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# NON-ARCHIMEDEAN (UNIFORMLY) CONTINUOUS MEASURES ON HOMOGENEOUS SPACES

Luc Duponcheel

## I. Introduction

Let $\mathscr{K}$ be a complete non-archimedean valued field. Let $G$ be a locally compact zerodimensional group. If $G$ has an open compact $q$-free subgroup, where $q$ is the characteristic of the residue class field of $\mathscr{K}$, then for every closed subgroup $H$ of $G$ the homogeneous space of all left cosets of $H$ in $G$ has a $\mathscr{K}$-valued quasi-invariant measure $\mu$ (see [4]). Let $M^{\infty}(G/H)$ be the space of all measures on $G/H$. In this paper we will prove the following result: an element $\lambda$ of $M^{\infty}(G/H)$ translates continuously for the strong (resp. weak) topology on $M^{\infty}(G/H)$ if and only if $\lambda = f\mu$ where $f$ is a bounded uniformly continuous (resp. continuous) function on $G/H$.

## II. Functions and measures

Let $\mathscr{K}$ be a *complete non-archimedean valued field.* Let $X$ be a *locally compact zerodimensional space*. Let $B_c(X)$ be the ring of all *open compact* subsets of $X$. If $A$ is an element of $B_c(X)$ then $\xi(A)$ will be the $\mathscr{K}$-valued *characteristic function* of $A$. Let $BC(X)$ be the space of all *bounded continuous* $\mathscr{K}$-valued functions on $X$. With the norm $\|f\| =: \sup_{x \in X}|f(x)|$, $(f \in BC(X))$, $BC(X)$ becomes a non-archimedian Banach space over $\mathscr{K}$. With the seminorms $p_x(f) =: |f(x)|$, $(f \in BC(X); x \in X)$, $BC(X)$ becomes a non-archimedean locally convex space over $\mathscr{K}$. The topology (resp. uniformity) on $BC(X)$ induced by the norm $|\ |$ will be called the *uniform* topology (resp. uniformity). The topology (resp. uniformity) on $BC(X)$ induced by the seminorms $p_x$ $(x \in X)$ will be called the *pointwise* topology (resp. uniformity). Let $BUC(X)$ be the space of all *bounded* $\mathscr{U}$*-uniformly continuous* $\mathscr{K}$-valued functions on $X$ where $\mathscr{U}$ is a non-archimedean uniformity on $X$ compatible with the topology on $X$. Let $C_{\infty}(X)$ be the space of all continuous $\mathscr{K}$-valued functions on $X$ which *vanish at infinity*. Both $BUC(X)$ and $C_{\infty}(X)$ are subspaces of $BC(X)$. In the same way as $BC(X)$, $BUC(X)$ and $C_{\infty}(X)$ can be made into a non-archimedean Banach space and a locally convex space over $\mathscr{K}$. Let $M^{\infty}(X)$ be the space of all *bounded additive* $\mathscr{K}$-valued functions on $B_c(X)$. The elements of $M^\infty(X)$ will be called *measures* on $X$. With the norm $\|\lambda\| =: \sup_{A \in B_c(X)} |\lambda(A)|$ $(\lambda \in M^\infty(X))$, $M^\infty(X)$ becomes a non-archimedean Banach space over $\mathscr{K}$. $(M^\infty(X), \|\ \|)$ is the *dual space* of $(C_\infty(X), \|\ \|)$. The duality is given by

$$(f, \lambda) \to \int_X f \mathrm{d}\lambda = \int_X f(x) \mathrm{d}\lambda(x) \ (f \in C\infty(X), \lambda \in M^\infty(X)).$$

Let $\lambda$ be an element of $M^\infty(X)$. If for every $x$ in $X$ we define $N_\lambda(x) =: \inf_{A \ni x} \sup_{B \subset A} |\lambda(B)|$ then for every $A$ in $B_c(X)$ we have $\sup_{B \subset A} |\lambda(B)| = \sup_{x \in A} N_\lambda(x)$. In particular we have $\|\lambda\| = \sup_{x \in X} N_\lambda(x)$. With the seminorms $q_x(\lambda) =: N_\lambda(x)$ $(x \in X, \lambda \in M^\infty(X))$, $M^\infty(X)$ becomes a non-archimedean locally convex space over $\mathscr{K}$. The topology (resp. uniformity) on $M^\infty(X)$ induced by the norm $\|\ \|$ will be called the *strong* topology (resp. uniformity). The topology (resp. uniformity) on $M^\infty(X)$ induced by the seminorms $q_x$ $(x \in X)$ will be called the *weak* topology (resp. uniformity).

If $\lambda$ is an element of $M^\infty(X)$ and $g$ is an element of $BC(X)$ we can define an element $g\lambda$ of $M^\infty(X)$ by setting:

$$\int_X f \mathrm{d} g\lambda =: \int_X fg \mathrm{d}\lambda \ (f \in C_\infty(X)).$$

It is not difficult to see that $N_{g\lambda}(x) = |g(x)| N_\lambda(x)$.

III. Proposition (1): *Let $\lambda$ be an element of $M^\infty(X)$ with $\inf_{x \in X} N_\lambda(x) \geqslant m > 0$. The map $g \to g\lambda$ is a linear homeomorphism from $BC(X)$ with the uniform (resp. pointwise) topology onto a closed subspace of $M^\infty(X)$ with the strong (resp. weak) topology.*

Proof: It is clear that the map $g \to g\lambda$ from $BC(X)$ with the uniform (resp. pointwise) topology to $M^\infty(X)$ with the strong (resp. weak) topology is a linear homeomorphism. We only need to prove that its image is closed.

For the the uniform topology on $BC(X)$ and the strong topology on $M^\infty(X)$ this is trivial.

For the pointwise topology on $BC(X)$ and the weak topology on $M^\infty(X)$ this runs as follows: let $(f_\alpha \lambda)_{\alpha \in I}$ be a net in $M^\infty(X)$ converging weakly to $\mu$. It is clear that the net $(f_\alpha)_{\alpha \in I}$ in $BC(X)$ converges pointwise to a function $f$. We have

$$N_{f(x)\lambda - \mu}(x) \leqslant \max\left[ N_{f(x)\lambda - f_\alpha \lambda}(x), N_{f_\alpha \lambda - \mu}(x) \right] \ (\alpha \in I, x \in X).$$

Therefore

$$N_{f(x)\lambda-\mu}(x)=0\ (x\in X).$$

$$|f(x)|\leqslant\frac{1}{m}|f(x)|N_\lambda(x)=\frac{1}{m}N_{f(x)\lambda}(x)$$

$$\leqslant\frac{1}{m}\max\left[N_{f(x)\lambda-\mu}(x),\,N_\mu(x)\right]\leqslant\frac{N_\mu(x)}{m}\tag{1}$$

$$|f(x)-f(y)|\leqslant\frac{1}{m}|f(x)-f(y)|N_\lambda(y)=\frac{1}{m}N_{f(x)\lambda-f(y)\lambda}(y)$$

$$\leqslant\frac{1}{m}\max\left[N_{f(x)\lambda-\mu}(y),\,N_{f(y)\lambda-\mu}(y)\right]$$

$$=\frac{1}{m}N_{f(x)\lambda-\mu}(y).\tag{2}$$

From the inequalities (1) and (2) it is easy to conclude that $f\in BC(X)$. (Notice that the function $N_{f(x)\lambda-\mu}$ is uppersemicontinuous and $N_{f(x)\lambda-\mu}(x)=0$). Now

$$N_{f\lambda-\mu}(x)\leqslant\max\left[N_{f\lambda-f(x)\lambda}(x),\,N_{f(x)\lambda-\mu}(x)\right]=0$$

and we may conclude that $\mu=f\lambda$ and we are done.

## IV. Groups and homogeneous spaces

Let $G$ be a *locally compact zerodimensional group*. For every *open compact subgroup* $K$ of $G$ we can define an equivalence relation on $G$ by setting:

$$s\sim t\Leftrightarrow sK=tK\ (\text{resp. } s\sim t\Leftrightarrow Ks=Kt)\ (s,t\in G).$$

Using those equivalence relations we can define the so called *left* (resp. *right*) *group uniformity* on $G$ which is compatible with the topology on $G$.

Let $BLUC(G)$ (resp. $BRUC(G)$) be the space of all *bounded left* (resp. *right*) *uniformily continuous* $\mathcal{K}$-valued functions on $G$. If $f$ is an element of $BC(G)$ and $s$ is an element of $G$, we can define an element $R_sf$ (resp. $L_sf$) of $BC(G)$ by setting:

$$R_sf(t)=:f(ts)\ (\text{resp. } L_sf(t)=:f(st))\ (t\in G).$$

If $f$ is an element of $BC(G)$ then the functions $s\to L_sf$ and $s\to R_sf$ from $G$ to $BC(G)$ are continuous for the pointwise topology on $BC(G)$.

An element $f$ of $BC(G)$ is an element of $BLUC(G)$ (resp. $BRUC(G)$)

if and only if the function $s \to R_s f$ (resp. $s \to L_s f$) from $G$ to $BC(G)$ is continuous for the uniform topology on $BC(G)$. It is important to notice that $C_\infty(G)$ is a subspace of $BLUC(G)$ (resp. $BRUC(G)$).

Let $H$ be a closed subgroup of $G$. Let $G/H$ be the set of all *left cosets* of $H$ in $G$. Let $\pi$: $G \to G/H$ be the *natural quotient map* from $G$ to $G/H$. With the quotient topology $G/M$ also becomes a locally compact zerodimensional space. $G$ has an *action* on $G/H$ by setting:

$$s(\pi(t)) =: \pi(st) \; (s \in G, \pi(t) \in G/H).$$

For every open compact subgroup $K$ of $G$ we can define an equivalence relation on $G/H$ by setting:

$$\pi(s) \sim \pi(t) \Leftrightarrow \pi(Ks) = \pi(Kt) \; (\pi(s), \pi(t) \in G/H).$$

Using those equivalence relations we can define the so called *homogeneous uniformity* on $G/H$ which is compatible with the topology on $G/H$. Let $BUC(G/H)$ be the space of all *bounded uniformly continuous* $\mathscr{K}$-valued functions on $G/H$. If $f$ is an element of $BC(G/H)$ and $s$ is an element of $G$ we can define an element $L_s f$ of $BC(G/H)$ by setting:

$$L_s f(\pi(t)) =: f(\pi(st)) \; (\pi(t) \in G/H).$$

If $f$ is an element of $BC(G/H)$ then the function $s \to L_s f$ from $G$ to $BC(G/H)$ is continuous for the pointwise topology on $BC(G/H)$. An element $f$ of $BC(G/H)$ is an element of $BUC(G/H)$ if and only if the function $s \to L_s f$ from $G$ to $BC(G/H)$ is continuous for the uniform topology on $BC(G/H)$. It is important to notice that $C_\infty(G/H)$ is a subspace of $BUC(G/H)$.

## V. Quasi-invariant measures on homogeneous spaces

Let $G$ be a *locally compact zerodimensional group*. Let $H$ be a *closed subgroup* of $G$. Let $G/H$ be the *homogeneous space* of all left cosets of $H$ in $G$. We suppose that $G$ has an open compact *q-free* subgroup where $q$ is the characteristic of the *residue class field* of $\mathscr{K}$. In that case $G$ has an *invariant measure m*. $H$ also has an *invariant measure n*. Let $\Delta$ be the *modular function* on $G$. Let $\delta$ be the *modular function* on $H$. Let $f$ be an element of $C_\infty(G)$. We can define an element $f^b$ of $C_\infty(G/H)$ by setting:

$$f^b(\pi(s)) =: \int_H f(st)\mathrm{d}t \; (\pi(s) \in G/H).$$

The map $f \to f^b$ from $C_\infty(G)$ to $C_\infty(G/H)$ is linear and continuous and using duality we can define a linear and continuous map $\lambda \to \lambda^\#$ from

$M^\infty(G/H)$ to $M^\infty(G)$. We have:

$$\int_G f d\lambda^\# = \int_{G/H} f^b d\lambda.$$

For every $s$ in $G$ we have $N_\lambda(\pi(s)) = N_{\lambda^\#}(s)$.

A *quasi-invariant measure* on $G/H$ is a measure $\mu$ such that $\mu^\# = \rho m$, where $\rho$ is an invertible element of $BRUC(G)$. Such a measure does always exist and it is unique up to an invertible element of $BUC(G/H)$. We can even say more: for every open compact subgroup $K$ of $G$ there exists a quasi-invariant measure $\mu$ on $G/H$ with $\mu^\# = \rho m$ where $|\rho| \equiv 1$ and $\rho$ is constant on the right cosets of $K$ (see [4]).

## VI. (Uniformly) continuous measures

Let $\lambda$ be an element of $M^\infty(G/H)$ and $s$ an element of $G$. We can define an element $\lambda_s$ of $M^\infty(G/H)$ by setting:

$$\int_{G/H} f d\lambda_s =: \int_{G/H} L_s f d\lambda \ (f \in C_\infty(G/H)).$$

An element $\lambda$ of $M^\infty(G/H)$ is called a *continuous* (resp. *uniformly continuous*) *measure* if the function $s \to \lambda_s$ from $G$ to $M^\infty(G/H)$ is weakly (resp. strongly) continuous.

It is clear that the function $s \to \lambda_s$ from $G$ to $M^\infty(G/H)$ is weakly (resp. strongly) continuous if and only if the function $s \to (\lambda^\#)_s$ from $G$ to $M^\infty(G)$ is weakly (resp. strongly) continuous. In order to find the continuous (resp. uniformly continuous) measures on $G/H$ it suffices to find the continuos (resp. uniformly continuous) measures on $G$.

Let $(K_\alpha)_{\alpha \in I}$ be a fundamental system of open compact $q$-free subgroups of $G$. We can define functions $(u(K_\alpha))_{\alpha \in I}$ from $G$ to $\mathscr{K}$ by setting

$$u(K_\alpha) =: \frac{1}{m(K_\alpha)} \xi(K_\alpha).$$

VII. Proposition (2): *Let $\lambda$ be an element of $M^\infty(G)$. Let $m$ be an invariant measure on $G$. $\lambda$ is a uniformly continuous (resp. continuous) measure if and only if $\lambda = fm$ for some element $f$ of $BRUC(G)$ (resp. $BC(G)$).*

Proof: (1) If $\lambda = fm$ for some element $f$ of $BRUC(G)$ (resp. $BC(G)$) then the function $s \to \lambda_s$ from $G$ to $M^\infty(G)$ is strongly (resp. weakly) continuous as can easily be verified.

(2) Suppose now that the function $s \to \lambda_s$ from $G$ to $M^\infty(G)$ is strongly (resp. weakly) continuous. For every $f$ in $C_\infty(G)$ we can define an element

$f * \lambda$ of $M^{\infty}(G)$ by setting:

$$f * \lambda = \int_G f(s)\lambda_s \mathrm{d}s$$

$$\left(\text{resp. } \int_G g\mathrm{d}f * \lambda = \int_G f(s)\left[\int_G g d\lambda_s\right]\mathrm{d}s \; g \in C_\infty(G)\right).$$

We will prove that $u(K_\alpha) * \lambda$ converges strongly (resp. weakly) to $\lambda$. $(*)$

From this it follows (using Proposition 1) that it suffices to prove that for every $f$ in $C_\infty(G)$ there exists an element $F$ of $BRUC(G)$ (resp. $BC(G)$) with $f * \lambda = Fm$. Define

$$F(s) =: \int_G \Delta(t)^{-1} f(st^{-1})\mathrm{d}\lambda(t)$$

$$\|F\| = \sup_{s\in G}|F(s)| = \sup_{S\in G}\left|\int_G \Delta(t)^{-1} f(st^{-1})\mathrm{d}\lambda(t)\right|$$

$$\leqslant \|\lambda\| \sup_{s\in G}\sup_{t\in G}|\Delta(t)^{-1} f(st^{-1})| \leqslant \|\lambda\|\,\|f\|.$$

$$\|L_t F - F\| = \sup_{s\in G}|F(ts) - F(s)|$$

$$= \sup_{s\in G}\left|\int_G \Delta(q)^{-1}[f(tsq^{-1}) - f(sq^{-1})]\mathrm{d}\lambda(q)\right|$$

$$\leqslant \|\lambda\| \sup_{s\in G}\sup_{q\in G}|\Delta(q)^{-1}[f(tsq^{-1}) - f(sq^{-1})]|$$

$$\leqslant \|\lambda\|\,\|L_t f - f\|.$$

It is clear that $F$ is an element of $BRUC(G)$ (in particular it is an element of $BC(G)$). Let $g$ be an element of $C_\infty(G)$.

$$\int_G gdf * \lambda = \int_G\int_G f(s)g(t)\mathrm{d}\lambda_s(t)\mathrm{d}s$$

$$= \int_G\int_G f(s)g(st)d\lambda(t)\mathrm{d}s$$

$$= \int_G\int_G f(s)g(st)\mathrm{d}sd\lambda(t)$$

$$= \int_G\int_G \Delta(t)^{-1} f(st^{-1})g(s)\mathrm{d}sd\lambda(t)$$

$$= \int_G g(s)\left[\int_G \Delta(t)^{-1} f(st^{-1})\,d\lambda(t)\right]\mathrm{d}s$$

$$= \int_G g(s)F(s)\mathrm{d}s$$

$$= \int_G g\,\mathrm{d}Fm$$

and we may conclude that $f * \lambda = Fm$. We still need to prove $(*)$:
- For the strong topology this runs as follows:

$$\forall \epsilon > 0\ \exists K_\alpha \vdash \forall K_\beta \subset K_\alpha\colon \sup_{s\in K_\beta}\|\lambda_s - \lambda\| \leqslant \epsilon.$$

Using the inequality

$$\|u(K_\beta) * \lambda - \lambda\| = \left\| \frac{1}{m(K_\beta)} \int_{K_\beta} (\lambda_s - \lambda)\mathrm{d}s \right\| \leqslant \sup_{s\in K_\beta}\|\lambda_s - \lambda\|$$

we see that

$$\forall \epsilon > 0\ \exists K_\alpha \vdash \forall K_\beta \subset K_\alpha\colon \|u(K_\beta) * \lambda - \lambda\| \leqslant \epsilon \text{ and we are done.}$$

- For the weak topology this runs as follows:

$$\forall t \in G\ \forall \epsilon > 0\ \exists K_\alpha \vdash \forall K_\beta \subset K_\alpha\colon \sup_{s\in K_\beta} N_{\lambda_s - \lambda}(t) \leqslant \epsilon.$$

If $p \in K_\beta t$ (say $p = q^{-1}t$, $q \in K_\beta$) then

$$N_{\lambda_s-\lambda}(p) = N_{\lambda_s-\lambda}(q^{-1}t)$$

$$= N_{\lambda_{sq}-\lambda_q}(t) \leqslant \max\left(N_{\lambda_{sq}-\lambda}(t), N_{\lambda_q-\lambda}(t)\right)$$

and we see that

$$\forall t \in G\ \forall \epsilon > 0\ \exists K_\alpha \vdash \forall K_\beta \subset K_\alpha\colon \inf_{A\ni t} \sup_{s\in K_\beta} \sup_{p\in A} N_{\lambda_s-\lambda}(p) \leqslant \epsilon$$

thus

$$\forall t \in G\ \forall \epsilon > 0\ \exists K_\alpha \vdash \forall K_\beta \subset K_\alpha\colon \inf_{A\ni t} \sup_{s\in K_\beta} \sup_{B\subset A} |\lambda_s(B) - \lambda(B)| \leqslant \epsilon.$$

Using the inequality

$$N_{u(K_\beta)*\lambda-\lambda}(t) = \inf_{A\ni t} \sup_{B\subset A} \left|\left(u(K_\beta) * \lambda - \lambda\right)(B)\right|$$

$$\leqslant \inf_{A \ni t} \sup_{B \subset A} \sup_{s \in K_\beta} |\lambda_s(B) - \lambda(B)|$$

we may conclude that

$$\forall t \in G \, \forall \epsilon > 0 \, \exists K_\alpha \vdash \forall K_\beta \subset K_\alpha \colon N_{u(K_\beta) * \lambda - \lambda}(t) \leqslant \epsilon$$

and we are done.

VIII. Remark: *The function $f$ of the foregoing proposition is the uniform (resp. pointwise) limit of the functions $(F_\alpha)_{\alpha \in I}$ defined by*:

$$F_\alpha(s) =: \frac{\lambda(K_\alpha s)}{m(K_\alpha s)} \quad (s \in G,\ \alpha \in I).$$

Proof: $\lambda = \lim_{\alpha \in I} u(K_\alpha) * \lambda = \lim_{\alpha \in I} F_\alpha m = (\lim_{\alpha \in I} F_\alpha) m$ with

$$F_\alpha(s) = \int_G \Delta(t)^{-1} u(K_\alpha)(st^{-1}) \mathrm{d}\lambda(t)$$

$$= \frac{1}{m(K_\alpha)} \int_G \Delta(t)^{-1} \xi(K_\alpha s)(t) \mathrm{d}\lambda(t)$$

$$= \frac{1}{m(K_\alpha)} \int_G \Delta(s)^{-1} \xi(K_\alpha s)(t) \mathrm{d}\lambda(t)$$

$$= \frac{\lambda(K_\alpha s)}{m(K_\alpha)\Delta(s)} = \frac{\lambda(K_\alpha s)}{m(K_\alpha s)}.$$

IX. Theorem: *Let $\lambda$ be an element of $M^\infty(G/H)$. Let $\mu$ be a quasi-invariant measure on $G/H$. $\lambda$ is a uniformly continuous (resp. continuous) measure if and only if $\lambda = f\mu$ for some element of $f$ of $BUC(G/H)$ (resp. $BC(G/H)$).*

Proof: Let $\rho$ be the invertible function of $BRUC(G)$ with $\mu^\# = \rho m$. The function $s \to \lambda_s$ is strongly (resp. weakly) continuous if and only if the function $s \to (\lambda^\#)_s$ is strongly (resp. weakly) continuous.

(a) If the function $s \to (\lambda^\#)_s$ from $G$ to $M^\infty(G)$ is strongly (resp. weakly) continuous there exists an element $g$ of $BRUC(G)$ (resp. $BC(G)$) with $\lambda^\# = gm$. It is easy to see that $\lambda = f\mu$ where $f(\pi(s)) =: [g(s)/\rho(s)]$ is an element of $BUC(G/H)$ (resp. $BC(G/H)$).

(b) If there exists an element $f$ of $BUC(G/H)$ (resp. $BC(G/H)$) with $\lambda = f\mu$ then it is easy to see that $\lambda^\# = gm$ where $g(s) = \rho(s) f(\pi(s))$ is an element of $BRUC(G)$ (resp. $BC(G)$) and therefore we see that the function $s \to (\lambda^\#)_s$ from $G$ to $M^\infty(G)$ is strongly (resp. weakly) continuous.

X. REMARK: *The function $f$ of the foregoing theorem is the uniform (resp. pointwise) limit of the functions $(F_\alpha)_{\alpha \in I}$ defined by*:

$$F_\alpha(\pi(s)) =: \frac{\lambda(\pi(K_\alpha s))}{\mu(\pi(K_\alpha s))} (\pi(s) \in G/H, \alpha \in I).$$

PROOF: Let $\nu$ be a quasi-invariant measure on $G/H$ with $\nu^{\#} = \rho m$ where $|\rho| \equiv 1$ and $\rho$ is constant on every right coset of every $K_\alpha (\alpha \in I)$. If $\lambda = g\nu$ then $g$ is the uniform (resp. pointwise) limit of the functions $(G_\alpha)_{\alpha \in I}$ where

$$G_\alpha(\pi(s)) = \frac{\lambda^{\#}(K_\alpha s)}{m(K_\alpha s)\rho(s)}.$$

Now

$$\lambda^{\#}(K_\alpha s) = \int_{G/H} \xi(K_\alpha s)^b d\lambda = \int_{G/H}\int_H \xi(K_\alpha s)(tr)\mathrm{d}r\mathrm{d}\lambda(\pi(t))$$

$$= n(K_\alpha s \cap H)\int_{G/H} \xi(\pi(K_\alpha s))(\pi(t))\mathrm{d}\lambda(\pi(t))$$

$$= n(K_\alpha s \cap H)\lambda(\pi(K_\alpha s))$$

and

$$m(K_\alpha s)\rho(s) = \int_G \xi(K_\alpha s)(t)\rho(t)\mathrm{d}t$$

$$= \int_{G/H}\int_H \xi(K_\alpha s)(tr)\mathrm{d}r\mathrm{d}\pi(t)$$

$$= n(K_\alpha s \cap H)\int_{G/H} \xi(\pi(K_\alpha s))(\pi(t))\mathrm{d}\pi(t)$$

$$= n(K_\alpha s \cap H)\nu(\pi(K_\alpha s))$$

so we may conclude that

$$G_\alpha(\pi(s)) = \frac{\lambda(\pi(K_\alpha s))}{\nu(\pi(K_\alpha s))}.$$

If $\mu = h\nu$ for some invertible element $h$ of $BUC(G/H)$ then, in the same way, $h$ is the uniform limit of the functions $(H_\alpha)_{\alpha \in I}$ where

$$H_\alpha(\pi(s)) = \frac{\mu(\pi(K_\alpha s))}{\nu(\pi(K_\alpha s))}.$$

If $\lambda = f\mu = (g/h)\mu$ then we see that $f$ is the uniform (resp. pointwise) limit of the functions $(F_\alpha)_{\alpha \in I}$ where

$$F_\alpha(\pi(s)) = \frac{G_\alpha(\pi(s))}{H_\alpha(\pi(s))} = \frac{\lambda(\pi(K_\alpha s))}{\mu(\pi(K_\alpha s))}$$

and were are finished.

## XI. Final remark

The most important results of this paper can be reformulated as follows:

*Let $\mu$ be a quasi-invariant measure on $G/H$. The map $f \to f\mu$ from $BUC(G/H)$ with the uniform topology (resp. $BC(G/H)$ with the pointwise topology) to $M^\infty(G/H)$ with the strong (resp. weak) topology is a linear homeomorphism onto a closed subspace of $M^\infty(G/H)$. This subspace consists exactly of those measures wich translate continuously for the strong (resp. weak) toplogy on $M^\infty(G/H)$.*

We can always find a quasi-invariant measure $\mu$ on $G/H$ with $N_\mu \equiv 1$. In that case the map $f \to f_\mu$ from $BUC(G/H)$ to $M^\infty(G/H)$ is a linear isometry.

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