{m}) \cdot \mathcal{O}_{X'}$ is the sheaf of ideals defining E. In particular $X'_1 \simeq E \simeq \mathbb{P}^{r-1}(k)$ where k is the residue field of A. There is an exact sequence of sheaves on E:

$$0 \to I^n/I^{n+1} \to \mathcal{O}_{X'}/I^{n+1} \to \mathcal{O}_{X'}/I^n \to 0.$$

Furthermore, $F|_E \simeq \mathcal{O}_E(-s)$ and $I^n/I^{n+1} \simeq \mathcal{O}_E(n)$. Therefore, tensoring with F gives an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^{r-1}(k)}(n-s) \to F_{n+1} \to F_n \to 0.$$

But $F_1 \simeq \mathcal{O}_{\mathbb{P}^{r-1}(k)}(-s)$ so $H^q(X'_1, F_1) = 0$ in the range $q > 0$ and $s < r$ by Serre's theorem on the cohomology of projective space ([5] p. 225). Then by induction, $H^q(X'_n, F_n) = 0$ so that $H^q(X', F)^{\wedge}_{\mathfrak{m}} = 0$. But $H^q(X', F)$ is a finitely generated A-module so it follows that $H^q(X', F) = 0$.

Now consider the following situation. Let A be a regular noetherian ring and $p \in X = \operatorname{Spec} A$ be a prime ideal of height r. Let π: $X' \to X$ be the blowing-up of X with center $Y = V(p)$ and let $E = \pi^{-1}(Y)$ be the exceptional divisor.

4.2. Proposition: *With the above notations, $H^q(X', \mathcal{O}[sE]) = 0$ for $q > 0$ and $s < r$.*

PROOF: Let $M = H^q(X', \mathcal{O}[sE])$. Then M is a finitely generated A-module and Supp $(M) \subseteq V(p)$. Thus $M = 0$ if and only if $M_p = 0$. Consider the flat base change map.

$$\begin{tikzcd} X'' \arrow[r, "v"] \arrow[d, "\eta"'] & X' \arrow[d, "\pi"] \\ \operatorname{Spec} A_p \arrow[r, "u"'] & X \end{tikzcd}$$

Then $M_p \simeq H^q(X'', v^*\mathcal{O}[sE])$ ([5], p. 255). But blowing-up commutes with flat base change, so X'' is the blow-up of Spec A_p at the maximal ideal pA_p and the result then follows from Proposition 4.1.

5. Main theorem

5.1. THEOREM: *Let f: $X \to Y$ be a weakly subtransversal morphism of nonsingular algebraic k-varieties. Then the scheme $X \times_Y X \simeq (f\times f)^{-1}(\Delta_Y)$ is reduced.*

PROOF: Let π: $(X \times X)' \to X \times X$ be the blow-up of $X \times X$ along the diagonal and let E denote the exceptional divisor. Since $Z(f)\backslash E \simeq (f \times f)^{-1}(\Delta_Y)\backslash \Delta_X$ ([13], p. 63) it is only necessary to study $(f\times f)^{-1}(\Delta_Y)$ along the diagonal of X. Thus let a be a closed point of X and consider two cases.

Case 1. $\pi^{-1}(a, a) \cap Z(f) = \emptyset$. In this case it follows from Lemma 1.2 of Roberts [13] that the tangent map $df(a)$: $T_a X \to T_{f(a)} Y$ is injective. Thus f^*: $\mathfrak{m}_{f(a)}/\mathfrak{m}^2_{f(a)} \to \mathfrak{m}_a/\mathfrak{m}^2_a$ is surjective so by Nakayama's lemma $f^*\mathfrak{m}_{f(a)} = \mathfrak{m}_a$. Hence $(f\times f)^{-1}(I_{\Delta_Y,(f(a),f(a))}) = I_{\Delta_X,(a,a)}$ and the result is proved in case 1.

Case 2. $\pi^{-1}(a, a) \cap Z(f) \neq \emptyset$. By localizing at a and $f(a)$ we may reduce to the situation in which $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ where A and B are regular local rings of dimension $n = \dim X$ and $p = \dim Y$ respectively. Then let $x_1,\dots,x_n$ be a regular system of parameters for A and $y_1,\dots,y_p$ a regular system of parameters for B. Define Φ_j in $\Gamma((X \times X)', \mathcal{O}_{(X\times X)'})$ by $\Phi_j = \pi^*((f^*y_j \otimes 1 - 1 \otimes (f^*y_j))$ for $1 \leqslant j \leqslant p$. Since f is weakly subtransversal, Lemma 3.2 implies that $J = (\Phi_1,\dots,\Phi_p)$. $\mathcal{O}_{(X\times X)'}$ is a reduced sheaf of ideals. Let $R = R(\Phi_1,\dots,\Phi_p)$ be the sheaf of relations of the sections $\Phi_1,\dots,\Phi_p$.

5.2. LEMMA: $H^1((X \times X)', R) = 0$.

Assuming the lemma for the moment, we will complete the proof of the theorem. From the lemma there is an exact sequence $\Gamma((X\times X)', \mathcal{O}^p_{(X\times X)'}) \to \Gamma((X \times X)', J) \to 0$. Let $F_j = f^*y_j \otimes 1 - 1 \otimes f^*y_j$ and suppose

$G \in \operatorname{rad}(F_1,\dots,F_p)$. Then π^*G is a global section of J since J is reduced. Thus $\pi^*G = \sum_{j=1}^{p} C_j \Phi_j = \sum_{j=1}^{p} C_j \pi^* F_j$ where C_j is a global section of $\mathcal{O}_{(X\times X)'}$. But $\Gamma((X\times X)', \mathcal{O}_{(X\times X)'}) \simeq \Gamma(X\times X, \mathcal{O}_{(X\times X)})$ so $G \in (F_1,\dots,F_p)\cdot \mathcal{O}_{(X\times X)}$ and the theorem is proved.

It remains to prove the lemma. Let $u_j = x_j \otimes 1 - 1 \otimes x_j$ and $u'_j = 1 \otimes x_j$ for $1 \leqslant j \leqslant n$. Then $u_1,\dots,u_n, u'_1,\dots,u'_n$ generate the maximal ideal in $\mathcal{O}_{(X\times X),(a,a)}$ while $u_1,\dots,u_n$ generate the ideal $I_{\Delta_X,(a,a)}$. The blowing-up of $(X\times X), (a,a))$ along $(\Delta_X, (a,a))$, i.e. $(X\times X)'$, is a closed subscheme of $\mathbb{P}^{n-1}_{X\times X}$. Its ideal is defined by the functions $u_i t_j - t_i u_j$ for $1 \leqslant i, j \leqslant n$. Let W_i be the affine open subset of $(X\times X)'$ determined by $t_i \neq 0$. Then on W_i the divisor E is defined by u_i and the subtransversality assumption implies that the functions $\varphi_j^{(i)} = \Phi_j/u_i$ $(1 \leqslant j \leqslant p)$ define an $\mathcal{O}|_{W_i}$ sequence. Therefore, on W_i there is an exact Koszul resolution

$$0 \to \mathcal{O}|_{W_i} \xrightarrow{d_p^{(i)}} \mathcal{O}^{\binom{p}{p-1}}_{|W_i} \xrightarrow{d_{p-1}^{(i)}} \dots \to \mathcal{O}^{\binom{p}{2}}_{|W_i} \xrightarrow{d_2^{(i)}} \mathcal{O}|^{p}_{W_i} \xrightarrow{d_1^{(i)}} \mathfrak{a}_i \to 0 \tag{5.1}$$

where $\mathcal{O}$ is the structure sheaf of $(X\times X)'$ and if $\{e_{\alpha_1\dots\alpha_k}: 1 \leqslant \alpha_1 < \dots < \alpha_k \leqslant p\}$ is a basis for $\mathcal{O}^{\binom{p}{k}}$ then the differential $d_k^{(i)}: \mathcal{O}^{\binom{p}{k}}_{|W_i} \to \mathcal{O}^{\binom{p}{k-1}}_{|W_i}$ is defined by

$$d_k^{(i)}\left(e_{\alpha_1\dots\alpha_k}\right) = \sum_{r=1}^{k} (-1)^{r-1} \varphi_{\alpha_r}^{(i)} e_{\alpha_1\dots\hat{\alpha}_r\dots\alpha_k}. \tag{5.2}$$

Also $\mathfrak{a}_i = (\varphi_1^{(i)},\dots,\varphi_p^{(i)})\cdot \mathcal{O}_{|W_i}$. Since $u_i \varphi_k^{(i)} = \Phi_k = u_j \varphi_k^{(j)}$ on $W_i \cap W_j$, one concludes that on $W_i \cap W_j$

$$d_k^{(i)} = \left(u_j/u_i\right) d_k^{(j)}. \tag{5.3}$$

We would like to glue the sequence in (5.1) into a sequence of bundle maps on $(X\times X)'$. If $\{s_i^{(k)}\}$ is a section of the k^{th} bundle with transition functions $\{h_{ij}^{(k)}\}$, then (5.3) implies that

$$d_k^{(i)}\left(s_i^{(k)}\right) = \left(\left(u_j/u_i\right) h_{ij}^{(k)}\right) d_k^{(j)}\left(s_j^{(k)}\right). \tag{5.4}$$

Thus, in order for the $d_k^{(i)}$ to glue together into a bundle map it must be true that $h_{ij}^{(k)} = (u_i/u_j) h_{ij}^{(k-1)}$. Taking $h_{ij}^{(1)} = 1$ we conclude that there is a resolution of J by locally free sheaves:

$$0 \to \mathcal{O}[(p-1)E] \xrightarrow{d_p} \mathcal{O}^{\binom{p}{p-1}}[(p-2)E] \xrightarrow{d_{p-2}} \dots$$

$$\dots \mathcal{O}^{\binom{p}{3}}[2E] \xrightarrow{d_3} \mathcal{O}^{\binom{p}{2}}[E] \xrightarrow{d_2} \mathcal{O}^{p} \xrightarrow{d_1} J \to 0 \tag{5.5}$$

where $d_1(h_1,\ldots,h_p)=\Sigma_{k=1}^{p}h_k\Phi_k$, and the other d_k are defined locally as in (5.1).

Now split the resolution of J into a sequence of short exact sequences.

$$0 \to K^1 \to \mathcal{O}^{\binom{p}{2}}[E] \to R \to 0$$

$$\vdots$$

$$0 \to K^q \to \mathcal{O}^{\binom{p}{q+1}}[qE] \to K^{q-1} \to 0$$

$$\vdots$$

$$0 \to \mathcal{O}[(p-1)E] \to \mathcal{O}^p[(p-2)E] \to K^{p-3} \to 0$$

For $q > 1$, the q^{th} short exact sequence has a long exact cohomology sequence which contains the following segment:

$$H^q\left((X\times X)', \mathcal{O}^{\binom{p}{q+1}}[qE]\right) \to H^q((X\times X)', K^{q-1})$$

$$\to H^{q+1}((X\times X)', K^q) \to H^{q+1}\left((X\times X)', \mathcal{O}^{\binom{p}{q+1}}[qE]\right).$$

According to Proposition 4.2, since codim $\Delta_X = n$, we will have $H^q((X\times X)', \mathcal{O}[qE]) = 0$ and $H^{q+1}((X\times X)', \mathcal{O}[qE]) = 0$ for all $n > q > 0$. But the vanishing is also true for $q \geqslant n$ since $H^q((X\times X)', \cdot)$ vanishes identically for $q \geqslant n = \dim X$ when X is affine. Thus we will get an isomorphism $H^1((X\times X)', R) \simeq H^{p-1}((X\times X)', \mathcal{O}[(p-1)E]) = 0$ which proves Lemma 5.2 and hence the theorem.

Combining Theorem 5.1 with Theorem 3.3 one obtains immediately the equality of weak normality and Lipschitz saturation for generic projections.

5.3. Theorem: *Let $X \subseteq \mathbb{P}^N$ be nonsingular. If L is a linear subspace of $\mathbb{P}^N$ of codimension $m+1$ ($m \geqslant$ dim X) and f_L: $X \to \mathbb{P}^m$ is the corresponding projection, let $Y_L = f_L(X)$. If L is chosen in general position, then $Y_L{}^* = \tilde{Y}_L$.* (*As in Theorem* 3.3, *if char k* > 0, *it may be necessary to reimbed X before applying the projection.*)

Proof: The normalization of Y_L is just X and f_L is the normalization map. According to Theorem 3.3 and Lemma 3.2, if L is chosen in general position then f_L is weakly subtransversal so that Theorem 5.1 applies to show that $I_L = (f_L\times f_L)^{-1}(I_{\Delta_{Y_L}})$ is a reduced sheaf of ideals. Thus $I_L \subseteq \bar{I}_L \subseteq \mathrm{Rad}(I_L) \subseteq I_L$. Hence $\bar{I}_L = I_L$ and the equality of $Y_L{}^*$ and $\tilde{Y}_L$ follows from Definition 2.1.

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