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SELF-DUALITY OF KÄHLER SURFACES

Mitsuhiro Itoh

1. Introduction and main theorems

On an oriented Riemannian 4-manifold (M, g) the star operator $*$ is defined on the space of 2-forms Λ^2 by

$$\alpha \wedge * \beta = (\alpha, \beta)_g \, dv_g, \quad \alpha, \beta \in \Lambda^2. \quad (1.1)$$

The operator $*$ depends essentially only on the conformal structure given by the metric g . Since $* \circ * = id$ on Λ^2 , Λ^2 splits into eigenspaces as $\Lambda^2 = \Lambda^2_+ + \Lambda^2_-$, where Λ^2_+ and Λ^2_- are the eigenspaces corresponding to eigenvalues $+1$ and -1 , respectively. A 2-form which belongs to Λ^2_+ (respectively, to Λ^2_-) is called self-dual (respectively, anti-self-dual).

Let W be Weyl's conformal curvature tensor of g . Then W is regarded as an End(TM) -valued 2-form.

DEFINITION [1]: An oriented Riemannian 4-manifold is called self-dual (respectively, anti-self-dual) if W is self-dual (respectively, anti-self-dual) as a 2-form.

The first Pontrjagin number $p_1(M)$ is written by

$$p_1(M) = 1/4 \pi^2 \int_M (|W_+|^2 - |W_-|^2) \, dv_g \quad (1.2)$$

with respect to the self-dual part W_+ and the anti-self-dual part W_- of the Weyl's tensor W . Since the signature $\tau(M)$ is given by $\tau(M) = 1/3 p_1(M)$, a compact, self-dual (respectively, anti-self-dual) 4-manifold (M, g) has nonnegative (respectively, nonpositive) signature and $\tau(M) = 0$ if and only if (M, g) is conformally flat [1].

It is known [1] that a self-dual Riemannian 4-manifold (M, g) admits a holomorphic Penrose fibering, that is, there exists a $P_1(\mathbb{C})$ -bundle over M whose canonical almost complex structure is integrable. Further its Penrose fibering is a Kähler manifold if and only if (M, g) is conformally equivalent to the 4-sphere S^4 or the complex projective plane $P_2(\mathbb{C})$ with standard metrics [8].

Note that a self-dual 4-manifold is anti-self-dual if the orientation is reversed.

On a compact, self-dual Riemannian 4-manifold of positive scalar curvature, moduli space of irreducible self-dual Yang-Mills connections admits a structure of manifold of certain dimension [1]. A similar statement is also obtained in [9] with respect to moduli space of irreducible anti-self-dual Yang-Mills connections on a compact Kähler surface (M, g) of positive scalar curvature, whether (M, g) is self-dual or not. Therefore it seems to be an interesting problem to characterize geometrically self-dual 4-manifolds and anti-self-dual 4-manifolds. In this article we discuss geometrical descriptions of such 4-manifolds whose metrics are Kähler.

Let (M, g) be a Kähler surface. The complex structure J induces canonically an orientation $\{x^1, x^2, x^3, x^4\}$, where $z^1 = x^1 + \sqrt{-1}x^2$, $z^2 = x^3 + \sqrt{-1}x^4$ are local holomorphic coordinates of M .

It is shown by a straight computation that a Kähler surface of constant holomorphic curvature is self-dual with respect to the canonical orientation. The following shows that the converse is also true if we restrict a Kähler surface to be Einstein.

THEOREM 1: *Let (M, g) be a Kähler surface. If it is self-dual with respect to the canonical orientation and it is Einstein, then it is of constant holomorphic curvature.*

This is shown in §2 by the aid of curvature conditions in Lemma 2.2. The following theorem is known further with respect to self-dual Kähler surfaces.

THEOREM A [5]: *Every self-dual Kähler surface with constant scalar curvature is locally symmetric.*

Moreover, each self-dual metric is characterized in terms of Bochner curvature tensor, introduced by Bochner ([14]).

THEOREM B [13]: *Let (M, g) be a Hermitian manifold of complex dimension two. Then the Bochner curvature tensor of g is the anti-self-dual part of the Weyl's conformal curvature tensor W of g .*

The complete characterization of compact self-dual Kähler surfaces is obtained in the following

THEOREM C [2,3,5]: *Let (M, g) be a compact, self-dual Kähler surface. Then (M, g) is either a space of constant holomorphic curvature ($P_2(\mathbb{C})$, a compact quotient of unit disk D^2 or a Kählerian flat torus T^2) or a compact quotient of a product space of $P_1(\mathbb{C})$ and the Poincaré disk D^1 with metrics of opposite curvature.*

REMARKS: (i) Theorem C was obtained in [3] by B.Y. Chen in terms of Bochner-Kähler metrics (i.e., Kähler metrics whose Bochner curvature

tensor vanishes) and independently by Derdzinski in [5] using Theorem A. Also Bourguignon verified this theorem by the aid of theorems with respect to harmonic curvature tensor [2].

(ii) Derdzinski obtained an example of non-compact, self-dual Kähler surface which is not locally symmetric [4].

Now we consider anti-self-dual Kähler surfaces. The following theorem characterizes these surfaces in terms of scalar curvature (refer to Problem 41, Problem section in [16]).

THEOREM 2: *Let (M, g) be a Kähler surface. Then it is anti-self-dual if and only if its scalar curvature vanishes everywhere.*

REMARK: From this theorem we claim that the total scalar curvature of a compact, anti-self-dual Kähler surface is necessarily zero and its Ricci form is anti-self-dual and is harmonic as a 2-form. Further we have another topological restriction $c_1(M)^2[M] \leq 0$.

The following gives a complete classification of compact, conformally flat Kähler surfaces, that is, compact, anti-self-dual Kähler surfaces whose signature is zero.

THEOREM 3: *Let (M, g) be a compact, conformally flat Kähler surface. Then (M, g) is either a Kählerian flat torus or a Kählerian ruled surface of genus $k (\geq 2)$.*

Since each compact, anti-self-dual Kähler surface (M, g) satisfies that $\tau(M) \leq 0$ and $c_1(M)^2[M] \leq 0$, we have the following by the aid of Theorem 2 together with the classification of complex surfaces.

THEOREM 4: *Let (M, g) be a compact, anti-self-dual Kähler surface. Then (M, g) is necessarily one of the following*

- (i) a Kählerian flat torus,
- (ii) a Kähler surface covered by a K 3 surface with a Ricci flat metric,
- (iii) a Kählerian ruled surface of genus $k (\geq 2)$ and
- (iv) a Kähler surface which is obtained by blowing up either $P_2(\mathbb{C})$ at least 10 times, a ruled surface of genus 0 at least 9 times or a ruled surface of genus $k (\geq 1)$ at least once.

In §2 we state local properties of self-dual Kähler surfaces and of anti-self-dual Kähler surfaces. We discuss in §3 global aspects of the anti-self-duality, from which Theorem 4 is deduced.

2. Local properties of (anti-)self-dual Kähler surfaces

We recall at first the definition of Weyl's conformal curvature tensor.

The Weyl's conformal curvature tensor W of a Riemannian 4-manifold (M, g) is written as [6]

$$\begin{aligned} & g(W(X_1, X_2)X_3, X_4) \\ &= g(R(X_1, X_2)X_3, X_4) \\ &\quad - \frac{1}{2}\{g(X_1, X_4)R^1(X_2, X_3) - g(X_2, X_4)R^1(X_1, X_3) \\ &\quad + R^1(X_1, X_4)g(X_2, X_3) - R^1(X_2, X_4)g(X_1, X_3)\} \\ &\quad + \frac{1}{6}\rho\{g(X_1, X_4)g(X_2, X_3) - g(X_2, X_4)g(X_1, X_3)\} \end{aligned} \quad (2.1)$$

where R , R^1 and ρ are the Riemannian curvature tensor, the Ricci tensor and the scalar curvature of g respectively, that is, R is defined by $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z$ with respect to the Levi-Civita connection ∇ , and R^1 and ρ are defined by $R^1(X, Y) = \sum_{i=1}^4 g(R(e_i, X)Y, e_i)$ and $\rho = \sum_{i=1}^4 R^1(e_i, e_i)$ respectively, where $\{e_1, e_2, e_3, e_4\}$ is an orthonormal basis.

Suppose that (M, g) is a Kähler surface. We shall begin with (anti-)self-duality condition of W of the Kähler metric g in terms of complex 2-forms. Before stating the condition we give a characterization of (anti-)self-dual 2-forms by the aid of complex 2-forms.

PROPOSITION 2.1 [9]: *A 2-form α is self-dual if and only if (1,1)-part of α is proportional to the Kähler form Ω , and a 2-form β is anti-self-dual if and only if β is of type (1,1) which satisfies that $(\beta, \Omega)_g = 0$.*

REMARK: The Kähler form Ω is a self-dual form and if a real form of type (1,1) $\sigma = \sqrt{-1} \sum \sigma_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$ satisfies $\sum g^{\alpha\bar{\beta}} \sigma_{\alpha\bar{\beta}} = 0$, then σ is anti-self-dual, where $(g^{\alpha\bar{\beta}})$ is the inverse of the component matrix of g .

Let $\{E_1, E_2\}$ be a local unitary basis. Unless otherwise stated, Greek indices $\alpha, \beta, \gamma, \dots$ run from 1 to 2, while Latin capitals A, B, C, \dots run over 1, 2, $\bar{1}$ and $\bar{2}$. We set $g_{AB} = g(E_A, E_B)$, $R_{ABCD} = g(R(E_A, E_B)E_C, E_D)$, $R^1_{AB} = R^1(E_A, E_B)$ and $W_{ABCD} = g(W(E_A, E_B)E_C, E_D)$.

Since $\Omega(E_1, E_{\bar{1}}) = \Omega(E_2, E_{\bar{2}})$ and $\Omega(E_1, E_{\bar{2}}) = 0$, we have from Proposition 2.1 that W is self-dual if and only if

$$W_{1\bar{1}AB} = W_{2\bar{2}AB} \quad \text{and} \quad W_{1\bar{2}AB} = 0 \quad (2.2)$$

and W is anti-self-dual if and only if

$$W_{12AB} = 0 \quad \text{and} \quad W_{1\bar{1}AB} + W_{2\bar{2}AB} = 0, \quad (2.3)$$

for any A and B .

Since g is a Kähler metric, the components of R satisfy that $R_{\alpha\beta CD} = R_{CD\alpha\beta} = 0$, $R_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\gamma\bar{\beta}\alpha\bar{\delta}} = R_{\gamma\bar{\delta}\alpha\bar{\beta}}$ and $\overline{R_{\alpha\bar{\beta}\gamma\bar{\delta}}} = R_{\beta\bar{\alpha}\delta\bar{\gamma}}$, and the components of R^1 and ρ are given by $R^1_{\alpha\bar{\beta}} = \sum_{\gamma} R_{\alpha\bar{\beta}\gamma\bar{\gamma}}$ and $\rho = 2\sum_{\alpha} R^1_{\alpha\bar{\alpha}} = 2\sum_{\alpha,\beta} R_{\alpha\bar{\alpha}\beta\bar{\beta}}$.

With respect to a self-dual Kähler surface, we obtain the following lemma.

LEMMA 2.2: *Let (M, g) be a Kähler surface. Then, it is self-dual if and only if the components of the Riemannian curvature tensor R satisfy that*

$$\begin{aligned} R_{1\bar{1}1\bar{1}} + R_{2\bar{2}2\bar{2}} - 4R_{1\bar{1}2\bar{2}} &= 0 \text{ (that is, } \rho = 12R_{1\bar{1}2\bar{2}}) \\ R_{1\bar{1}1\bar{2}} = R_{2\bar{2}1\bar{2}} \text{ and } R_{1\bar{2}1\bar{2}} = R_{2\bar{2}1\bar{1}} &= 0 \end{aligned} \tag{2.4}$$

for any unitary basis $\{E_1, E_2\}$.

PROOF OF LEMMA 2.2: We have from formula (2.1) that

$$W_{1\bar{1}1\bar{1}} = R_{1\bar{1}1\bar{1}} - R^1_{1\bar{1}} + \frac{1}{6}\rho \text{ and } W_{2\bar{2}2\bar{2}} = R_{2\bar{2}2\bar{2}}.$$

Suppose that (M, g) is self-dual. Then from (2.2) we have $\frac{1}{6}\rho = 2R_{2\bar{2}1\bar{1}}$, which is the first formula. The second of (2.4) is obtained from $W_{1\bar{1}1\bar{2}} = W_{2\bar{2}1\bar{2}}$. To show the last we define a new unitary basis $\{E_1^\theta, E_2^\theta\}$ with real parameter θ , by $E_1^\theta = \cos \theta E_1 + \sin \theta E_2$ and $E_2^\theta = -\sin \theta E_1 + \cos \theta E_2$. Since $\rho = 12R_{1\bar{1}2\bar{2}}$ holds also for this basis, by differentiating this with respect to θ twice and setting $\theta = 0$ we have that $2(R_{1\bar{1}1\bar{1}} + R_{2\bar{2}2\bar{2}} - 4R_{1\bar{1}2\bar{2}} - R_{1\bar{2}1\bar{2}} - R_{2\bar{2}1\bar{1}}) = 0$. Then we have that $R_{1\bar{2}1\bar{2}} + R_{2\bar{2}1\bar{1}} = 0$, that is, $R_{1\bar{2}1\bar{2}}$ is pure imaginary for $\{E_1, E_2\}$. For a new unitary basis $\{e^{\sqrt{-1}\pi/4}E_1, E_2\}$ $R_{1\bar{2}1\bar{2}}$, which is also pure imaginary, is reduced to a real number. Hence we have that $R_{1\bar{2}1\bar{2}} = 0$ for $\{E_1, E_2\}$.

Conversely suppose that (M, g) satisfies (2.4). Then $W_{1\bar{1}1\bar{1}} = W_{2\bar{2}2\bar{2}}$ and $W_{1\bar{1}1\bar{2}} = W_{2\bar{2}1\bar{2}}$ hold from the first equalities of (2.4). Since other equalities $W_{1\bar{1}AB} = W_{2\bar{2}AB}$ and $W_{1\bar{2}AB} = 0$ are easily obtained, W is self-dual from (2.2).

PROOF OF THEOREM 1: Since g is Einstein, $R^1_{\alpha\bar{\beta}} = R_{1\bar{1}\alpha\beta} + R_{2\bar{2}\alpha\beta} = \rho/4\delta_{\alpha\beta}$ for any unitary basis $\{E_1, E_2\}$. Then we have from the above lemma that

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = 0 \text{ except for } R_{1\bar{1}1\bar{1}} = R_{2\bar{2}2\bar{2}} = 2R_{1\bar{1}2\bar{2}}.$$

Therefore R has the form $R_{\alpha\bar{\beta}\gamma\bar{\delta}} = c(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\gamma\beta})$ where $c = g(R(E_1, \overline{E_1})E_1, \overline{E_1})$, which may be a local function. With respect to local holomorphic coordinates z^1 and z^2 the components of R can be written as

$$g(R(\partial/\partial z\alpha, \partial/\partial \bar{z}\beta, \partial/\partial z\gamma, \partial/\partial \bar{z}\delta)) = c(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}})$$

where $g_{\alpha\bar{\beta}} = g(\partial/\partial z\alpha, \partial/\partial \bar{z}\beta)$. Covariant-differentiating this with respect

to $\partial/\partial z^\varepsilon$ and applying the second Bianchi's identity $\nabla_\varepsilon R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \nabla_\alpha R_{\varepsilon\bar{\beta}\gamma\bar{\delta}}$ we conclude that c must be constant. Hence from Proposition 7.6 in [10] (M, g) is a Kähler surface of constant holomorphic curvature.

In the last half part of this section we shall show Theorem 2

PROOF OF THEOREM 2: Let (M, g) be an anti-self-dual Kähler surface. Then from (2.3) $W_{12AB} = 0$ for any A and B . If we set $A = \bar{1}$ and $B = \bar{2}$ in W_{12AB} , then $W_{12\bar{1}\bar{2}} = \frac{1}{2}(R_{1\bar{1}}^1 + R_{2\bar{2}}^1) - \frac{1}{6}\rho = \frac{1}{6}\rho$, hence we have that $\rho = 0$. Suppose conversely that the scalar curvature ρ vanishes identically. Since $R_{12CD} = 0$, $W_{12CD} = -\frac{1}{2}(g_{1D}R_{C2}^1 - g_{2D}R_{C1}^1 + R_{1D}^1g_{C2} - R_{2D}^1g_{C1})$. If we set $C = \bar{1}$ and $D = \bar{2}$ in this representation, then $W_{12\bar{1}\bar{2}} = \frac{1}{2}(R_{1\bar{1}}^1 + R_{2\bar{2}}^1) = \frac{1}{4}\rho = 0$. We have easily that $W_{12CC} = 0$ and $W_{1212} = 0$. That $W_{1\bar{1}CD} + W_{2\bar{2}CD} = 0$ for any C and D is shown as follows. For $C = 1$ and $D = 2$ we have that $W_{1\bar{1}12} + W_{2\bar{2}12} = -\frac{1}{2}(R_{21}^1 - R_{12}^1) = 0$. We also obtain that $W_{1\bar{1}1\bar{1}} + W_{2\bar{2}1\bar{1}} = R_{1\bar{1}1\bar{1}} + R_{2\bar{2}1\bar{1}} - \frac{1}{2}(R_{1\bar{1}}^1 + R_{1\bar{1}}^1) = 0$ and $W_{1\bar{1}1\bar{2}} + W_{2\bar{2}1\bar{2}} = R_{1\bar{1}1\bar{2}} + R_{2\bar{2}1\bar{2}} - \frac{1}{2}(R_{21}^1 + R_{1\bar{2}}^1) = 0$. Similarly we have that $W_{1\bar{1}CD} + W_{2\bar{2}CD} = 0$ for $(C, D) = (\bar{1}, \bar{2}), (\bar{1}, 1)$ and $(\bar{1}, 2)$.

Hence W is anti-self-dual.

3. Global aspects of anti-self-dual Kähler surfaces

Let (M, g) be a compact oriented Riemannian 4-manifold. Then its signature $\tau(M)$ is written as $\tau(M) = b^+ - b^-$ where b^+ and b^- are given by the dimension of the space of real, self-dual harmonic 2-forms and the dimension of the space of real, anti-self-dual harmonic 2-forms, respectively.

Now let (M, g) be a compact Kähler surface. Let Δ and \square be the real Laplace-Beltrami operator and the complex Laplace-Beltrami operator defined on the space $\Gamma(\Lambda^k)$ of smooth k -forms, respectively. We notice that $\Delta = 2\square$ and \square preserves type of k -forms.

LEMMA 3.1: *Let (M, g) be a compact Kähler surface of nonnegative scalar curvature. Then each holomorphic form of type $(2,0)$ is parallel. Moreover if the scalar curvature is positive at some point, then the geometric genus is equal to zero and $b^+ = 1$.*

PROOF: Let σ be a holomorphic form of type $(2,0)$. Note that σ is a global section of the canonical line bundle K . From Proposition 5.5 in [12] and $\square = \bar{\square}$, $\sigma = 1/2\sum\sigma_{\mu\nu}dz^\mu\wedge dz^\nu$ is $\bar{\square}$ -harmonic. From the following formula, similar to Theorem 6.1 in [12]

$$(\bar{\square}\sigma)_{\mu\nu} = -\sum g^{\alpha\bar{\beta}}\nabla_{\bar{\beta}}\nabla_{\alpha}\sigma_{\mu\nu} + \rho/2\sigma_{\mu\nu}, \quad (3.1)$$

we have that

$$0 = \int_M \left(\sum g^{\mu\bar{\epsilon}} g^{\nu\bar{\lambda}} g^{\alpha\bar{\beta}} \nabla_{\alpha} \sigma_{\mu\nu} \overline{\nabla_{\beta} \sigma_{\epsilon\lambda}} \right) dv_g + \int_M \frac{\rho}{2} \left(\sum g^{\mu\bar{\epsilon}} g^{\nu\bar{\lambda}} \sigma_{\mu\nu} \overline{\sigma_{\epsilon\lambda}} \right) dv_g. \tag{3.2}$$

Here ρ denotes the scalar curvature of g . Since ρ is nonnegative, $\nabla_{\alpha} \sigma_{\mu\nu} = 0$. Therefore σ is indeed parallel because $\nabla_{\bar{\alpha}} \sigma_{\mu\nu} = 0$. If ρ is positive at some point, then $\sigma = 0$, that is, the geometric genus $p_g = \dim H^0(M, K)$ is zero. To verify $b^+ = 1$ it suffices to show that each real, self-dual harmonic 2-form is proportional to the Kähler form Ω . Let ϕ be a self-dual, real harmonic 2-form. Then from Proposition 2.1 ϕ is written by $\phi = \phi^{2,0} + (\phi^{2,0})^- + a\Omega$ where $\phi^{2,0}$ is a form of type (2,0) and a is a real smooth function. Since $\square\phi = 0$, $\phi^{2,0}$ and $a\Omega$ are \square -harmonic. From the above we have that $\phi^{2,0} = 0$. Since $\square(a\Omega) = (\square a)\Omega$, a must be constant. Thus the lemma is verified.

REMARK: From this lemma, every compact, anti-self-dual Kähler surface has trivial canonical line bundle if $p_g > 0$.

The following is given as a remark in §1. We give a proof here.

LEMMA 3.2: *Every compact, anti-self-dual Kähler surface satisfies that $c_1(M)^2[M] \leq 0$ ($= 0$ if and only if g is Ricci flat) and its Ricci form γ is anti-self-dual and harmonic.*

PROOF: Since the scalar curvature vanishes, γ is anti-self-dual from Remark of Proposition 2.1. Then from (1.1) we have $\gamma \wedge \gamma = -|\gamma|_g^2 dv_g$, hence $c_1(M)^2[M] = -1/(4\pi^2) \int_M |\gamma|_g^2 dv_g$. With respect to the formal adjoint ∂^* of ∂ we have that $(\partial^* \gamma)_{\alpha} = \sum g^{\sigma\bar{\tau}} \nabla_{\sigma} R^1_{\alpha\bar{\tau}} = \nabla_{\alpha}(\rho/2) = 0$. Hence we have the lemma.

REMARK: By a slight consideration we have the following. Let (M, g) be a compact Kähler surface whose total scalar curvature is zero. Then K_M admits a hermitian metric whose Ricci form is anti-self-dual and also $c_1(M)^2[M] \leq 0$.

PROOF OF THEOREM 3: Since a conformally flat, compact Kähler surface (M, g) is also self-dual and has zero signature, Theorem 3 is a corollary of Theorem C. In fact, if (M, g) is not a flat torus, then it is a compact quotient of $P_1(\mathbb{C}) \times D^1$ with the metric. By an easy argument (M, g) is a holomorphic bundle over a complex curve (C_1, g_1) , $C_1 = D^1/\Gamma$ of genus k (≥ 2) with fibre $(P_1(\mathbb{C}), g_2)$ whose projection is a Kählerian submersion, that is, (M, g) is a Kählerian ruled surface of genus k .

REMARKS: (i) From this theorem and Theorem C we obtain that on each conformally flat, compact Kähler surface which is not flat there is one

parameter family $\{g_t\}$ of Kähler metrics of constant scalar curvature c_t , where c_t takes any real value.

(ii) We can exhibit an example of Kählerian ruled surface of genus ≥ 2 which is a nontrivial holomorphic $P_1(\mathbb{C})$ -bundle.

PROOF OF THEOREM 4: As was shown above, a compact, anti-self-dual Kähler surface (M, g) has topological restrictions $3\tau(M) = c_1(M)^2[M] - 2c_2(M)[M] \leq 0$ and $c_1(M)^2[M] \leq 0$. Then compact, anti-self-dual Kähler surfaces are divided into four classes. If $\tau(M) = 0$ and $c_1(M)^2[M] = 0$, then $W = 0$ and $R^1 = 0$, hence g is flat, that is (M, g) is a Kählerian flat torus. In the similar manner we obtain that (M, g) with $c_1(M)^2[M] = 0$ and $\tau(M) < 0$ is covered by a K^3 surface of a Ricci flat metric. If $\tau(M) = 0$ and $c_1(M)^2[M] < 0$, then (M, g) is conformally flat, but not flat. Then from Theorem 3 (M, g) is a Kählerian ruled surface of genus k (≥ 2). Now let (M, g) be a compact, anti-self-dual Kähler surface with $\tau(M) < 0$ and $C_1(M)^2[M] < 0$. If there is a non-trivial holomorphic 2-form on M , then from Lemma 3.1 it is parallel, hence it never vanishes, so the canonical line bundle K is trivial. As a consequence we have $c_1(M) = 0$. But this contradicts to $c_1(M)^2[M] < 0$. Therefore the geometric genus p_g is zero. Let M_0 be a relatively minimal complex surface such that M is obtained by blowing up M_0 . Then p_g of M_0 is also zero. By Kodaira's classification theorem [11] M_0 is either $P_2(\mathbb{C})$ or a ruled surface of genus k (i.e., a holomorphic $P_1(\mathbb{C})$ -bundle over a complex curve of genus k). Since $c_2 = 3$ and $c_1^2 = 9$ for $P_2(\mathbb{C})$, and $c_2 = 4(1 - k)$ and $c_1^2 = 8(1 - k)$ for a ruled surface of genus k , and blowing up one point increases c_2 one and decreases τ and c_1^2 one, M is obtained by blowing up $P_2(\mathbb{C})$ at least 10 times, a ruled surface of genus 0 at least 9 times or a ruled surface of genus k (≥ 1) at least once.

REMARKS: (i) On each ruled surface there exists a Kähler metric of positive scalar curvature [15]. Every compact complex surface, obtained by blowing up a ruled surface several times admits a Hodge metric whose total scalar curvature is positive [15].

(ii) On the other hand, each ruled surface M of genus $k \geq 2$ over a complex curve can be endowed with a Kähler metric of negative scalar curvature under a certain condition. By its definition, M is a holomorphic $P_1(\mathbb{C})$ -bundle over D^1/Γ of genus k , where $\Gamma \subset \text{Hol}(D^1)$. Let $\pi; D^1 \rightarrow D^1/\Gamma$ be the covering map. Since π^*M over D^1 is trivial as a smooth bundle and D^1 is Stein, π^*M is also trivial as a holomorphic bundle. Then there is a homomorphism $\rho; \Gamma \rightarrow \text{Hol}(P_1(\mathbb{C}))$ such that $M = D^1 \times_{\rho} P_1(\mathbb{C})$. Since $\text{Hol}(D^1) \subset \text{Aut}(D^1, \bar{g}_1)$ with respect to the Poincare metric \bar{g}_1 , D^1/Γ admits a metric g_1 , locally isomorphic to \bar{g}_1 . Assume that $\Gamma/\text{Ker } \rho$, isomorphic to $\text{Im } \rho$, is a finite subgroup. Then $g_2 = \sum_{\rho(\alpha) \in \text{Im } \rho} \rho(\alpha)^* \bar{g}_2$ defines a Γ -invariant Kähler metric on $P_1(\mathbb{C})$, where \bar{g}_2 is a standard metric. Then M admits a one parameter family of Kähler

metrics $\{g_t\}$ ($t > 0$), locally isometric to $\bar{g}_1 \oplus t\bar{g}_2$. If t tends to the infinity, then the scalar curvature becomes negative everywhere. Note that for sufficiently large t g_t is considered as a Hodge metric. Therefore by the same argument in the proof of Proposition 3 in [16], on any compact complex surface \tilde{M} , obtained by blowing up M there is a Hodge metric of negative total scalar curvature.

(iii) Let M be a ruled surface of genus $k \geq 2$ with $\# \text{Im } \rho < +\infty$. Then any complex surface \tilde{M} , obtained by blowing up M , admits a Kähler metric whose total scalar curvature is zero by the aid of the above remarks.

(iv) Let M be a complex surface, obtained by blowing up either $P_2(\mathbb{C})$ k times ($k \leq 9$) or $P_1(\mathbb{C}) \times P_1(\mathbb{C})$ j times ($j \leq 8$). Then $c_1(M)^2[M] \geq 0$. Therefore by Remark of Lemma 3.1 M can not admit any Kähler metric whose total scalar curvature is zero.

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