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## ON THE FIXED PART OF CERTAIN LINEAR SYSTEMS ON SURFACES

Xavier Benveniste

First we recall what is a numerically positive divisor:

**DEFINITION:** Let  $V$  be a smooth projective variety and  $D$  be a divisor on  $V$ . We say that  $D$  is numerically positive if for any curve  $C$  on  $V$  we have  $D \cdot C \geq 0$ .

Let  $R$  be a numerically positive divisor on a smooth projective surface over an algebraically closed field of any characteristic such that  $R^2 > 0$ ; let  $\mathcal{A}$  be the set of irreducible curves  $C$  such that  $R \cdot C = 0$  (we will show that  $\mathcal{A}$  is finite). The aim of this paper is to prove the following result:

**PROPOSITION:** *Let  $\xi$  be a connected component of  $\bigcup_{C \in \mathcal{A}} C$ ; let  $Z$  be the fundamental cycle associated to  $\xi$ . If  $H^1(Z, \mathcal{O}_Z) = 0$ , then  $\xi$  is not a fixed component of  $|nR|$  for sufficiently large  $n$ .*

Let us make a few comments. The first one is that the condition  $H^1(Z, \mathcal{O}_Z) = 0$  characterizes rational singularities. The second is that this proposition generalizes slightly a well known theorem of Zariski ([1] theorems 6-1, 6-2).

**PROOF:** Let  $\xi$  be as in the proposition and let  $(C_i)_{i \in \{1, \dots, m\}}$  the set of irreducible components of  $\xi$ . Then:

**LEMMA 1:** *The set  $\mathcal{A}$  is finite and the bilinear symmetric form defined by the matrix  $(C_i \cdot C_j)_{i, j \in \{1, \dots, m\}}$  is negative definite.*

**PROOF:** We shall show that if  $(E_i)_{i \in \{1, \dots, n\}}$  is a finite family of elements of  $\mathcal{A}$ , then the classes  $[E_i]$  of the  $E_i$  are  $\mathbb{Q}$ -linearly independent in  $NS(S) \otimes_{\mathbb{Z}} \mathbb{Q}$ . Assume the contrary. Any relation of dependence between the  $[E_i]$  in  $NS(S) \otimes_{\mathbb{Z}} \mathbb{Q}$  can be written:

$$\sum_{i \in I} a_i [E_i] \sim \sum_{i \in I'} a_i [E_i],$$

where  $I$  and  $I'$  are two non-void disjoint subsets of  $\{1, \dots, n\}$  and, for

any  $i \in I \cup I'$ ,  $a_i \in \mathbb{N} - \{0\}$ . Because  $R \cdot \sum_{i \in I} a_i E_i = 0$ , the index theorem on  $S$  implies:

$$\left( \sum_{i \in I} a_i E_i \right)^2 \leq 0.$$

But

$$\left( \sum_{i \in I} a_i E_i \right)^2 = \left( \sum_{i \in I} a_i E_i \right) \cdot \left( \sum_{i \in I'} a_i E_i \right) \geq 0.$$

Again by the index theorem on  $S$  we have:

$$\sum_{i \in I} a_i [E_i] \sim 0.$$

So

$$\forall i \in I \cup I', \quad a_i = 0.$$

If we observe that  $rk_Z(NS(S))$  is finite we have the result. The fact that the bilinear form defined by the matrix  $(C_i \cdot C_j)_{i,j \in \{1, \dots, m\}}$  is definite negative, is because the intersection form on  $NS(S)$  is negative definite on the orthogonal of  $R$ .  $\square$

We recall now that the fundamental cycle  $Z$  associated to  $\xi$  is defined by the following condition:

It is the “smallest” effective divisor with support in  $\xi$  such that:

$$\forall i \in \{1, \dots, m\}, \quad Z \cdot C_i \leq 0.$$

This is the definition of Artin in [2].

**LEMMA 2:** Let  $D = \sum_{i=1}^n a_i C_i$  and  $\mathcal{L}$  an invertible sheaf on  $S$  such that for any  $i \in \{1, \dots, m\}$ ,  $\deg_{C_i}(\mathcal{L}) \geq 0$ . Then

$$H^1(D, \mathcal{O}_D \otimes \mathcal{L}) = 0.$$

**PROOF:** The proof can be found in [2] Lemma 5, but here we give an elementary proof. First of all we observe that all  $C_i$  are rational smooth curves because we have a surjective map:

$$\mathcal{O}_Z \rightarrow \mathcal{O}_{C_i} \rightarrow 0$$

for each  $i \in \{1, \dots, m\}$ , and  $H^1$  is a right exact functor in this case. Now we distinguish two cases.

*1st case:* Assume that we have proved the result for all divisors of the form  $nZ$  with  $n \in \mathbb{N} - \{0\}$ ; because there exists an integer  $n$  such that

$$nZ \geq D,$$

we have a surjective map  $\mathcal{O}_{nZ} \rightarrow \mathcal{O}_D \rightarrow 0$ , we get a surjective map

$$\mathcal{O}_{nZ} \otimes \mathcal{L} \rightarrow \mathcal{O}_D \otimes \mathcal{L} \rightarrow 0.$$

This gives the result because  $H^1$  is right exact.

*2nd case:* We can assume  $D = nZ$  and shall prove the result by induction on  $n$ . If  $n = 1$  it is very easy to see that there exists a sheaf of finite length  $\mathcal{F}$  and an exact sequence

$$0 \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_Z \otimes \mathcal{L} \rightarrow \mathcal{F} \rightarrow 0.$$

Because  $H^1(Z, \mathcal{O}_Z) = H^1(Z, \mathcal{F}) = 0$  we have the result. Assume we have the result for  $n$ ; we shall prove it for  $n + 1$ . We have the exact sequence:

$$0 \rightarrow \mathcal{O}_Z \otimes \mathcal{L}(-nZ) \rightarrow \mathcal{O}_{(n+1)Z} \otimes \mathcal{L} \rightarrow \mathcal{O}_{nZ} \otimes \mathcal{L} \rightarrow 0.$$

We observe that:

$$\forall i \in \{1, \dots, m\}, \deg_{C_i}(\mathcal{L}(-nZ)) = \deg_{C_i}(\mathcal{L}) - nZ \cdot C_i \geq 0.$$

This implies that:

$$H^1(\mathcal{O}_Z \otimes \mathcal{L}(-nZ)) = H^1(\mathcal{O}_{nZ} \otimes \mathcal{L}) = 0. \quad \square$$

**LEMMA 3:** *There exists an effective divisor  $L$  such that the rational map  $\varphi_L$  defined by  $|L|$  is a birational morphism on its image from  $S$  to a surface  $Y$  such that  $\varphi_L: S - \xi \xrightarrow{\sim} Y - \{P\}$ , where  $P$  is a closed point of  $Y$  and:*

$$\forall n \in \mathbb{N} - \{0\}, \quad H^1(S, \mathcal{O}_S(nL)) = 0.$$

**PROOF:** This lemma can also be found in [2] minus the last global condition which could be easily obtained. Here we prefer to give another proof. Let  $H$  be a very ample divisor on  $S$  such that:

$$\begin{aligned} \forall n \in \mathbb{N} - \{0\}, \quad & H^1(S, \mathcal{O}_S(nH)) = 0, \\ \forall i \in \{1, \dots, m\}, \quad & H \cdot C_i = db_i, \end{aligned}$$

where  $b_i \in \mathbb{N}$  and  $d = |\det(C_i \cdot C_j)|$ . Because the bilinear form defined by the matrix  $(C_i \cdot C_j)_{i,j \in \{1, \dots, m\}}$  is definite negative by Lemma 1, there

exists an effective divisor  $D = \sum_{i=1}^m a_i C_i$  with  $a_i \in \mathbb{N}$  such that:

$$\forall i \in \{1, \dots, m\}, \quad H \cdot C_i = -D \cdot C_i.$$

We consider the exact sequence

$$0 \rightarrow \mathcal{O}_S(H) \rightarrow \mathcal{O}_S(H + D) \rightarrow \mathcal{O}_D(H + D) \rightarrow 0.$$

Observe that  $\mathcal{O}_D(H + D) = \mathcal{O}_D$ . Hence we have the exact sequences:

$$H^0(\mathcal{O}_S(H + D)) \rightarrow H^0(\mathcal{O}_D) \rightarrow 0,$$

$$0 \rightarrow H^1(\mathcal{O}_S(H + D)) \rightarrow H^1(\mathcal{O}_D).$$

This implies by Lemma 2 that

$$H^1(\mathcal{O}_S(H + D)) = 0.$$

On the other hand the constant function equal to 1 belongs to  $H^0(\mathcal{O}_D)$  so the linear system  $|H + D|$  does not have any fixed component in  $\xi$ . Because  $H$  is very ample it is clear that  $L = H + D$  satisfies the conditions of Lemma 3.  $\square$

**LEMMA 4:** *There exists  $n_0 \in \mathbb{N}$  such that for  $n \geq n_0$  there exists  $r(n) \in \mathbb{N}$  such that for  $r \geq r(n)$ ,  $|rL + nR|$  doesn't have  $\xi$  as fixed component.*

**PROOF:** Because we have  $R^2 > 0$ , if we replace  $R$  by a suitable multiple, we can assume  $R$  to be an effective divisor. So we can write  $R = F + G$  with  $F$  and  $G$  effective divisors such that all the irreducible components of  $F$  are contained in  $\xi$  and none of the irreducible components of  $G$  are in  $\xi$ . But we have the exact sequences for  $r \in \mathbb{N} - \{0\}$ ,

$$0 \rightarrow \mathcal{O}_S(rL) \rightarrow \mathcal{O}_S(rL + R) \rightarrow \mathcal{O}_R(rL + R) \rightarrow 0,$$

$$0 \rightarrow \mathcal{O}_G(rL + G) \rightarrow \mathcal{O}_R(rL + R) \rightarrow \mathcal{O}_F \rightarrow 0.$$

The exact sequence of cohomology gives:

$$H^0(\mathcal{O}_S(rL + R)) \rightarrow H^0(\mathcal{O}_R(rL + R)) \rightarrow 0.$$

If  $r$  is big enough, because of the vanishing theorem of Serre, we have:

$$H^1(\mathcal{O}_G(rL + G)) = 0,$$

so we have a surjective map:

$$H^0(\mathcal{O}_R(rL + R)) \rightarrow H^0(\mathcal{O}_F) \rightarrow 0.$$

But the constant function equal to 1 belongs to  $H^0(\mathcal{O}_F)$ , so we get the result.  $\square$

Now we show the proposition; we assume that there exist infinitely many  $n$  such that  $|nR|$  has  $\xi$  as a fixed component. For any  $n \in \mathbb{N} - \{0\}$  we write

$$nR \equiv M_n + F_n,$$

where  $M_n$  is the moving part of  $|nR|$ ,  $F_n$  its fixed part.

By Theorem 9-1 of [1] we have that for sufficiently large  $n$ ,  $R \cdot C = 0$  for any irreducible component  $C$  of  $F_n$ , because  $R$  is numerically positive. Replacing  $R$  by a suitable multiple we can assume that

$$R \equiv \Delta + F + G,$$

where  $\Delta$  is the moving part of  $|R|$ ,  $F + G$  its fixed part such that  $R \cdot F = R \cdot G = 0$ , and all the irreducible components of  $F$  are contained in  $\xi$ , while none of the irreducible components of  $G$  are contained in  $\xi$ . By the hypothesis for any  $i \in \{1, \dots, m\}$  we have  $C_i \cdot G = 0$ .

Because  $nR \equiv n\Delta + nF + nG$ , we can assume that  $\xi$  is a fixed component of  $|n(\Delta + F)|$  for infinitely many  $n$ . But if  $R' = \Delta + F$  we have

$$\forall i \in \{1, \dots, m\}, \quad R' \cdot C_i = (R - G) \cdot C_i = 0.$$

By the definition of  $\xi$  and applying the proof of Lemma 4, there exists  $r_0$  such that for  $r \geq r_0$ ,  $|rL + R'|$  doesn't have  $\xi$  as a fixed component.

Let  $\{P_1, \dots, P_s\}$  be the base points of  $|\Delta|$ . Then there exists a smooth irreducible curve  $E \in |L|$  such that:

$$\forall i \in \{1, \dots, s\}, \quad P_i \notin E,$$

$$E \cap \xi = \emptyset,$$

For any  $i, j \in \mathbb{N}$  we denote by  $H_{ij}$  the trace on  $E$  of the linear systems  $|iL + jR'|$  on  $S$ . Then it follows from the choice of  $E$  (recall that  $|R'|$  has only fixed components in  $\xi$ ) that:

$$(A) \quad H_{i_1 j_1} + H_{i_2 j_2} \subset H_{i_1 + i_2, j_1 + j_2} \quad \text{for any } i_1, i_2, j_1, j_2 \in \mathbb{N}$$

$$(B) \quad H_{10} \text{ and } H_{01} \text{ are free from base points.}$$

By the Theorem 4-2 (Relation 24) of [1] we have for a suitable integer  $N$ ,

$$\forall i, j \in \mathbb{N}, \quad j \geq N, \quad H_{ij} = H_{i, j-1} + H_{01}$$

It follows that  $|iL + jR'|$  is spanned by the following two subsystems

$$(1) \quad |iL + (j-1)R'| + |R'|, \quad |(i-1)L + jR'| + E,$$

where the second system has  $E$  as a fixed component. For fixed  $j$  the linear system  $|iL + jR'|$  has no base point if  $i$  is sufficiently large by the proof of Lemma 4. Let then  $j$  a fixed integer  $\geq N$  and let  $i$  be such that  $|iL + jR'|$  has no base points. If  $P$  is any base point of  $|R'|$  then  $P$  is also a base point of the first of the two linear systems (1).

Therefore  $P$  is not a base point of  $|(i-1)L + jR'|$ . Applying the same argument to this last system (if  $i-1 > 0$ ), we find that  $P$  is not a base point of  $|(i-2)L + jR'|$ . Ultimately we reach the conclusion that  $P$  is not a base point of  $|jR'|$  (if  $j \geq N$ ).  $\square$

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