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GLUINGS AND CLOSURES

Eric Degreef and René Fourneau

Abstract

The idea of the gluing of convexity structures is to endow the union of the universal sets with the least closure (with respect to the usual ordering of the transformations of a set) the restriction of which is greater, on each constituent, than the original closure. We describe here a construction of this closure, using transfinite induction, and we give some bounds on the k-Radon, Helly and Caratheodory numbers of the convexity structure obtained that way.

This idea was introduced by Sierksma in [8] for convexity structures $(X_i, \mathscr{C}_i)$, $(i = 1,\ldots,n)$, where $X_i \cap X_j = \emptyset$ for each $i \neq j$, by considering $(\cup_{i=1}^{n} X_i, +_{i=1}^{n}\mathscr{C}_i)$ with $+_{i=1}^{n}\mathscr{C}_i = \{\cup_{i=1}^{n} A_i \colon A_i \in \mathscr{C}_i$ for each $i = 1,\ldots,n\}$.

1. Notation and terminology

A hull on a set X is a transformation $\mathscr{H}$ of the power set $\mathscr{P}(X)$ of X such that:

(1) $A \subset \mathscr{H}(A)$, $\forall A \subset X$;

(2) $A \subset B \subset X \Rightarrow \mathscr{H}(A) \subset \mathscr{H}(B)$.

An idempotent hull on X well be called *a closure on X.*

To every hull $\mathscr{H}$ on X is associated a closure $\mathscr{H}$ which is the least closure greater than $\mathscr{H}$ (see [5]).

A convexity structure is a pair $(X, \mathscr{C})$ where X is a set and $\mathscr{C} \subset \mathscr{P}(X)$ is such that $\emptyset$, $X \in \mathscr{C}$ and $\cap\mathscr{F} \in \mathscr{C}$ for every $\mathscr{F} \subset \mathscr{C}$. If, moreover, $\cup\mathscr{T} \in \mathscr{C}$ for every chain $\mathscr{T} \subset \mathscr{C}$, $(X, \mathscr{C})$ is *an alignment.* The members of $\mathscr{C}$ are called *convex sets* and the closure operator associated to $\mathscr{C}$ is called *the $\mathscr{C}$-hull* and given by

$$\mathscr{C}(S) = \cap\{A \in \mathscr{C} : S \subset A\}, \quad \forall S \subset X.$$

The different numbers attached to a convexity structure are defined as usual.

The Caratheodory number of a convexity structure $(X, \mathscr{C})$ is defined as the least natural number n such that for each subset A of X,

$$\mathscr{C}(A) = \cup\{\mathscr{C}(T) : T \subset A, \#T \leqslant n\}.$$

The Helly number of a convexity structure $(X, \mathscr{C})$ is defined as the least natural number n such that, for each subset A of X with cardinality $(n+1)$, $\cap\{\mathscr{C}(A\setminus a)\colon a\in A\}\neq\emptyset$.

Let $(X, \mathscr{C})$ be a convexity structure. A partition $\{A_1, A_2, \ldots, A_k\}$ of a subset A of X, is called a k-Radon partition of A iff $\mathscr{C}(A_1)\cap\ldots\cap\mathscr{C}(A_k)\neq\emptyset$. For each subset A of X we define $\mathscr{R}_k(A)$ as the set of all k-Radon partitions of A. *The k-Radon number* of a convexity structure $(X, \mathscr{C})$ is defined as the least natural number n such that, for each subset A of X with cardinality $\geqslant n$, $\mathscr{R}_k(A)\neq\emptyset$. Observe that the 2-Radon number of a convexity structure $(X, \mathscr{C})$ is what is classically called the Radon number of $(X, \mathscr{C})$ (see e.g. [7]).

2. Gluings of convexity structures

2.1. PROPOSITION: *Let $(X_i, \mathscr{C}_i)$ be a convexity structure $(i=1,\ldots,n)$. The transformation of $\mathscr{P}(\cup_{i=1}^n X_i)$ defined by*

$$\mathscr{G}(S)=\bigcup_{i=1}^{n}\mathscr{C}_i(S\cap X_i),\quad \forall S\subset\bigcup_{i=1}^{n}X_i,$$

is the least hull on $\cup_{i=1}^n X_i$ the restriction of which is greater than $\mathscr{C}_i(.)$ on every $X_i (i=1,\ldots,n)$.

PROOF: We first prove that $\mathscr{G}$ is a hull on $X=\cup_{i=1}^n X_i$. For every $S\subset X$,

$$S=\bigcup_{i=1}^{n}S\cap X_i\subset\bigcup_{i=1}^{n}\mathscr{C}_i(S\cap X_i)=\mathscr{G}(S)$$

and, if $S\subset T$,

$$\mathscr{C}_i(S\cap X_i)\subset\mathscr{C}_i(T\cap X_i)\quad(i=1,\ldots,n),$$

hence

$$\mathscr{G}(S)=\bigcup_{i=1}^{n}\mathscr{C}_i(S\cap X_i)\subset\bigcup_{i=1}^{n}\mathscr{C}_i(T\cap X_i)=\mathscr{G}(T),$$

and $\mathscr{G}$ is a hull on X. Now, if $S\subset X_j$,

$$\mathscr{G}(S)=\mathscr{C}_j(S)\cup\bigcup_{\substack{i=1\\ i\neq j}}^{n}\mathscr{C}_i(S\cap X_i)\supset\mathscr{C}_j(S).$$

Therefore $\mathscr{G}|_{X_j}$ majorizes $\mathscr{C}_j$. Moreover, il $\mathscr{H}$ is a hull on X such that $\mathscr{H}|_{X_i}$ is greater than $\mathscr{C}_i$, $(i=1,\ldots,n)$,

$$\mathscr{H}(S)\supset\mathscr{H}(S\cap X_i)\supset\mathscr{C}_i(S\cap X_i),$$

hence

$$\mathscr{H}(S) \supset \bigcup_{i=1}^{n} \mathscr{C}_i(S \cap X_i) = \mathscr{G}(S).$$

2.2. Definition: Let $(X_i, \mathscr{C}_i)$ be a convexity structure $(i = 1, \ldots, n)$. On $X = \cup_{i=1}^{n} X_i$,

$$\mathscr{C} = \{ S \subset X : S \cap X_i \in \mathscr{C}_i, i = 1, \ldots, n \}$$

contains the intersection of every subfamily, i.e. is an intersectional family. The structure $(X, \mathscr{C})$ is said to be obtained by *gluing* the structures $(X_i, \mathscr{C}_i)$ ([1], [2], [3], [4], [8] for the case of disjoint constituents; [9]).

2.3. Remarks: The properties of gluings proved in [1] are in fact properties of $\mathscr{G}$. Henceforth, we shall use them freely.

The symbol "$\mathscr{G}$", as defined in 2.1, will be used throughout this paper.

2.4. Proposition: *Let $(X_i, \mathscr{C}_i)$ be a convexity structure $(i = 1, \ldots, n)$. The closure operator of the gluing $(X, \mathscr{C})$ of the X_i's is $\mathscr{G}$.*

Proof: First, $\mathscr{C}|_{X_j}$ is greater than $\mathscr{C}_j$: for any $S \subset X_j$,

$$\mathscr{C}(S) = \cap \{ T \in \mathscr{C} : T \supset S \}$$

and since $T \subset X$ belongs to $\mathscr{C}$ if and only if its trace on X_i belongs to $\mathscr{C}_i (i = 1, \ldots, n)$

$$\mathscr{C}(S) = \cap \{ T \subset X : T \cap X_i \in \mathscr{C}_i ; i = 1, \ldots, n;\ T \supset S \};$$

$$\supset \cap \{ T \subset X : T \cap X_j \in \mathscr{C}_j , T \supset S \}$$

this last set being just $\mathscr{C}_j(S)$ since it can be written as

$$\cap \{ T \subset X_j : T \in \mathscr{C}_j , T \supset S \}.$$

Moreover, if $\mathscr{F}$ is a closure on X, the restriction of which to X_i $(i = 1, \ldots, n)$ is greater than $\mathscr{C}_i(.)$, then for every $\mathscr{F}$-closed set S the following holds:

$$S = \mathscr{F}(S) = \mathscr{F}\left(\bigcup_{i=1}^{n} \mathscr{F}(S \cap X_i) \right) \supset \mathscr{F}\left(\bigcup_{i=1}^{n} \mathscr{C}_i(S \cap X_i) \right)$$

$$\supset \bigcup_{i=1}^{n} \mathscr{C}_i(S \cap X_i).$$

Therefore $S \cap X_i \supset \mathscr{C}_1(S \cap X_i)$ $(i = 1,\ldots,n)$, i.e. $S \cap X_i \in \mathscr{C}_i$. So $S \in \mathscr{C}$. Since the $\mathscr{F}$-closed sets are $\mathscr{C}$-closed, $\mathscr{F}$ is greater than $\mathscr{C}$.

2.5. THEOREM: *Let $(X_i, \mathscr{C}_i)$ be a convexity structure $(i = 1,\ldots,n)$. There is an ordinal μ such that, for every $S \subset \cup_{i=1}^{n} X_i$, $\mathscr{G}^{\mu}(S) = \mathscr{C}(S)$, where the $\mathscr{G}^{\mu}$-hulls are defined by transfinite induction as follows: $\mathscr{G}^1 = \mathscr{G}$ and, if the $\mathscr{G}^{\nu}$-hulls are defined for every ordinal $\nu < \mu$, set*

$$\mathscr{G}^{\mu}(S) = \mathscr{G}\left[\mathscr{G}^{\mu-1}(S)\right] \text{ if } \mu \text{ is not a limit ordinal,}$$

$$\bigcup_{1 \leqslant \nu < \mu} \mathscr{G}^{\nu}(S) \text{ if } \mu \text{ is a limit ordinal.}$$

PROOF: It suffices to use the results of [5], mainly 7.5.

2.6. PROPOSITION: *The gluing of finitely many alignments is again an alignment.*

PROOF: Recall, [6], that $(X, \mathscr{C})$ is an alignment if and only if the convex hull of a set is the union of the hulls of its finite subsets. The conjunction of Theorem 2 in [1], 2.4. and [5; 8.2] insures that this property is shared by $\mathscr{C} = \check{\mathscr{G}}$.

3. Invariants

3.1. THEOREM: *Let the $(X_i, \mathscr{C}_i)$ be convexity structures with k-Radon number $r_i^k (i = 1,\ldots,n)$. Then their gluing has a k-Radon number r_k satisfying*

$$r_k \leqslant \sum_{i=1}^{n} r_i^k - n + 1.$$

PROOF: Theorem 5 of [1] shows that if $S \subset \cup_{i+1}^{n} X_i$ has at least $\sum_{i=1}^{n} r_i^k - n + 1$ points, one can find a partition $\{A_1,\ldots,A_k\}$ of S such that

$$\mathscr{G}(A_1) \cap \ldots \cap \mathscr{G}(A_k) \neq \emptyset.$$

Therefore, since

$$\mathscr{C}(A_1) \cap \ldots \cap \mathscr{C}(A_k) \supset \mathscr{G}(A_1) \cap \ldots \cap \mathscr{G}(A_k),$$

$\{A_1,\ldots,A_k\}$ is a $\mathscr{C}$-k-Radon partition of S, establishing the theorem.

3.2. THEOREM: *Let $(X_i, \mathscr{C}_i)$ be convexity structures with Helly number h_i $(i = 1,\ldots,n)$. Then their gluing has a Helly number h satisfying*

$$h \leqslant \sum_{i=1}^{n} h_i.$$

PROOF: Theorem 6 of [1] shows that, for any $S \subset \cup_{i=1}^{n} X_i$ having at least $\Sigma_{i=1}^{n} h_i + 1$ points,

$$\bigcap_{s \in S} \mathscr{G}(S \setminus \{s\}) \neq \emptyset.$$

It suffices to remark that, for every $s \in S$,

$$\mathscr{C}(S \setminus \{s\}) \supset \mathscr{G}(S \setminus \{s\}).$$

3.3. REMARK: (1) Alas the Caratheodory number does not behave so well. The following example shows that the Caratheodory number may increase in a gluing. Consider in $\mathbb{R}^2$ the sets $X_1 = \{(0, 0),(-1, 0),(-2, 0)\}$ and $X_2 = \{(0, 0),(-1, 1), (0,1), (1,2), (1,-1), (0,-2)\}$ with the convexity structure induced by the usual $\mathbb{R}^2$ convexity structure. The subset $S = \{(-1, 1),(1, 2),(0, -2),(1, -1),(-2, 0)\}$ of $X_1 \cup X_2$ is such that $\mathscr{G}^2(S) = X_1 \cup X_2$, but it can be seen that $(-1, 0)$ does not belong to $\mathscr{G}^2(B)$ whatever be $B \subset S$ with cardinality less than 4.

We shall give a bound on the Caratheodory number of certain gluings in the last part of this paper.

(2) In [8] Sierksma showed the following comparable results on r, h and c: let $X_i \cap X_j = \emptyset$ for each $i, j = 1,\ldots,n$ with $i \neq j$ and let $(X_i, \mathscr{C}_i)$ be a convexity structure with Caratheodory number c_i, Helly number h_i and Radon number r_i; $i = 1, 2,\ldots,n$. Then the respective numbers c, h and r for the convex sum structure $(\cup_{i=1}^{n} X_i, +_{i+1}^{n} \mathscr{C}_i)$, with $+_{i=1}^{n} \mathscr{C}_i = \{\cup_{i=1}^{n} A_i | A_i \in \mathscr{C}_i \text{ for each } i = 1,\ldots,n\}$, satisfy:

$$c = \max_{1 \leqslant i \leqslant n} c_i$$

$$h = \sum_{i=1}^{n} h_i$$

$$r = \sum_{i=1}^{n} r_i - n + 1.$$

4. Order of a gluing of convexity structures

4.1. DEFINITION: If the $(X_i, \mathscr{C}_i)$ are convexity structures, we shall say that their gluing is of *order* α if α is the smallest ordinal such that $\mathscr{C}(S) = \mathscr{G}^{\alpha}(S)$ for each $S \subset \cup_{i+1}^{n} X_i$. Note that such an α exists by Theorem 2.5 above.

4.2. PROPOSITION: *Let $(X_i, \mathscr{C}_i)$ $(i = 1, 2)$ be convexity structures. For any ordinal μ and any subset S of $X_1 \cup X_2$, such that $\mathscr{G}^{\mu}(S) \notin \mathscr{C}$, either $\mathscr{G}^{\mu+1}(S) \in \mathscr{C}$ or $(\mathscr{G}^{\mu+1}(S) \cap X_1 \cap X_2) \setminus (\mathscr{G}^{\mu}(S) \cap X_1 \cap X_2) \neq \emptyset$.*

PROOF: Assume that $[\mathscr{G}^{\mu+1}(S) \cap X_1 \cap X_2] \backslash [\mathscr{G}^{\mu}(S) \cap X_1 \cap X_2] = [\mathscr{G}^{\mu+1}(S) \backslash \mathscr{G}^{\mu}(S)] \cap X_1 \cap X_2 = \emptyset$. We have to show that

$$\mathscr{G}^{\mu+1}(S) \cap X_i = [\mathscr{C}_1(\mathscr{G}^{\mu}(S) \cap X_1) \cup \mathscr{C}_2(\mathscr{G}^{\mu}(S) \cap X_2)] \cap X_i \in \mathscr{C}_i,$$

$(i = 1, 2)$. The proof will be detailed for $i = 1$; the case $i = 2$ is alike. First the very definition of $\mathscr{G}^{\mu+1}$ gives

$$\mathscr{G}^{\mu+1}(S) \cap X_1 = \mathscr{C}_1[\mathscr{G}^{\mu}(S) \cap X_1] \cup (\mathscr{C}_2[\mathscr{G}^{\mu}(S) \cap X_2] \cap X_1).$$

But, since $\mathscr{C}_2$ is a hull, we have

$$\mathscr{G}^{\mu}(S) \cap X_2 \cap X_1 \subset \mathscr{C}_2[\mathscr{G}^{\mu}(S) \cap X_2] \cap X_1$$

and,

$$\mathscr{C}_2[\mathscr{G}^{\mu}(S) \cap X_2] \cap X_1 \subset \mathscr{G}^{\mu+1}(S) \cap X_2 \cap X_1 = \mathscr{G}^{\mu}(S) \cap X_2 \cap X_1.$$

Consequently, the inclusion

$$\mathscr{G}^{\mu}(S) \cap X_2 \cap X_1 \subset \mathscr{C}_1[\mathscr{G}^{\mu}(S) \cap X_1]$$

leads to

$$\begin{aligned} \mathscr{G}^{\mu+1}(S) \cap X_1 &= \mathscr{C}_1[\mathscr{G}^{\mu}(S) \cap X_1] \cup [\mathscr{G}^{\mu}(S) \cap X_2 \cap X_1] \\ &= \mathscr{C}_1(S \cap X_1), \end{aligned}$$

i.e. $\mathscr{G}^{\mu+1}(S) \in \mathscr{C}_1$.

4.3. COROLLARY: *Let $(X_i, \mathscr{C}_i)$ $(i = 1, 2)$ be convexity structures and let $(X_1 \cup X_2, \mathscr{C})$ be their gluing. If $S \subset X_1 \cup X_2$ is such that $X_1 \cap X_2 \subset \mathscr{G}^{\mu}(S)$ for a certain ordinal μ, then $\mathscr{G}^{\mu+1}(S) = \mathscr{C}(S)$.*

PROOF: Indeed, if $\mathscr{G}^{\mu}(S) \in \mathscr{C}$, the result holds trivially; if not,

$$\begin{aligned} &[\mathscr{G}^{\mu+1}(S) \cap X_1 \cap X_2] \backslash [\mathscr{G}^{\mu}(S) \cap X_1 \cap X_2] \\ &\quad = [\mathscr{G}^{\mu+1}(S) \cap X_1 \cap X_2] \backslash (X_1 \cap X_2) = \emptyset, \end{aligned}$$

hence $\mathscr{G}^{\mu+1} \in \mathscr{C}$.

4.4. THEOREM: *Let $(X_i, \mathscr{C}_i)$ be convexity structures $(i = 1, 2)$ such that $\#(X_1 \cap X_2) = \alpha$. The order β of their gluing is at most the cardinal successor α^+ of α and, if α is infinite, β is less than α^+.*

PROOF: Let β be the order of the gluing and let $S \subset X_1 \cup X_2$ be such that $\mathscr{G}^{\mu}(S) \notin \mathscr{C}$ for every $\mu < \beta$. Proposition 4.2. insures that

$$\{\mathscr{G}^{\mu}(S) \cap X_1 \cap X_2 : \mu < \beta\}$$

is a chain isomorphic to β. In every $\mathscr{G}^{\mu}(S) \cap X_1 \cap X_2$ ($\mu \geqslant 1$, μ non-limit), select a point x^{μ} such that $x^{\mu} \notin \mathscr{G}^{\nu}(S)$ for every $\nu < \mu$. The set $A = \{x^{\mu} : \mu < \beta\} \subset X_1 \cap X_2$ is such that $A \cup \{\emptyset\}$ is isomorphic to β minus the limit points. The ordinal of this set is β or the ordinal predecessor of β, which has the same cardinal as β. Therefore, $\#\beta \leqslant \alpha^{+}$ with the strict inequality if α is infinite.

4.5. REMARK: Proposition 4.2. cannot be generalized to the gluing of finitely many convexity structures. In this case, on has to use a step by step method, adding one space at each step.

5. Finite order gluings of convexity structures

5.1. THEOREM: *Let the $(X_i, \mathscr{C}_i)$ be convexity structures with Caratheodory numbers $c_i (i = 1, \ldots, n)$. If their gluing has finite order α, it has a Caratheodory number $c \leqslant (\max_{i=1,\ldots,n} c_i)^{\alpha}$.*

PROOF: Let m be the maximum of $c_1, \ldots, c_n$ and let $S \subset \cup_{i=1}^{n} X_i$. By Theorem 3 of [1], $\mathscr{G}(S)$ is the union of all $\mathscr{G}(B)$, where $B \subset S$ has at most m points. Assume that $\mathscr{G}^k(S)$ is the union of all $\mathscr{G}^k(B)$ where $B \subset S$ has at most m^k points. Take $x \in \mathscr{G}^{k+1}(S) = \cup_{i=1}^{n} \mathscr{C}_i[\mathscr{G}^k(S) \cap X_i]$, say: x is in the i-th term. Then there exists a set $F \subset \mathscr{G}^k(S) \cap X_i$ with at most $c_i \leqslant m$ points, such that $x \in \mathscr{C}_i(F)$. For each $y \in F$, the inductive assumption on $\mathscr{G}^k$ gives us a set $F_y \subset S$ with at most m^k points, such that $y \in \mathscr{G}^k(F_y)$. Then $\cup_{y \in F} F_y \subset S$ has at most m^{k+1} points, and

$$x \in \mathscr{C}_i\left[\mathscr{G}^k\left(\bigcup_{y \in F} F_y\right) \cap X_i\right] \subset \mathscr{G}^{k+1}\left(\bigcup_{y \in F} F_y\right).$$

Consequently,

$$\mathscr{G}^{k+1}(S) \subset \cup\{\mathscr{G}^{k+1}(B) : B \subset S, \quad \#B \leqslant m^{k+1}\} = \mathscr{G}^{k+1}(S),$$

which ends the proof.

5.2. REMARKS: The preceding theorem establishes the existence of the Caratheodory number of a finite order gluing of convexity structures having a Caratheodory number; the bound which is produced is not claimed to be a sharp one. Other results on the Caratheodory number of a gluing can be found in [3] and [4], especially for the gluing of structures having finitely many points in common. Such a gluing has finite order by Theorem 4.4.

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