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EXAMPLES OF NON-AMPLE NORMAL BUNDLES

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§1. Introduction

Let $Z = Gr(1, \mathbb{P}^3_{\mathbb{C}})$ be the Grassmannian of 1 planes in $\mathbb{P}^3$, embedded as a quadric hypersurface in $\mathbb{P}^5$. In this note, I construct in Z two smooth surfaces, $X_3 = \mathbb{P}^2$ blown up at one point and $X_4 = \mathbb{P}^1 \times \mathbb{P}^1$, of degrees 3 and 4 respectively, such that for each, the normal bundle in Z is not ample. This is in apparent contradiction to a result of Papantonopoulou [4], stating that any smooth surface in Z, having a non-ample normal bundle, must be a linear $\mathbb{P}^2$.

I do know of other examples, other than linear $\mathbb{P}^2$'s, although, I will not discuss these, here. The reader is referred to Hartshorne [3] for the definition of ampleness.

In a future paper, I will discuss the motivation for these constructions. It is described, briefly, below, and involves a geometric interpretation of the tangent and cotangent bundles of Z; this generalizes to the r-dimensional quadric $Z^r \subset \mathbb{P}^{r+1}$. Also, the problem of which m-dimensional submanifolds $X^m \subset Z^r$ $(m \geqslant 2)$ have non-ample normal bundles reduces, largely, via hyperplane sections, to the study of smooth surfaces in Z^{r-m+2}, whose normal bundles are not ample.

We consider the 4-dimensional quadric, Z. Let $\mathbb{P}(T^*Z) = (T^*Z \setminus Z)/\mathbb{C}^*$. The sections of TZ induce a map

$$\phi : \mathbb{P}(T^*Z) \to \mathbb{P}^n.$$

Let $X \subset Z$ be a surface, and NX the normal bundle of X. Let ϕ' denote the restriction of ϕ to NX. The normal bundle, NX, is ample when ϕ' is finite to 1. So, NX can fail to be ample in 3 ways:

(a) $\dim \phi'(\mathbb{P}(N^*X)) = 2$

(b) $\dim \phi'(\mathbb{P}(N^*X)) = 3$ and ϕ' has an infinity of positive dimensional fibres, or

(c) $\dim \phi'(\mathbb{P}(N^*X)) = 3$ and ϕ' has only a finite number of positive dimensional fibres.

All these possibilities can occur. The possibility (a) happens when X is a linear $\mathbb{P}^2$ (cf [4]), and (b) when $X = X_4$, as in §3; in a future paper, we will see that for (a) and (b), that these are the only possibilities. FInally, the example $X = X_3$ in §4 shows that (c) can occur with a *single* positive dimensional fibre.

I would like to thank Andrew Sommese for suggesting that I look at this topic, and for ways of viewing the problem. Towards completing the characterization of X_4, I had helpful conversations with Gary Kennedy, Daniel Phillips and Avinash Sathaye. Also, the referee's suggestions have improved the presentation of this paper.

2. Notation and background material

(2.1) Let A be a submanifold of the manifold B. I denote the normal bundle of A in B as $N(B/A) := TB/TA$. Its dual, the conormal bundle, is

$$N^*(B/A) = \{\alpha \in T^*B : \alpha = 0 \text{ on } TA\}.$$

(2.2) Let $[z_0, \ldots, z_5]$ be homogeneous coordinates in $\mathbb{P}^5$, and let $\mathcal{O}(-1)$ be the tautological line bundle; it is the one for which the transition function from the patch $z_i \neq 0$ to the path $z_j \neq 0$ is given by multiplication by $z_j z_i^{-1}$. We consider the well-known Euler sequence (see e.g. [2] p. 409).

$$0 \to T^*(\mathbb{P}^5) \to \mathcal{O}(-1)^{\oplus 6} \to \mathcal{O} \to 0.$$

If $\alpha = (\alpha_0, \ldots, \alpha_5) \in T_z^*(\mathbb{P}^5)$ is represented in $z_i \neq 0$ by $a = (a_0, \ldots, a_5) \in \mathbb{C}^6$, then in $z_j \neq 0$ α is represented by $z_j z_i^{-1} a$.

(2.3) Let Y be a projective subvariety of $\mathbb{P}^5$ and $E \to Y$ a holomorphic vector bundle spanned by global sections. Then, according to Gieseker ([1] Proposition 2.1), E is not ample precisely when there exists a curve $C \subset Y$ and a trivial bundle $\mathcal{O}_C \subset E^*|_C$.

A "line" in $\mathbb{P}^5$ denotes a linear $\mathbb{P}^1$.

§3. The surface X_4

Let $\gamma_1, \gamma_2 : \mathbb{P}^1 \to \mathbb{P}^5$

$$\gamma_1(s, t) = (s^2, 2st, 2t^2, 0, 0, 0)$$

$$\gamma_2(s, t) = (0, 0, 0, s^2, 2st, 2t^2).$$

Let $X = X_4$ be the scroll surface in $\mathbb{P}^5$ determined by γ_1 and γ_2, i.e. X is the union of the lines obtained by joining corresponding points of γ_1 and γ_2. Since $\gamma_1(\mathbb{P}^1)$ and $\gamma_2(\mathbb{P}^1)$ are contained in disjoint linear spaces, X is smooth. Also, the equation for $Z \subset \mathbb{P}^5$ is $z_0 z_5 - z_1 z_4 + z_2 z_3 = 0$, so that $X \subset Z$, as is easily checked.

Consider the exact sequence of conormal spaces (see (2.1) for notation)

$$0 \to N^*(\mathbb{P}^5/Z) \to N^*(\mathbb{P}^5/X) \to N^*(Z/X) \to 0.$$

Let ℓ be any line of the ruling of X. We construct a "section"

$$\sigma: \ell \to N^*(\mathbb{P}^5/X)|_\ell$$

which is both well defined and nowhere zero, modulo $N^*(\mathbb{P}^5/Z)$. This, then, determines a nowhere zero section

$$\sigma: \ell \to N^*(Z/X)|_\ell$$

i.e. a trivial bundle $\mathcal{O}_\ell \subset N^*(Z/X)|_\ell$ so that, by (2.3), $N(Z/X)$ is not ample. We remark that $N(Z/X)$ is spanned by global sections since TZ is spanned by global sections (Z is a homogeneous space), and $N(Z/X)$ is a quotient bundle of TZ.

Construction of σ: Let $\mathbb{C} \times \mathbb{P}^1 \hookrightarrow X$ be the patch $(t, [\lambda, \mu]) \mapsto (\lambda, 2t\lambda, 2t^2\lambda, \mu, 2t\mu, 2t^2\mu)$. The tangent $\mathbb{P}^2$ to X at $(t, [\lambda, \mu])$ is spanned by $(1, 2t, 2t^2, 0, 0, 0)$, $(0, 0, 0, 1, 2t, 2t^2)$ and $(0, \lambda, 2t\lambda, 0, \mu, 2t\mu)$.

Consider the line ℓ corresponding to some fixed t. On the patch $\lambda \neq 0$, let $\sigma = (2t^2, -2t, 1, 0, 0, 0)$. Then, $\sigma \in N^*_{(t,[\lambda,\mu])}(\mathbb{P}^5/X)$ since σ vanishes on the tangent $\mathbb{P}^2$ to X. On $\mu \neq 0$, by (2.2), $\sigma = \mu\lambda^{-1}(2t^2, -2t, 1, 0, 0, 0)$. But, $N^*(\mathbb{P}^5/Z)$ is spanned by $(z_5, -z_4, z_3, z_2, -z_1, z_0) = (2t^2\mu, -2t\mu, \mu, 2t^2\lambda, -2t\lambda, \lambda)$. Thus $\sigma = -(0, 0, 0, 2t^2, -2t, 1)$ modulo $N^*(\mathbb{P}^5/Z)$ and is, therefore, a global nowhere zero section of $N^*(Z/X)$ over ℓ. Q.E.D.

Except for linear $\mathbb{P}^2$'s and up to an automorphism of Z, X_4 is the unique surface in Z with a non-ample normal bundle, and which contains a positive dimensional family of curves, along each of which $N(Z/X)$ is not ample. The proof of this will appear elsewhere.

§4. The surface X_3

Let δ_1, δ_2: $\mathbb{P}^1 \to \mathbb{P}^5$,

$$\delta_1(s, t) = (s, t, 0, 0, 0, 0)$$

$$\delta_2(s, t) = (0, 0, 0, s^2, st, t^2)$$

and $X = X_3 \subset Z$ the scroll surface determined by δ_1 and δ_2. The line $t = 0$ is the only curve $l \subset X$ for which there is a trivial bundle $\mathcal{O}_\ell \subset N^*(Z/X)|_\ell$.

I defer the proof of this to a future paper. The construction of σ is similar to that in §3 for X_4:

$$\text{on} \quad \lambda \neq 0, \quad \sigma = (0, 0, 0, 0, 0, 1), \quad \text{and}$$

$$\text{on} \quad \mu \neq 0, \quad \sigma = -(0, 0, 1, 0, 0, 0).$$

References

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