S. RALLIS

Injectivity properties of liftings associated to weil representations

*Compositio Mathematica*, tome 52, n° 2 (1984), p. 139-169

<http://www.numdam.org/item?id=CM_1984__52_2_139_0>
INJECTIVITY PROPERTIES OF LIFTINGS ASSOCIATED TO WEIL REPRESENTATIONS

S. Rallis

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Introduction

We have presented in [R-2] an approach to studying the space of cusp forms of classical groups of type Sp or O(Q). This approach uses the lifting theory of representations between dual pairs arising from oscillator representations. We have given a natural decomposition of the space of cusp forms on $L^2_{cusp}(Sp,\mathcal{A})$ into orthogonal pieces $X_i$ where if $X_i \neq \{0\}$, then $X_i$ "lifts" to a space of cusp forms on $O(Q_i,\mathcal{A})$, ($Q_i$, a quadratic form of dimension $2\lambda$ with split part $2\lambda - 2i$). We have shown in [R-2] several cases where the Howe duality conjecture is valid for the lifting from the space $X_i$ to $L^2_{cusp}(O(Q_i,\mathcal{A}))$; indeed this says that the lifting on the set of automorphic representations occurring in $X_i$ is an injective mapping and that multiplicities are preserved under the lifting. It is reasonable to expect that the methods of [R-2] can be extended so that this conjecture (or maybe a minor variant of it) is valid for each $X_i$. However, what is not considered in [R-2] is to give a precise description of the representations occurring in $X_i$ or the representations occurring in the image of $X_i$. We note that the first such instance where such a precise description is given at the global level is for the pair $GL_2(\mathcal{A})$ and $D_\mathcal{A}$, $D$ a quaternion algebra over $K$, relative to the lifting between $L^2(D_K\backslash D_\mathcal{A})$ and $L^2_{cusp}(GL_2(\mathcal{A}))$ given in [J-L]. Indeed the lifting map is shown to be injective, and the image is characterized as the set of cuspidal representations of $GL_2(\mathcal{A})$ whose components at the ramified primes of $D$ are

This research was supported, in part, by NSF Grant MCS 78-02414.
square integrable representations. The next set of cases considered were
the Saito-Shintani base change, the symmetric square map from \( GL_2 \) to
\( GL_3 \), and the Shimura lifting from \( GL_2 \) to its two-fold cover. In each of
these instances a comparison of traces formula is proved; from this the
existence of a global (and then a local) lifting is deduced with specific
information given on the injectivity of the map. Moreover there is very
fine information characterizing the representations occurring in the image
of the lifting map.

We are interested in considering the same questions for a general dual
reductive pair. In principle, it is straightforward to prove the existence
and injectivity of the lifting as shown by the various cases of the Howe
conjecture in [R-2]. However, to give an adequate characterization of the
image of a lift, it seems mandatory to use some type of trace formula.
Indeed the basic question that initiated this study was to determine what
isometry properties the lifting map possessed when it was given in
concrete terms by a kernel function; the fundamental question is to have a
method of relating the Hilbert inner product of two "lifted" modular
forms \( \ell_{\psi, \psi} \), to the initial Hilbert inner product of the \( \psi_i \). To this end we
have discovered an inner product formula of the following form:

\[
\langle \ell_{\psi, \psi}, \ell_{\psi', \psi'} \rangle_{O(\mathbb{Q})} = \langle \psi_i \ast \mathcal{L}(\varphi, \varphi') | \psi'_i \rangle_{Sp_n(\mathbb{A})},
\]

where \( \mathcal{L}(\varphi, \varphi') \) is an \( L^1 \) function of \( Sp_n(\mathbb{A}) \) (depending on \( \varphi \) and \( \varphi' \)) and
\( \Delta \) is a certain involution on \( L^2_{cusp}(Sp_n(\mathbb{A})) \). The import of such a formula
is that to test the nonvanishing of \( \ell_{\psi, \psi} \), it suffices to test the nonvanishing
of \( \langle \psi_i \ast \mathcal{L}(\varphi, \varphi') | \psi'_i \rangle \). What is remarkable is that this term can be
effectively computed! Indeed we note that \( \mathcal{L}(\varphi, \varphi) \) will be a matrix
coefficient of a particular global Weil representation; then with mild
restrictions \( \langle \psi_i \ast \mathcal{L}(\varphi, \varphi) | \psi'_i \rangle \) will be an Euler product in which each
local integral will be of the form

\[
\int_{Sp_n(K_v)} \langle q_{v, \psi} \ast G_v | q_{v, \psi} \rangle_v \langle (\psi_i)_v \ast G_v | (\psi_i)_v \rangle_v dG_v,
\]

where \( \langle q_{v, \psi} \ast G_v | q_{v, \psi} \rangle_v \) is the matrix coefficient of a local Weil representation
and \( \langle (\psi_i)_v \ast G_v | (\psi_i)_v \rangle_v \) is the matrix coefficient of a local component
of \( \psi_i \). In particular we note that the Euler product will, in fact,
decompose into two pieces: i.e., a product \( L_1 \cdot L_2 \), where \( L_1 \) is the special
value of a Langlands \( L \) function associated to the representation \( \Pi \)
(which carries \( \psi_i \)) and \( L_2 \) is a finite product of local integrals as above
(over those primes where the local component \( (\Pi)_v \) is not a spherical
representation). It is then easy to see that \( L_1 \neq 0 \) and that the nonvanishing
of \( L_2 \) depends on whether \( (\Pi)_v \) will "occur" in the associated local
Weil representation. Thus, in principle, we have reduced the question of
determining the image of a lift to determining locally which representations occur in the Weil representation. We note that we do not expect this always to be the case (see [P]); however for $\text{Sp}_n$ large compared to $O(Q)(\mathbb{A})$, it should be true, as we show in this paper for the family of cases $\text{Sp}_1 \times O(Q)$ where $\dim Q \geq 8$.

We note that as a special consequence of the above work, we are able to construct global automorphic representations $\sigma$ of $O(Q)(\mathbb{A})$ appearing in $L^2_{\text{cusp}}(O(Q)(\mathbb{A}))$ such that $\sigma_{\infty} =$ the component at some Archimedean prime “$\infty$” will be a representation of $O(Q)(\mathbb{R})$ with nonvanishing Lie algebra cohomology (see [B-W] and [M-R] for this general problem).

On the other hand, we know from the inner product formula shown above that a type of formal comparison of traces of Hecke operators is possible. That is, we recall from [R-1] that there exists a surjective homomorphism $\omega_{Q_v}$ of the local Hecke algebra $H(O(Q_v)/\mathbb{K}_v)$ to $H(\text{Sp}_1(K_v)//\text{Sp}_1(K_v))$ which is compatible with the local Weil representation $\pi_{Q_v}$, i.e., $\pi_{Q_v}(\omega_Q(f)) = \pi_{Q_v}(f)$ for all $f \in H(O(Q_v)//\mathbb{K}_v)$. Then for the following classical case, we have a comparison of trace formula. Namely, let $S_{4t}(\mathbb{Z})$ be the space of cusp forms of weight $4t$ for the group $SL_2(\mathbb{Z})$. Let $Q$ be a unimodular quadratic form $(8|m, m = \dim Q)$ and let $F_Q = \text{the finite dimensional space of functions on the double coset space } O(Q)(\mathbb{Q})BO(Q)(\mathbb{A})/U_Q$, where $U_Q = \text{stabilizer in } O(Q)(\mathbb{A}) \text{ of the standard lattice } \mathbb{Z}^m$. Then we know that the lifting map defines a linear injection of $S_{4t}(\mathbb{Z})$ to $F_Q$, which is compatible with the local maps $\omega_{Q_v}$ (for all finite $v$ in $\mathbb{Q}$). Moreover we have the formula

$$T_r(\rho(f_v)|_{X_Q}) = \text{Tr}(\rho(\omega_Q(\tilde{f}_v)))$$

for all $f_v \in H(O(Q_v)/\mathbb{K}_v)$ (here $\rho$ denotes the respective representations of local Hecke algebras on the given spaces of cusp forms), and $X_Q =$ the image of the lifting.

Also we note another possible use of the above work. Namely if we assume that $\psi_i$ is an eigenfunction of $L(\varphi, \varphi')$ (which is the case when $\psi_i$ lies in an irreducible component of $L^2_{\text{cusp}}(\text{Sp}_n(\mathbb{A}))$), then the inner product gives, in principle, a way to determine the algebraic nature of the ratio

$$\frac{\langle \ell_{\varphi, \psi_i} | \ell_{\varphi', \psi_i}' \rangle}{\langle \psi_i | \psi_i' \rangle}.$$  

Indeed in Remark 2.4, we see that, for the pair $(\text{Sp}_1, O(Q))$ with certain restrictions on $Q$, this ratio (aside from a universal constant not depending on the $\psi_i$) can be expressed as a ratio of special values of $L$ functions associated to the representation space $\Pi$ of $\psi_i$. We can view such a formula as a possible generalization of the exact formulae of Waldspurger and Kohnen-Zagier ([W] and [K-Z]) relating the ratios of Peters-
son inner products of half-integral weight modular forms to integral weight modular forms (via the Shimura lifting).

We organize the paper in the following fashion.

In § 0 we present preliminary definitions and notation. We also recall certain results in [R-S].

In § 1 we consider the lifting $\ell$ from $L_{cusp}^2(\text{Sp}_k(\mathbb{A}))$ and $L_{cusp}^2(\text{Sp}_{k'}(\mathbb{A}))$ to the space of smooth automorphic forms $S(\text{O}(Q)(K) \backslash \text{O}(Q)(\mathbb{A}))$ (for a fixed form $Q$). Then it is possible to compute the Petersson inner product of $\ell_{\psi, \chi}$ and $\ell_{\psi', \chi'}$ if $k + k' < m/4$ ($m = \dim Q$) on $\text{O}(Q)(\mathbb{A})$. In Proposition 1.1 (I), we show that the lifts $\ell_{\psi, \chi}$ and $\ell_{\psi', \chi'}$ are perpendicular if $k \neq k'$. Thus the main case of interest is when $k = k'$, and we prove in Proposition 1.1 (II) the inner product formula alluded to above. The key point in proving this formula is the use of Siegel’s formula and the determination of the set of double cosets of the form $\text{Sp}_k \times \text{Sp}_{k'} \backslash \text{Sp}_{k+k'}/\text{P}_{k+k'}$. We note that we have used a similar such decomposition in [R-2] as one of the main technical tools in proving the local Howe duality conjecture. The construction of $\mathcal{L}(\psi, \chi')$ as a matrix coefficient of an appropriate Weil representation, which is factorizable as a product of matrix coefficients of the associated local Weil representations, is given in the proof of Proposition 1.1 and Remark 1.2. Then we look at the local factors of $\mathcal{L}(\psi, \chi)$ which are of spherical type and determine in Remark 1.3 the corresponding local factor of the inner product $\langle \psi_1 \ast \mathcal{L}(\psi, \chi') | \psi_2 \rangle$. Moreover in § 4 we give the details of this computation when $k = 1$. Then as an application of this factorization property of $\langle \psi_1 \ast \mathcal{L}(\psi, \chi') | \psi_2 \rangle$ we determine in Corollary 1 to Proposition 1.1 the representations occurring in $I_1(\mathbb{Q})$ ($\mathbb{Q}$ anisotropic over $K$ with $m > 6$).

Here we use also certain results from [R-S], i.e., the possible discrete series representations of $\text{SL}_2(\mathbb{R})$ that occur in the Weil representation for the pair $(\text{O}(\mathbb{Q}), \text{Sp}_1 = \text{SL}_2)$. In Theorem 1.6 we show, by dualizing the above Corollary, that the lifting map is injective between $R_\iota(\mathbb{Q})$ and $I_1(\mathbb{Q})$ (see § 0 for precise definitions); moreover using the results of [R-2], we show that the global Howe duality conjecture is valid for this range of cases. Finally in Remark 1.7 we show how to construct global cuspidal automorphic representations of $\text{O}(\mathbb{Q})(\mathbb{A})$ such that a local component at $\{\infty\}$ has nonvanishing Lie algebra cohomology of a certain prescribed level.

In § 2 we give another application of Proposition 1.1. Namely we prove the “Comparison of Trace” Theorem (Theorem 2.1) discussed above. We also indicate in Remark 2.3 the arithmetic nature of the ratio $\langle \ell_{\psi, \psi_1} | \psi_2 \rangle /\langle \psi_1 | \psi_2 \rangle$ in terms of special values of classical-type zeta functions and $L$-functions of Langlands type.

Sections 3 and 4 are devoted to proving convergence of certain integrals in § 1 and the determination of local factors discussed above.

We would like to thank John Millson, whose incisive questions initiated part of this study.
§ 0. Notation and preliminaries

(I) Let $k$ be a local field of characteristic 0. We fix a nontrivial additive character $\tau$ on $k$. Let $\langle \cdot, \cdot \rangle_k$ be the usual Hilbert symbol on $k$. Let $dx$ be a Haar measure on $k$ which is self dual relative to $\tau$. We let $||$ be an absolute value of $k$.

If $k$ is a nonarchimedean field, we let $\mathfrak{o}_k = \text{ring of integers of } k$, $\mathfrak{p}_k = \text{maximal ideal in } \mathfrak{o}_k$, and $q = \text{cardinality of } \mathfrak{o}_k/\mathfrak{p}_k$.

(II) Let $K$ be a number field (i.e., finite degree extension of $\mathbb{Q}$, the rational numbers). Let $A_K$ be the corresponding adelic group. Then embed $K$ as a discrete subring in $A_K$. Let $K_v$ be the completion of $K$ relative to a prime $v$ in $K$. Let $\tau$ be a nontrivial character on $A_K$ which equals 1 on $K$; then there exist compatible characters $\tau_v$ on $K_v$ (for all primes $v$ in $K$) such that $\tau(X) = \prod_v \tau_v(X_v)$. Let $dX$ be the measure (Tamagawa measure) on $A_K$ such that the group $A_K/K$ is self dual relative to $\tau$ and $A_K/K$ has mass 1. When the context is clear, we drop $K$ in $A_K$ and use just $A$ for $A_K$.

(III) Let $Q$ be a nondegenerate quadratic form on $K^m$. Let $Q_v$ be the corresponding local versions on $K_v^m$. If $Q_v$ is a totally split form which is the direct sum of $r$ hyperbolic planes, then we let $Q_v = H_r$. Let $O(Q)$ be the orthogonal group of $Q$. Then we can form the corresponding adelic group $O(Q)(A)$ and the corresponding local orthogonal groups $O(Q_v)$ of $Q_v$ at $K_v$. Let $O(Q)(K)$ be the $K$ rational points in $O(Q)$ and embed $O(Q)(K)$ into $O(Q)(A)$ in the standard way. Choose a Tamagawa measure on the quotient $O(Q)(K)/O(Q)(A)$ as given in [A].

Similarly let $A$ be a nondegenerate alternating form on $K^{2n}$. Let $Sp_{2n}$ be the corresponding symplectic group and $Sp_{2n}(A)$, $Sp_{2n}(K_v)$ the associated adelic and local objects. Let $Sp_{2n}(K) = \text{the } K \text{ rational points in } Sp_{2n}$ and embed $Sp_{2n}(K)$ into $Sp_{2n}(A)$ again in the standard fashion and choose a Tamagawa measure on the quotient $Sp_{2n}(K)/Sp_{2n}(A)$ as given in [A].

(IV) If $\tilde{G}_A$ is a global group, then we denote the space of cusp forms on $\tilde{G}_K \setminus \tilde{G}_A$ by $L^2_{\text{cusp}}(\tilde{G}_A)$. We note (by our convention) that if $\tilde{G}_K \setminus \tilde{G}_A$ is compact, then $L^2_{\text{cusp}}(\tilde{G}_A) = \{ \text{all functions } f \perp \text{ constants} \}$. We know that $L^2_{\text{cusp}}(\tilde{G}_A)$ is discretely decomposable as a $\tilde{G}_A$ module, and each unitary irreducible representation occurring in $L^2_{\text{cusp}}(\tilde{G}_A)$ has a finite multiplicity.

We denote

$$\langle f_1 | f_2 \rangle_{\tilde{G}_A} = \int_{\tilde{G}(K) \setminus \tilde{G}_A} f_1(g) \overline{f_2(g)} dg,$$

where $dg$ is some Tamagawa measure on the quotient $\tilde{G}(K) \setminus \tilde{G}_A$.

(V) We consider the following Weil representation. Namely, we fix a nondegenerate form $Q$ on $K^m$ and a nondegenerate alternating form $A$ on $K^{2n}$. Then we consider the alternating form $Q \otimes A$ on $M_{m,2n}(K)$. To
this form we associate a Weil representation of the two-fold cover $\tilde{\text{Sp}}_{m,n}$ of $\text{Sp}_{mn}$ given in [R-2]. Then we restrict the representation to a subgroup $\tilde{\text{Sp}}_n \times \text{O}(Q)$ = inverse image of $\text{Sp}_n \times \text{O}(Q)$ in $\tilde{\text{Sp}}_{mn}$. If $m$ is even, then we know that we get an honest representation of $\text{Sp}_n \times \text{O}(Q)$. This we call the Weil representation $\pi_Q$ of $\text{Sp}_n \times \text{O}(Q)$ on the space $S[M_{mn}(k)]$. In fact, we recall formulae for generators of $\text{Sp}_n \times \text{O}(Q)$ in $\pi_Q$ in §1 of [R-2] where $Q = Q_0$, a local form of $Q$.

(VI) Using the local data in (V), we can define a global Weil representation $\pi_Q$ of $\text{Sp}_n(\mathbb{A}) \times \text{O}(Q)(\mathbb{A})$ on the space $S[M_{mn}(\mathbb{A})]$ (for details, see [R-2]). That is, to every $\varphi \in S[M_{mn}(\mathbb{A})]$, we can construct a function $(\varphi, (G, g) = \pi_Q(G, g)(\varphi)(\xi)$.

Then $\theta_\varphi$ is an automorphic function on $\text{Sp}_n(\mathbb{A}) \times \text{O}(Q)(\mathbb{A})$. That is, $\theta_\varphi(\gamma_1 G, \gamma_2 g) = \theta_\varphi(G, g)$ for all $\gamma_1 \in \text{Sp}_n(\mathbb{K}), \gamma_2 \in \text{O}(Q)(\mathbb{K})$. Moreover, $\theta_\varphi$ is a slowly increasing function on $\text{Sp}_n(\mathbb{K}) \times \text{O}(Q)(\mathbb{K}) \backslash \text{Sp}_n(\mathbb{A}) \times \text{O}(Q)(\mathbb{A})$ in the sense of [A].

(VII) We construct here representatives for the maximal parabolic subgroups of $\text{Sp}_n$.

$\text{Sp}_n$ has parabolics given by $P_{n-i}, i = 0, \ldots, n - 1$, where

$$P_{n-i} = \text{Sp}_i \times G\ell_{n-i} \times U_i^n$$

with

$$\text{Sp}_i = \left\{ \begin{bmatrix} A & 0 & B & 0 \\ 0 & I & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \middle| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \text{Sp}_i \right\}$$

and

$$G\ell_{n-i} = \left\{ \begin{bmatrix} I & 0 \\ 0 & A \\ 0 \end{bmatrix} \middle| A \in G\ell_{n-i} \right\}$$

and

$U_i^n$ is the semidirect product $U_i^{n,1} \times U_i^{n,2}$,
where
\[
U^{n,1}_i = \begin{bmatrix}
I & 0 & 0 \\
T & I & 0 \\
0 & I & -T'
\end{bmatrix} | T \in M_{n-i,i}(k)
\]
and
\[
U^{n,2}_i = \begin{bmatrix}
I & 0 & T \\
T' & X & 0 \\
0 & I & X
\end{bmatrix} | T \in M_{i,n-i}(k), X \in \text{Sym}_{n-i}(k)
\]

(VIII) We recall here Theorem 1.2.1 of [R-2]: \(L^2_{\text{cusp}}(O(Q(A)))\) is an orthogonal direct sum of

\[R_1(Q) \oplus R_2(Q) \oplus \ldots \oplus R_m(Q)\]

where \(R_i(Q) = \{ f \in L^2_{\text{cusp}} | \langle \theta_{\psi}(G, g), f(g) \rangle = 0 \text{ for all } G \in \text{Sp}_1(A) \text{ and all } \psi \in S[M_{m1}(A)] \} \) in \(L^2_{\text{cusp}}\) and inductively \(R_i(Q) = \{ f \in L^2_{\text{cusp}} | \langle \theta_{\psi}(G, g), f(g) \rangle = 0 \text{ for all } G \in \text{Sp}_i(A) \text{ and all } \psi \in S[M_{m\ell}(A)] \} \) with \(\ell \leq i\) in \(L^2_{\text{cusp}}\). Similarly \(L^2_{\text{cusp}}(\text{Sp}_n(A))\) has an orthogonal decomposition

\[I_1(Q) \oplus I_2(Q \oplus H_1) \oplus \ldots \oplus I_{r-1}(Q \oplus H_r)\]

where the spaces \(I_i\) are defined in a similar manner as the \(R_i\) above. (Here \(Q\) is a nondegenerate anisotropic form over \(K\) and is possibly zero.)

(IX) We use the parametrization of the representations of \(\text{SL}_2(\mathbb{R})\) as given in [L]. Then \(H_0^+\) and \(H_0^-\) are the mock discrete series and \(D_m^\pm\) is the family of discrete series with \(m \geq 2\).

(X) We let \(e(\ldots)\) be the exponential function \(e\) raised to the \((\ldots)\) power.

(XI) We let
\[
w = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix} \in \text{Sp}_n(Q) \text{ and } w^{-1} = w_1.
\]

(XII) We let
\[
\Omega_i = w \begin{bmatrix}
I & T_i \\
0 & I
\end{bmatrix} \in \text{Sp}_{2n}(Q)
\]
where

\[ T_i = \begin{bmatrix} I_{n-i} & 0 \\ 0 & 0 \\ I_{n-i} & 0 \\ 0 & 0 \end{bmatrix}, \]

\[ I_{n-i} = (n-i) \times (n-i) \text{ identity matrix.} \]

§ 1. An inner product formula

One of the main goals mentioned in [R-2] is to have an effective comparison of Hecke operators on the spaces \( L^2_{\text{cusp}}(\ ) \) and \( R_\ell \) (also \( L^2_{\text{cusp}}(\ ) \) and \( S_{\ell}(\ ) \)). However prior to having such a statement, we must determine a relation between the inner product of the lift of cusp forms on \( O(Q)(\mathbb{A}) \) to the inner product of the initial cusp forms themselves on \( \text{Sp}_k(\mathbb{A}) \).

Indeed we fix cusp forms \( \psi_1 \) and \( \psi_2 \) on the groups \( \text{Sp}_k(\mathbb{A}) \) and \( \text{Sp}_{k'}(\mathbb{A}) \) respectively. Then we consider the associated lifts

\[ \ell_{k,\psi_1}^k(g) = \int_{\text{Sp}_k(\mathbb{A})} \theta_{\psi_1}(G, g) \psi_1(G) \; dG \]

and

\[ \ell_{k',\psi_2}^{k'}(g) = \int_{\text{Sp}_{k'}(\mathbb{A})} \theta_{\psi_2}(G, g) \psi_2(G) \; dG. \]

Then in order to compute the inner product of these two functions, we assume that at least one is a cusp form, and hence we have

\[ \langle \ell_{k,\psi_1}^k, \ell_{k',\psi_2}^{k'} \rangle_{O(Q)(\mathbb{A})} \]

\[ = \int_{\text{Sp}_k \times \text{Sp}_{k'}(\mathbb{A}) \setminus \text{Sp}_k \times \text{Sp}_{k'}(\mathbb{A})} \psi_1(G) \overline{\psi_2((G')^{\delta_0})} \mathcal{T}_Q(\varphi \otimes \overline{\varphi}') \]

\[ (G, G') \; dG \; dG', \quad (1-1) \]

\((G')^{\delta_0} \) defined below) where \( \mathcal{T}_Q \) is the functional on \( S[M_{m,k+k'}(\mathbb{A})] = S[M_{m,k}(\mathbb{A})] \otimes S[M_{m,k'}(\mathbb{A})] \) given by (provided we have convergence of such an integral)

\[ \mathcal{T}_Q(\psi)(G) = \int_{O(Q)(K) \setminus O(Q)(\mathbb{A})} \theta_{\psi}(G, g) \; dg, \]
with $\psi \in S[M_{m,k+k'}(A)]$, $G \in \text{Sp}_{k+k'}(A)$. In the above formula (1-1), we consider $\text{Sp}_k \times \text{Sp}_{k'}$ as a subgroup of $\text{Sp}_{k+k'}$ via the obvious diagonal embedding map.

**Proof of (1-1):** We first write down the integral for the inner product in (1-1), and we have

$$
\langle \xi^k \psi_1 | \xi'^{k'} \psi_2 \rangle = \int_{O(Q)(K) \setminus O(Q)(A)} \int_{\text{Sp}_k \times \text{Sp}_{k'}(K) \setminus \text{Sp}_k \times \text{Sp}_{k'}(A)} \psi_1(G)\psi_2(G') \times \sum_{(\xi \xi') \in M_{m,k+k'}(K)} \pi_Q(G, g)(\varphi)(\xi)\pi_Q(G', g)(\varphi')(\xi')
$$

$$
dGdG' d\gamma.
$$

But then we know that $\pi_Q(G, g)(\varphi')(X) = \pi_Q((G')^{\Delta_0}, g)(\varphi')(X)$ for all $\varphi' \in S[M_{m,k}(A)]$, $X \in M_{m,k}(A)$ and

$$
(G')^{\Delta_0} = \begin{bmatrix} I_k & 0 \\ 0 & -I_{k'} \end{bmatrix} G' \begin{bmatrix} I_k & 0 \\ 0 & -I_{k'} \end{bmatrix}.
$$

Thus combining the terms in the above series via this latter rule and changing the order of integration we get (1-1). We note that it is possible to switch the order of integration when $\mathcal{I}_Q(\psi)$ is an absolutely convergent integral (i.e., this happens when $Q$ is anisotropic or when $\dim(Q) > \text{rank}(Q) + (k + k') + 1$ if $Q$ is isotropic); then $\mathcal{I}_Q(\psi)$ is a slowly increasing function on its restriction to $\text{Sp}_k \times \text{Sp}_{k'}(A)$ and is integrated against the product of cusp forms $\psi_1 \cdot \psi_2$.

On the other hand, we can infer something about the functional $\psi \to \mathcal{I}_Q(\psi)(G)$, $G \in \text{Sp}_{k+k'}(A)$. Namely we can apply Siegel's formula to $\mathcal{I}_Q$ (when $\dim Q > 2(k + k') + 2$), and we have

$$
\mathcal{I}_Q(\psi)(G) = \sum_{\gamma \in P_{k+k'}(K) \setminus \text{Sp}_{k+k'}(K)} \pi_Q(\gamma G)(\psi)(0).
$$

But we know that $\text{Sp}_k(K) \times \text{Sp}_{k'}(K)$ operating on the coset space $P_r(K) \setminus \text{Sp}_k(K)$ ($r = k + k'$) has a finite number of orbits. Thus if $\Omega_i$ is a set of representatives of such orbits and $(\text{Sp}_k \times \text{Sp}_{k'})(\Omega_i)$ is the stabilizer of $P_r(K) \cdot \Omega_i$ in $\text{Sp}_k \times \text{Sp}_{k'}(K)$, then we see easily that (see § 3 for the
proof of convergence in (1-2))

\[ \langle \tilde{\rho}_{\varphi, \psi_1} \| \tilde{\rho}_{\varphi, \psi_2} \rangle_{O(\Omega)}(\mathbf{A}) \]

\[ = \sum_{\Omega_i} \left( \int_{\text{Sp}_k \times \text{Sp}_{k'}(\mathbb{A}) \backslash \text{Sp}_k \times \text{Sp}_{k'}(\mathbf{A})} \psi_1(G) \overline{\psi_2((G')^{\Delta_0})} \right) \]

\[ \times \pi_Q(\Omega_i(G, G'))(\varphi \otimes \overline{\varphi}')(0) d\sigma_i(G, G') \]  \hspace{1cm} (1-2)

Thus we have completed the first reduction using Siegel's formula to express \( \langle \tilde{\rho}_{\varphi, \psi_1} \| \tilde{\rho}_{\varphi, \psi_2} \rangle \) as a finite sum of adelic integrals over \( \text{Sp}_k \times \text{Sp}_{k'}(\mathbf{A}) \).

Then we know that if \( k \geq k' \), then \( \text{Sp}_k \times \text{Sp}_{k'}(\Omega_i) \) is isomorphic to the group \((i = 0, \ldots, k')\) (see (XII of §0)

\[ \Delta_i \times (S_i \times R_i) \]

where

\[ \Delta_i = \{ (g, g^{\Delta_i}) | g \in \text{Sp}_i(K) \} \]

and

\[ S_i = GL_{k-i} \times U_i \]

\[ R_i = GL_{k'-i} \times U_i^{k'} \]

(for this, see similar computations given in [R-2]). We recall that \( G^{\Delta_i} = ((G')^{-1})^{\Delta_0} \). We let \( G^{\Delta} = (G')^{-1} \).

Thus the second reduction step is to evaluate the adelic integrals in (1-2). We have that

\[ \int_{\text{Sp}_k \times \text{Sp}_{k'}(\Omega_i) \backslash \text{Sp}_k \times \text{Sp}_{k'}(\mathbf{A})} \psi_1(G) \overline{\psi_2((G')^{\Delta_0})} \pi_Q(\Omega_i(G, G')) \]

\[ = (\varphi \otimes \overline{\varphi}')(0) dG dG' \]

\[ \int_{\Delta_i \times \tilde{S}_i(\mathbf{A}) \times \tilde{R}_i(\mathbf{A}) \backslash \text{Sp}_k \times \text{Sp}_{k'}(\mathbf{A})} \pi_Q(\Omega_i(G, G'))(\varphi \otimes \overline{\varphi}')(0) \]

\[ \left\{ \int_{U_i^{k}(K) \backslash U_i^{k}(K) \times U_i^{k}(K) \backslash U_i^{k}(\mathbf{A})} \psi_1(uG) \overline{\psi_2(uG')} dudu' \right\} d\sigma(G, G'), \]

where \( \tilde{S}_i(\mathbf{A}) = GL_{k-i}(K) \times U_i^k(\mathbf{A}) \) and \( \tilde{R}_i(\mathbf{A}) = GL_{k'-i}(K) \times U_i^{k'}(\mathbf{A}) \)

and \( d\sigma \) is a suitably normalized measure on the quotient \( \Delta_i \times \tilde{S}_i \times \tilde{R}_i \backslash
\[ S_{p_k} \times S_{p_k}(A). \] Here \( U^k \times U^{k'}(A) \) fixes \( \Omega \), i.e. \( \pi_Q(\Omega, (x, y))(\psi)(0) = \pi_Q(\Omega, \psi)(0) \) for all \( \psi \in S[M_{m, k + k'}(A)] \) (i.e. using factorizability of \( \pi_Q \) into local \( \pi_Q \), and the validity of the statement for local Weil representations \( \pi_{Q_v} \)).

Thus we have that if \( k > k' \) or \( i > 0 \) (if \( k = k' \)), then (since \( \psi_1 \) and \( \psi_2 \) are cusp forms) \( \langle \ell^{k}_{\psi, \psi_1} | \ell^{k'}_{\psi, \psi_2} \rangle \equiv 0 \). Thus if \( i = 0 \) and \( k = k' \), the only adelic integral that we must consider is

\[
\int_{\Delta_0 \setminus S_{p_k} \times S_{p_k}(A)} \psi_1(G) \psi_2(G') \pi_Q(\Omega_0(G, G'))(\varphi \otimes \overline{\varphi'})(0) d\sigma(G, G').
\]

(1-3)

But it is possible to telescope the integration above so that (1-3) equals

\[
\int_{\Delta_0(A) \setminus S_{p_k} \times S_{p_k}(A)} \pi_Q(\Omega_0(G, G'))(\varphi \otimes \overline{\varphi'})(0)
\]

\[
\left( \int_{\Delta_0(K) \setminus \Delta_0(A)} \psi_1(hG) \psi_2(h^k(G')^\frac{k}{2}) dh \right) d\sigma(G, G').
\]

(Here again we use the factorizability properties of the Weil representation to get \( \pi_Q(\Omega_0(g, g^k))(\psi)(0) = \pi_Q(\Omega_0)(\psi)(0) \) for all \( \psi \in S[M_{m, k}(A)] \).)

But then we note that the map \( S_{p_k}(A) \rightarrow \Delta_0(A) \setminus S_{p_k}(A) \times S_{p_k}(A) \) given by \( g \mapsto (g, 1) \Delta_0(A) \) is a homomorphism of topological spaces carrying the Haar measure on \( S_{p_k}(A) \) onto the measure \( d\sigma \) (given above) on \( \Delta_0(A) \setminus S_{p_k}(A) \times S_{p_k}(A) \). Thus we can rewrite (1-3) as

\[
\int_{S_{p_k}(A)} \pi_Q(\Omega_0(G, 1))(\varphi \otimes \overline{\varphi'})(0) \langle \psi_1 \ast G | \psi_2^\Delta \rangle_{S_{p_k}(A)} d\omega(G),
\]

where

\[
\langle \psi_1 \ast G | \psi_2^\Delta \rangle_{S_{p_k}(A)} = \int_{S_{p_k}(K) \setminus S_{p_k}(A)} \psi_1(hG) \psi_2^{\Delta}(h) dh.
\]

Thus we define for any pair \((\varphi, \varphi')\) in \( S[M_{m, k}(A)] \times S[M_{m, k}(A)] \) a function on \( S_{p_k}(A) \),

\[
\mathcal{L}(\varphi, \varphi')(G) = \pi_Q(\Omega_0(G, 1))(\varphi \otimes \overline{\varphi'})(0).
\]

Then we have the following Proposition determining \( \langle \ell^k_{\psi, \psi_1} | \ell^{k'}_{\psi, \psi_2} \rangle \).

**Proposition 1.1:** Assume that \( m > 2(k + k') + 2 \). Assume \( \ell^k_{\psi, \psi_1} \) or \( \ell^{k'}_{\psi, \psi_2} \) is a cusp form. Then

(I) Let \( k \neq k' \). Then \( \langle \ell^k_{\psi, \psi_1} | \ell^{k'}_{\psi, \psi_2} \rangle \equiv 0 \).
(II) If \( k = k' \), then

\[
\langle \ell^k, \psi_1 \rangle \psi_1 = \langle \psi_1 \ast \mathcal{L}(\varphi, \varphi') \rangle \psi_2
\]

where \( \mathcal{L}(\varphi, \varphi') \) is an \( L^1 \) function on \( \text{Sp}_k(A) \).

**PROOF:** We recall the construction of the Tamagawa measure on \( \text{Sp}_n(A) \). Locally we fix for each prime \( v \) in \( K \) a Haar measure \( d\sigma_v \) on \( \text{Sp}_n(K_v) \) such that if \( v \) is finite, then

\[
\int_{\text{Sp}_n(K_v)} d\sigma_v = q^{-\dim(\text{Sp}_n)} \# \left[ \text{Sp}_n(\mathcal{O}_{K_v}/\mathcal{O}_{K_v}) \right].
\]

Then the Tamagawa measure on \( \text{Sp}_n(A) \) is the unique measure \( d\sigma \) on \( \text{Sp}_n(A) \) such that \( d\sigma \) induces on each product

\[
\text{Sp}_n(A_s) = \prod_{v \in S} \text{Sp}_n(\mathcal{O}_{K_v}) \times \prod_{v \in S} \text{Sp}_n(K_v)
\]

\( (S = \text{all primes in } K \text{ except a finite number}) \), the product measure \( \prod_v d\sigma_v \). Thus to test whether \( \mathcal{L}(\varphi, \varphi') \) is an \( L^1 \) function on \( \text{Sp}_n(A) \), it suffices to consider \( \varphi = \prod \varphi_v \), \( \varphi' = \prod \varphi'_v \) and to show that for the filtered set

\[
\psi_S(G) = \pi_Q(\Omega(G, 1))(\varphi \otimes \varphi')(0),
\]

where \( G \in \text{Sp}_n(A_S) \),

\[
\lim_S \int_{\psi_S(G)} |\varphi| d\sigma < \infty.
\]

On the other hand, we know that

\[
\pi_Q(\Omega_0(G, 1))(\varphi \otimes \varphi')(0) = \int_{M_n(A) \times M_n(A)} \tau \left[ \text{tr} \left( \mu' Q \sigma \right) \right] \pi_Q(G)(\varphi)(\mu) \pi_Q(G)(\varphi')(\sigma) d\mu d\sigma,
\]

while the latter term is (up to a nonzero scalar) of the form

\[
\int_{M_n(A)} \pi_Q(G)(\varphi)(\mu) \pi_Q(w)(\varphi')(\mu) d\mu.
\]

But then we can express the above integral as an absolutely convergent product

\[
\prod_{v} \int_{M_n(K_v)} \pi_Q(G_v)(\varphi_v)(\mu_v) \pi_Q(w_v)(\varphi_v')(\mu_v) d\mu_v.
\]
But on the other hand, we know that for almost all primes \( v \) in \( K \), \( Q_v \) is isotropic (here \( m > 4 \)) and the Weil representation \( \pi_{Q_v} \) of \( \text{Sp}_k(K_v) \) on \( S[M_{mk}(K_v)] \) has the characteristic function \( \chi_v \) of the lattice \( M_{mk}(O_v) \) in \( M_{mk}(K_v) \) as an \( \text{Sp}_k(O_v) \) fixed vector. Thus for almost all primes \( v \) in \( K \) (let \( R \) be this set),

\[
c_v = \int_{\text{Sp}_k(O_v)} \left( \int_{M_{mk}(K_v)} \pi_{Q_v}(k_v)(\chi_v)(\mu_v)\bar{\chi}_v(\mu_v) d\mu_v \right) d\sigma_v
\]

\[
= q^{-\dim(\text{Sp}_v)} \# \left[ \text{Sp}_k(O_v)/\text{Sp}_k(\pi O_v) \right] \text{vol}(M_{mk}(O_v)).
\]

Hence we have

\[
\int_{\text{Sp}_k(A_v)} |\psi_S(G)| d\sigma \leq \left( \prod_{v \in (S \cap R)} c_v \right) \prod_{v \notin S \cap R} \left\{ \int_{\text{Sp}_k(K_v)} \left| \pi_{Q_v}(g_v)(\varphi_v)(\mu_v) \bar{\varphi}_v(\mu_v) \right| d\mu_v d\sigma_v \right\}.
\]

(\(*\))

But then following the general arguments of \([H-M]\) we have that

\[
\int_{M_{mk}(K_v)} \pi_Q(g_v)(\psi_v)(\mu_v) \tilde{\psi}_v(\mu_v) d\mu_v
\]

for

\[
g_v = \begin{bmatrix} D & 0 \\ 0 & D^{-1} \end{bmatrix}
\]

with \( D = \text{diag}(a_1, \ldots, a_k) \) is majorized by

\[
|a_1|^{-m/2} \ldots |a_k|^{-m/2}.
\]

Thus since \( m > 4k + 2 \), we have that in (\(*\))

\[
\prod_{v \notin S \cap R} \int_{\text{Sp}_k(K_v)} \left( \int |\pi_{Q_v}(g_v)(\varphi_v)(\mu_v)| |\pi_{Q_v}(w)\bar{\varphi}_v(\mu_v)| d\mu_v \right) d\sigma_v
\]

is majorized independent of the set \( S \). (Here we use the polar decomposition of \( \text{Sp}_k(K_v) \) relative to \( \text{Sp}_k(O_v) \) and the diagonals.)

Thus we have by monotonic convergence that

\[
\lim_S \int_{\text{Sp}_k(A_v)} |\psi_S(G)| d\sigma_S(G)
\]
converges to
\[ \int_{\text{Sp}_k(A)} |\psi(G)| \, d\sigma(G). \]
Thus \( \psi \) is an \( L^1 \) function on \( \text{Sp}_k(A) \).

**Q.E.D.**

**REMARK 1.2:** In the course of the proof of Proposition 1.1, we have shown that \( \mathcal{L}(\varphi, \varphi') \) is, up to a nonzero scalar multiple (independent of \( \varphi \) and \( \varphi' \)),

\[ \langle \pi_Q(G) \varphi | \pi_Q(w_1) \varphi' \rangle_{L^2(M_{mk}(A))}, \]

\( \langle \cdot, \cdot \rangle \) Hilbert product on \( L^2(\ldots) \) and see (XI) of §0) which is a matrix coefficient of the Weil representation of \( \text{Sp}_k(A) \times \text{O}(Q)(A) \) on \( S[M_{mk}(A)] \). Moreover, assuming that \( \varphi = \otimes \varphi_v \) and \( \varphi' = \otimes \varphi'_v \), we show that

\[ \langle \pi_Q(G) \varphi | \pi_Q(w_1) \varphi' \rangle = \prod_{v \in K} \langle \pi_Q(G_v) \varphi_v | \pi_Q(w_1) \varphi'_v \rangle. \]

Then it is possible to draw certain interesting consequences from Proposition 1.1 and Remark 1.2.

Indeed we see directly that the subspace of \( L^2_{\text{cusp}}(\text{Sp}_n(A)) \) where each irreducible component maps to a nonzero subspace in \( L^2_{\text{cusp}}(\text{O}(Q)(A)) \) via the lifting map can be characterized as the closure of \( \{ \mathcal{L}(\varphi, \varphi')(L^2_{\text{cusp}}(\text{Sp}_n(A))) \} \) (\( \varphi, \varphi' \) vary in \( S[M_{mn}(A)] \)) in \( L^2_{\text{cusp}}(\text{Sp}_n(A)) \). Thus it becomes a matter of determining the range of the operators \( \mathcal{L}(\varphi, \varphi') \) acting on \( L^2_{\text{cusp}}(\text{Sp}_n(A)) \). But using Remark 1.2 and the proof of Proposition 1.1, we see that if \( \psi_1 \) and \( \psi_2^\Delta \) lie in the same irreducible component \( \Pi_1 \), then

\[ \langle \psi_1 * \mathcal{L}(\varphi, \varphi') | \psi_2^\Delta \rangle = \prod_{v \in K} \int_{\text{Sp}_n(K_v)} \langle \pi_Q(G_v) \varphi_v | \pi_Q(w_1) \varphi'_v \rangle \]

\[ \times \langle (\psi_1)_v * G_v(\psi_2^\Delta)_v \rangle_{\Pi_v} d\sigma_v(G_v) \]

if \( \varphi = \otimes_{v \in K} \varphi_v \) and \( \varphi' = \otimes_{v} \varphi'_v \) are factorizable functions. Thus the essential **local question** is to determine which local unitary representations of \( \text{Sp}_n(K_v) \) have the property that the space \( A(\delta) \) of matrix coefficients of \( \delta \) can be nonsingularly paired with the space \( A(\pi_Q) \), the space of matrix coefficients of the local Weil representation \( \pi_Q \) of \( \text{Sp}_n(K_v) \) on \( S[M_{mn}(K_v)] \), via the bilinear form

\[ A(\delta) \otimes A(\pi_Q) \rightarrow C \]
given by

\[ f_1 \otimes f_2 \mapsto \int_{\operatorname{Sp}_n(K_v)} f_1(x) f_2(x) d\sigma_v(x). \]

However we know that if \( A(\pi_{Q_v}) \subseteq L^1(\operatorname{Sp}_n(K_v), d\omega_v) \), then a necessary and sufficient condition for the above form to be zero (or, simply put, \( A(\pi_{Q_v}) \subseteq \text{Annihilator}(\delta) = \{ f \in L^1|\pi_{\delta}(f) \equiv 0 \} \)) is that \( \operatorname{Hom}_{\operatorname{Sp}_n(K_v)}(\pi_{Q_v}, \delta) = \{0\} \) (where \( \delta \) = the contragredient representation of \( \delta \)). We shall use this criterion below to determine the nonvanishing properties of the inner product

\[ \langle \psi_1 \ast \mathcal{L}(\varphi, \psi') | \psi_2^2 \rangle. \]

**REMARK 1.3:** We can compute

\[ \int_{\operatorname{Sp}_n(K_v)} \langle \pi_{Q_v}(G_v) \varphi_v | \pi_{Q_v}(w_1) \varphi_v' \rangle \langle (\psi_1)_v \ast G_v | (\psi_1)_v \rangle \Pi_v d\sigma_v(G_v) \]

explicitly in the case when \( \psi_1 \) is an \( \operatorname{Sp}_k(\mathcal{O}_v) \) invariant vector (spherical-type local component) of an irreducible representation \( \pi_v \) (\( v \) finite) and \( \varphi_v = \varphi_v' = \text{characteristic function of the lattice } M_{mk}(\mathcal{O}_v) \) in \( M_{mk}(K_v) \).

Indeed, if

\[ G_v = \begin{bmatrix} D & 0 \\ 0 & D^{-1} \end{bmatrix} \]

where \( D \) is the diagonal matrix

\[ \begin{bmatrix} \pi_{m_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \pi_{m_k} \end{bmatrix}, \]

then the above integral equals the infinite sum

\[ \sum_{D \in \Sigma} \omega_{\Pi_v}(D) r(D) \text{vol}(D) \langle \text{det } D | \Delta_Q \rangle \]

(\( \Delta_Q \) = discriminant of \( Q \)), where \( \omega_{\Pi_v} \) is the spherical function associated to the representation \( \Pi_v \) (i.e. \( \omega_{\Pi_v}(g) = \langle (\psi_1)_v \ast g_v | (\psi_1)_v \rangle_{\Pi_v} \), \( r(D) = q^{-(m_1 + \ldots + m_k) \cdot m/2} \), and vol\( (D) \) is the volume of the double coset \( \operatorname{Sp}_k(\mathcal{O}_v)D \operatorname{Sp}_k(\mathcal{O}_v) \) relative to \( d\sigma_v \) and \( \Sigma \) runs through the set of distinct \( \operatorname{Sp}_k(\mathcal{O}_v) \) double cosets in \( \operatorname{Sp}_k(K_v) \). Then it is possible to compute the infinite sum as a rational function (see § 4), and we get for \( k = 1 \) that the sum equals

\[
\frac{(1 + \langle \Delta_Q | \pi \rangle q^{-(m/2) - 1}) (1 - \langle \Delta_Q | \pi \rangle q^{-m/2})}{(1 - \langle \Delta_Q | \pi \rangle q^{-\sigma - (m/2) - 1}) (1 - \langle \Delta_Q | \pi \rangle q^{\sigma - (m/2) - 1})}
\]
where $\sigma$ is the complex number which determines the representation $\Pi_\nu$, i.e. $\Pi_\nu$ is equivalent to the unitary representation contained in the induced representation of $\text{Sp}_1(K_v)$ induced from the parabolic subgroup

$$\begin{bmatrix} \alpha & \beta \\ 0 & \alpha^{-1} \end{bmatrix} \mapsto |\alpha|^{s+1}.$$ 

In particular, noting that $\Pi_\nu$ must be unitary, we have that the Euler product

$$\prod_v \frac{(1 - \langle \Delta Q | \pi \rangle q^{-(m/2)-1}) (1 - \langle \Delta Q | \pi \rangle q^{-m/2})}{(1 - \langle \Delta Q | \pi \rangle q^{-(m/2)-1}) (1 - \langle \Delta Q | \pi \rangle q^{-(m/2)-1})}$$

(where the product runs over all finite primes $v$ in $K$ where $\text{Sp}_1(K_v)$ has a nonzero fixed vector in $\Pi_\nu$) is a nonzero function (i.e. $\sigma_v$ satisfies $|\text{Re}(\sigma_v)| < 1$).

Thus with the above computation, we can obtain the following consequence of Proposition 1.1.

For the remainder of this section, we assume that $K$ is a totally real field over $\mathbb{Q}$.

**Corollary 1 to Proposition 1.1:**

Let $Q$ be anisotropic form over $K$ with $m > 6$. Then

$$I_1(Q) = \bigoplus_{\Pi \in X_Q} \Pi$$

where $X_Q = \{\text{all cuspidal } \text{Sp}_1(\mathbb{A}) \text{ representations } \Pi \text{ satisfying (i) if } v \text{ is a prime where } Q_v \text{ is anisotropic, then the local factor } \Pi_v \text{ of } \Pi \text{ occurs discretely in the Weil representation } \pi_{-Q_v}; \text{ (ii) if } v \text{ is a real prime where } Q_v \text{ is isotropic of signature } (a_v, 1) (1, a_v) \text{ resp. and if } \Pi_v \text{ admits a highest weight vector, then } \Pi_v \text{ is either } (-) \text{ mock discrete series } (D_r^-, \text{ resp.}) \text{ if } Q_v \text{ is positive definite (negative definite).} \}$

**Remark 1.4:** We note that since $m > 6$, then all the primes where $Q_v$ is anisotropic are Archimedean real primes. Thus we can characterize the set $X_Q$ in a simpler fashion as follows: $X_Q = \{\text{all cuspidal } \text{Sp}_1(\mathbb{A}) \text{ representations } \Pi \text{ satisfying (i) if } v \text{ is a prime where } Q_v \text{ is anisotropic, then } \Pi_v \text{ is a discrete series of the form } D_r^- \text{ with } r \geq m/2 (D_r^+, \text{ resp.}, \text{ with } r \geq m/2) \text{ if } Q_v \text{ is positive definite (negative definite).} \}$

(i) if $v$ is a prime where $Q_v$ is anisotropic, then $\Pi_v$ is a discrete series of the form $D_r^-$ with $r \geq m/2$ ($D_r^+$, resp., with $r \geq m/2$) if $Q_v$ is positive definite (negative definite).

(ii) if $v$ is a real prime where $Q_v$ has signature $(a_v, 1) (1, a_v)$ and if $\Pi_v$ admits a highest weight vector, then $\Pi_v$ is a discrete series of the form $D_i^-$ with $i \geq 2$ or a mock discrete series $H_0^- (D_i^+, \text{ resp. with } i \geq 2 \text{ or a mock discrete series } H_0^+)$.


PROOF: We must show two statements. First, if \( \Pi \) occurs in \( X_Q \), then for some \( \psi \in \Pi \), \( \zeta_{\psi, \psi} \neq 0 \) (for a suitable choice of \( \varphi \) in \( S[\mathcal{M}_{m1}(A)] \)). Secondly, if \( \Pi \) does not occur in \( X_Q \), we must show \( \zeta_{\psi, \psi} \equiv 0 \) for all \( \psi \) in \( \Pi \) and \( \varphi \) in \( S[\mathcal{M}_{m1}(A)] \).

From Remark 1.3 (noting that \( \varphi \) and \( \varphi' \) are factorizable), we have for any cuspidal representation \( \Pi \) occurring in \( L^2_{\text{cusp}}(\text{Sp}_1(A)) \) that \( (\psi_1, \psi_2 \text{ are elements of } \Pi) \)

\[
\langle \psi_1 \ast \mathcal{L}(\varphi, \varphi') | \psi_2^\Delta \rangle = \prod_{v \in \mathcal{R}'} \frac{1 + \langle \Delta_Q \pi \rangle q_v^{-(m/2)-1}}{1 - \langle \Delta_Q \pi \rangle q_v^{-(m/2)-1}} \times \left\{ \prod_{v \in \mathcal{R}''} \int_{\text{Sp}_1(K_v)} \langle \pi_{\mathfrak{o}_v}(G_v) \varphi_v | \pi_{\mathfrak{o}_v}(w_1) \varphi_v' \rangle \right. \\
\times \left. \left\langle (\psi_1)_v \ast G_v | (\psi_2^\Delta)_v \right\rangle \mathrm{d} \sigma_v(G_v) \right\}
\]

where \( \mathcal{R}' = \text{set of all primes in } K \text{ which are finite and where } (\psi_1)_v \text{ and } (\psi_2^\Delta)_v \) are \( \text{Sp}_1(\mathcal{O}_v) \) invariant vectors in \( \Pi_v \) and \( \varphi_v = \varphi_v' = \text{characteristic function of the lattice } \mathcal{M}_{m1}(\mathcal{O}_v) \). We note that \( \mathcal{R}'' \), the remaining set of primes in \( K_v \), is a finite set. In any case, we note that the non-vanishing of \( \langle \psi_1 \ast \mathcal{L}(\varphi, \varphi') | \psi_2^\Delta \rangle \) depends only on the set \( \mathcal{R}'' \) and the corresponding local integrals appearing in its product.

Then we let \( v \in \mathcal{R}'' \) and suppose \( v \) is finite. By using the tensor product properties of the Weil representation, it is possible to write \( \langle \pi_{Q_v}(g_v) \varphi_v | \pi_{Q_v}(w_1) \varphi_v' \rangle \) as a product

\[
\langle \pi_{Q_v}(g_v) \varphi_v^1 | \pi_{Q_v}(w_1) \varphi_v^{1'} \rangle \langle \pi_{Q_v}(g_v) \varphi_v^2 | \pi_{Q_v}(w_1) \varphi_v^{2'} \rangle,
\]

where \( Q_v \) splits as an orthogonal direct sum \( (Q_1) \oplus (Q_2) \) with \( (Q_1) \) a split form and \( (Q_2) \) an anisotropic form (of dimension 0, 2, or 4) and \( \varphi_v = \varphi_v^1 \otimes \varphi_v^2, \varphi_v' = \varphi_v^{1'} \otimes \varphi_v^{2'} \). But this means that the local integral is given by integrating \( \langle \pi_{Q_v}(g_v) \varphi_v^1 | \pi_{Q_v}(w_1) \varphi_v^{1'} \rangle \) against the product \( \langle \pi_{Q_v}(g_v) \varphi_v^2 | \pi_{Q_v}(w_1) \varphi_v^{2'} \rangle \langle (\psi_1)_v \ast g_v(\psi_2^\Delta)_v \rangle \). Choosing \( \varphi_v^2 \) and \( \varphi_v^{2'} \) appropriately, it is simply a matter (to show the non-vanishing of this local integral) of showing that \( \langle \pi_{Q_v}(g_v) \varphi_v^1 | \pi_{Q_v}(w_1) \varphi_v^{1'} \rangle \) is a sufficiently generic function, i.e., arbitrary of compact support. However we recall from \([R-2]\) that the family of functions

\[
g_v \rightarrow \pi_{(Q_1)_v}(\Omega_0 g_v)(\rho_v \otimes \tilde{\rho}_v')(0)
\]
(g_v \in \text{Sp}_2(K_v) and \rho_v, \rho'_v \text{ arbitrary functions in } S[M_{m_1}(K_v)], \text{ span}

\text{Ind}_{\text{Sp}_2}^{\text{Sp}_1} \left( \begin{array}{cc}
A & X \\
0 & (A')^{-1}
\end{array} \right) \rightarrow \det |A|^{-m'_v/2}.

(Here m' = dimension of the space on which (Q_1)_v lives). Then by suitable choice of \rho_v and \rho'_v and by restriction to \text{Sp}_1 \times \{1\}, we obtain a function g_v \sim \pi_{(Q_1)}(\Omega_0(g_v, 1)(\rho_v \otimes \overline{\rho'_v})(0)) whose integral against the product function above is nonzero!

Thus, so far, we have shown that if g \in any cuspidal irreducible representation, then \pi will appear in S_1(Q) if and only if \text{Hom}_{\text{Sp}_1(K_v)}(\pi_{Q_v}, \Gamma_1) \neq \{0\} for all Archimedean primes in K_v. Thus to finish the proof, we need only to determine the representations \chi of \text{Sp}_1(K_v) which satisfy \text{Hom}_{\text{Sp}_1(K_v)}(\pi_{Q_v}, \sigma_v) \neq 0. Indeed, if Q_v is anisotropic, then we know exactly which \sigma_v appear in \pi_{Q_v} (see [R-S]). If Q_v is isotropic and v is real, then we known from the computation given in § 4 (see also [R-S, § 5 and § 6]) that \text{Hom}_{\text{Sp}_1(R)}(\pi_{Q_v}, H_s) \neq 0 for any principal series representation H_s with |\text{Re}(\sigma)| < 1 (see comments below). Thus it suffices to check all representations of discrete series type (both usual discrete series and mock discrete series) for \text{Sp}_1(R). Here we use the results of [R-S]; indeed if b > 1 (where signature(Q) = (a, b)), then all discrete series representations of \text{Sp}_1(R) occur in \pi_{Q_v}. If b = 1, then only discrete series of the type D^+_0 occur in \pi_{Q_v}. We note here that in § 4, with b = 1, we will show that \text{Hom}_{\text{Sp}_1(R)}(\pi_{Q_v}, H_0^+) \neq 0 (where H_0^+ = (+) mock discrete series) and from [R-S, § 6], we have \text{Hom}_{\text{Sp}_1(R)}(\pi_{Q_v}, H_0^-) = \{0\} (where H_0^- = (−) mock discrete series).

Q.E.D.

REMARK 1.5: We note that the main obstacle to extending the above Corollary to the general \text{Sp}_n case consists of two technical points: (1) to establish a formula analogous to the one for Z(\omega, \lambda) (defined and derived in § 4 for \text{Sp}_1) in order to prove the nonvanishing of the main part of the Euler product for \langle \psi_1 \bullet \mathcal{L}(\varphi, \varphi')|\psi_2 \rangle given above and (2) to characterize at the real Archimedean primes which unitary representations of \text{Sp}_n(R) occur in the Weil representation \pi_Q on S[M_{mn}(R)].

By dualizing the lifting problem above, we can now give a complete characterization of the lift from \text{O}(Q)(\mathbb{A}) to \text{Sp}_1(\mathbb{A}).

THEOREM 1.6: Let Q be an anisotropic form with m > 6. Then the lifting map from R_1(Q) into L_{\text{cusp}}^2(\text{Sp}_1(\mathbb{A})) has as an image the space I_1(−Q).

Moreover if Q is defined over \mathbb{Q}, then the lifting map from R_1(Q) to I_1(−Q) satisfies the global Howe duality conjecture and is multiplicity preserving in the sense of [R-2].
PROOF: The proof of the second statement follows from the first statement coupled with the results of [R-2, § 2].

On the other hand, we recall if $\psi$ is a cusp form on $Sp_1(\mathbb{A})$ and $f$ is a cusp form on $O(Q)(\mathbb{A})$ with

$$\ell_{\psi, \psi}(g) = \int_{Sp_1(K) \backslash Sp_1(\mathbb{A})} \psi(G) \theta_{\psi}(G, g) \, dG$$

and

$$\mathcal{L}_{\psi, f}(g) = \int_{O(Q)(K) \backslash O(Q)(\mathbb{A})} f(g) \theta_{\psi}(G, g) \, dg,$$

then for fixed $\varphi$ in $S[M_{m_1}(\mathbb{A})]$, the linear maps $\psi \rightarrow \ell_{\psi, \psi}$ and $f \rightarrow \mathcal{L}_{\psi, f}$ are "twisted" adjoint to each other, that is

$$\int_{O(Q)(K) \backslash O(Q)(\mathbb{A})} \ell_{\psi, \psi}(g) \overline{f(g)} \, dg = \int_{Sp_1(K) \backslash Sp_1(\mathbb{A})} \psi(G) \overline{\mathcal{L}_{\psi, f}(G \delta_0)} \, dG_0.$$ 

Then we recall that $\pi_{-Q}(G) = \pi_{G^0}(G^0)$ for all $G \in Sp_n(\mathbb{A})$. In particular this implies that $\beta \in I_1(-Q)^\perp$ if and only if $\beta_{G^0}^{\perp} (= \text{def} \beta(G^0))$ for $G \in Sp_n(\mathbb{A}) \in I_1(Q)^\perp$. Hence it follows by a standard argument that if $f \in R_1(Q)$, then $L_{\psi, f} \in I_1(-Q)$. (Here we use the inner product formula given above.) Similarly we have that if $\beta \in I_1(-Q)$, then $\ell_{\psi, \beta} \in R_1(Q)$; this implies that $I_1(-Q)$ equals the image of $R_1(Q)$ under lifting.

Q.E.D.

REMARK 1.7: We consider an automorphic cuspidal representation $\Pi$ in

$L^2_{\text{cusp}}(Sp_1(\mathbb{A}))$, where $\Pi_{\infty} = \otimes D_{m/2}$ (that is, at every real prime we specify the representation $D_{m/2}$). Such representations certainly exist in $L^2_{\text{cusp}}(Sp_1(\mathbb{A}))$. Then we consider a cuspidal representation $\sigma$ in

$L^2_{\text{cusp}}(O(Q)(\mathbb{A}))$ which maps onto $\Pi$ via the lifting map. We know from [R-2, § 2] that the local components $\sigma_v$ of $\sigma$ must satisfy

$$\text{Hom}_{Sp_1(K_v) \times O(Q_v)}(S[M_{m_1}(K_v)], \Pi_v \otimes \sigma_v) \neq 0$$

(all components of $\sigma$ are self-contragredient). Thus it follows that if $Q_v$ is anisotropic, then $\sigma_v = \text{trivial representation of } O(Q_v)$ and if $Q_v$ is isotropic, then $\sigma_v$ is the unique representation of $O(Q_v)$ which has the property that

$$H^*(LA(O(Q_v)), M_v, \sigma_v) \neq 0$$

where $H^* = \text{relative Lie algebra cohomology and } M_v = \text{maximal compact}$
subgroup of $O(Q)$. Thus we have constructed an automorphic cuspidal representation $\sigma$ of $O(Q)(A)$ such that for a real prime $v$ in $K$ where $Q_v$ is isotropic, $\sigma_v$ has a nonvanishing cohomology at level $b_v$.

**Remark 1.8:** The import of Corollary 1 to Proposition 1.1 and Theorem 1.6 is a complete characterization of the image of the lifting $\beta_1$ from $O(Q)(A)$ to $Sp_1(A)$; moreover, in the case that $Q$ is anisotropic over $Q$, we have that the lifting satisfies the global Howe duality conjecture and is multiplicity preserving. Moreover, we see that if $Q_1$ and $Q_2$ are any two anisotropic forms of equal dimension greater than 6 over $K$ and have the same signature at every real Archimedean prime of $K$, then $I_1(Q_1) = I_1(Q_2)$; that is, $I_1(Q)$ depends only on the signature of $Q$ at the Archimedean real primes of $K$.

§ 2. A “comparison of trace” formula

It is possible to draw a conclusion about classical cusp forms from the above work. Namely, we let $Q$ be a unimodular quadratic form over $\mathbb{Z}$ on $R^{8, t}$. In particular, this means that if we consider the genus of $Q$, we get a finite class of lattices $L$, on $Z^{8, t}$ such that the finite set of double cosets

$O(Q)(Q) \setminus O(Q)(A)/U_{Z^{8, t}}$,

$U_{Z^{8, t}} = \text{the stabilizer of } Z^{8, t} \text{ in } O(Q)(A)$,

is in one-one correspondence with the lattices $L_i = g_i(Z^{8, t})$, $g_i$ a coset representative above. Then we know that the associated family of $\theta$-series

$$\left( \theta_{L_i}(\tau) = \sum_{\xi \in L_i} \frac{\tau}{\pi \sqrt{-1} n(L_i)} Q(\xi), \right)$$

the norm of $L_i$ given in [E, § 8]}

belongs to the space of holomorphic modular forms $M_{4t}(SL_2(\mathbb{Z})) =$

$$\left\{ f: H \to \mathbb{C}, \text{ holomorphic } |f\left(\frac{az + b}{cz + d}\right) = (cz + d)^4 f(z) \right\}$$

with

$$\left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in SL_2(\mathbb{Z}).$$

Let $X_Q$ = the subspace of $M_{4t}(SL_2(\mathbb{Z}))$ spanned by the $\theta_{L_i}$ above.

We know that $SL_2(A) = Sp_1(A)$ satisfies the strong approximation Theorem over $Q$, the rationals. This implies that we can identify the
space $S_{4t}(SL_2(\mathbb{Z}))$, the space of cusp forms in $M_{4t}$, with
\[ \oplus \{ v \in \Pi | \Pi = \Pi_{\text{fin}} \otimes \Pi_{\infty} \text{ where } \Pi_{\infty} = D_{k}^{+} \}
\]
the component of $v$ in $\Pi_{\text{fin}}$ is fixed under $|K| = \prod_{\nu < \infty} SL_2(\mathcal{O}_K)$
via the map $\phi(g_{\infty})f(g_{\infty}, \sqrt{-1})^k = f_{\phi}(g_{\infty}(-1))$, where $f_{\phi} \in S_{4t}(SL_2(\mathbb{Z}))$
and $\phi \in L_{cusp}^2(Sp_1(\mathbb{A}))$. (Here $j$ is the usual canonical automorphy factor,
and $g \in Sp_1(\mathbb{A})$ has the decomposition $g = g_{Q} \cdot g_{O} \cdot g_{U}$, $g_{Q} \in Sp_1(\mathbb{Q})$, $g_{\infty} \in Sp_1(\mathbb{R})$, and $g_{U} \in K$.

Then we know that $UZ_{8t} = O(Q)(\mathbb{R})L$ where $L = \prod_{\nu < \infty} L_{\nu}$
with $L_{\nu} = \text{the stabilizer in } O(Q_{\nu})$ of the lattice $O_{\nu} \otimes \mathbb{Z}^{8t \nu}$. Then we have that
$F(O(Q)(\mathbb{Q}) \setminus O(Q)(\mathbb{A}) / U_{Z^{8t}}) = \{ f : O(Q)(\mathbb{A}) \to \mathbb{C} | f \text{ is invariant on the left and right by } O(Q)(\mathbb{Q}) \text{ and } U_{Z^{8t}}, \text{ resp.} \}$
is a finite dimensional space and, in fact, using the usual Tamagawa measure on $O(Q)(\mathbb{Q}) \setminus O(Q)(\mathbb{A})$,
we have $F \subset L^2(O(Q)(\mathbb{Q}) \setminus O(Q)(\mathbb{A}))$, and if $F = \{ f \in F | \langle f | 1 \rangle = 0 \}$, then
$F \subset L^2_{cusp}(O(Q)(\mathbb{A}))$. Hence $F$ can be identified to
\[ L^2_{cusp}(O(Q)(\mathbb{A}))^{UZ_{8t}}. \]
In particular, if $\sigma$ appears in $L^2_{cusp}$, then $\sigma$ has a nonzero $U_{Z^{8t}}$, invariant if
and only if $\sigma_{\infty} = 1_{O(Q)(\mathbb{R})}$ and $\sigma_{\text{fin}} = \otimes_{\nu < \infty} \sigma_\nu$ with $\sigma_{\nu} \neq 0$ for all $\nu$. Moreover
\[ \sigma^{UZ_{8t}} \]
is, at most, a one dimensional space. Hence $F$ is in bijective correspondence
with the set of cuspidal representations $\sigma$ in $L^2_{cusp}(O(Q)(\mathbb{A}))$ which satisfy
\[ \sigma^{UZ_{8t}} \neq 0. \]

Then we choose the function $\varphi \in S(M_{m_1}(\mathbb{A}))$ such that $\varphi_{\infty}(X) = e^{-\pi|X|}$
and $\varphi_{\nu}(X) = \text{characteristic function of } M_{m_1}(\mathcal{O}_\nu)$. Then using such a $\varphi$ and
choosing the characteristic function $\rho_\tau$ of the double coset $O(Q)(\mathbb{Q})g_{UZ_{8t}}$, we have that $f_{\varphi_{\nu}}(\tau)$ is a nonzero multiple of $\theta_{\nu}$ [R-3]. On the other hand, if we consider $F \cap R_{1}(Q)$, then we know that the lifting map $\beta_1$ is injective on this space. In particular, we let $\{ \psi_i \}$ be the basis of functions
in $L^2_{cusp}(Sp_1(\mathbb{A}))$ which lie in a unique $\Pi$ of $Sp_1(\mathbb{A})$ (which correspond to $S_{4t}(SL_2(\mathbb{Z}))$ above and which are eigenfunctions of the $\Delta$ operation on $Sp_1(\mathbb{A})$ (denote by $R_{\Delta}$, the operation $f(X) \to f(X^3)$). A simple computation shows that $R_{\Delta}$ acts trivially on $S_{4t}(\mathbb{Z})$. By the above arguments, it is easy to see that $\ell_{\varphi, \psi} \neq 0$. Hence $\ell_{\varphi, \psi}$ will lie in a unique irreducible component of $L^2_{cusp}(O(Q)(\mathbb{A}))$. On the other hand, using Proposition 1.1
\[ \langle \ell_{\varphi, \psi} | \ell_{\varphi, \psi} \rangle = \langle \psi_i * \mathcal{L}(\varphi, \psi) | \psi_j \rangle. \]
Then if $i \neq j$, we have $\ell_{q, \psi_i} \perp \ell_{q, \psi_j}$. Thus, in any case, we have that each $\ell_{q, \psi_i}$ lies in a distinct irreducible of $R_1(Q)$ and that $\ell_{q, \psi_i}$ span $\mathbb{F}$. Thus we have deduced the well known fact that $X_Q = M_{4t}(SL_2(\mathbb{Z}))$ (see [E]).

But using Proposition 1.1 and the results of [R-1], it is now possible to make a comparison of traces Theorem about Hecke operators on $S_{4t}$ and $F \cap R_1(Q)$. We recall the relevant results of [R-1]. If $v$ is a finite prime, we let $\mathcal{H}_1 = \mathcal{H}(O(Q_v) \backslash / L_v) =$ the algebra of $L_v$ biinvariant, compactly-supported functions on $O(Q_v)$ and $\mathcal{H}_2 = \mathcal{H}(Sp_1(K_v) \backslash / Sp_1(\mathbb{Q}_v)) =$ the algebra of $Sp_1(\mathbb{Q}_v)$ biinvariant, compactly-supported functions on $Sp_1(K_v)$. Then there exists a surjective homomorphism $\omega_{Q_v} : \mathcal{H}_1 \to \mathcal{H}_2$ such that the ideal

$$I_v(Q_v) = \{ \psi \in \mathcal{H}_1 \otimes \mathcal{H}_2 | \pi_{Q_v}(\psi) = 0 \}$$

is generated by $\{ f \otimes \Pi - \Pi f \otimes \omega(f) | f \in \mathcal{H}_1 \}$. In simple terms this means that for each $f \in \mathcal{H}_1$, $\pi_{Q_v}(f)(\varphi) = \pi_{Q_v}(\omega(f))(\varphi)$ for all $\varphi \in \mathcal{S}(M_{ml}(Q_v))$. Then we have the following Theorem.

**THEOREM 2.1:** Let $f \in \mathcal{H}_1$. Let $\rho_\ast$ denote the corresponding representation of $O(Q)$ or $Sp$ on the corresponding space of cusp forms. Then

$$\text{Trace}(\rho_{O(Q)}(f) |_{F \cap R_1(Q)}) = \text{Trace}(\rho_{Sp_1}(\omega_{Q_v}(f) \ast) |_{S_{4t}})$$

where $\ast$ denotes the operation $\varphi^\ast(x) = \varphi(x^{-1})$ on $\mathcal{H}_2$.

**REMARK 2.2:** Thus we have essentially a simple example of an abstract comparison of traces formula between the groups $Sp_1$ and $O(Q)$ for Hecke operators acting on the appropriate spaces of cusp forms. (See [La] for the general philosophy of comparison of traces relative to the base change operation.)

**PROOF OF THEOREM 2.1:** Here we embed the Hecke operators $f \in \mathcal{H}_1$ into $C_c^\infty(O(Q)(\mathbb{A}))$ in the standard fashion. Namely if we let $f = f_v$, the characteristic function of a double coset $L_v \xi_v L_v$, then we let the corresponding element in $C_c^\infty(O(Q)(\mathbb{A}))$ be

$$\frac{1}{\text{vol}(L_v \xi_v L_v)} f_v \otimes \prod_{w \neq v} \chi_w^\ast,$$

where $\chi_w = \text{characteristic function of the maximal compact subgroup } L_w$ in $O(Q_w)$.

Similarly we embed $\omega_{Q_v}(f) \in \mathcal{H}_2$ into $C_c^\infty(Sp_1(\mathbb{A}))$ in a similar fashion, but with one slight difference. That is, let $g_v$ be the characteristic function of a double coset $Sp_1(\mathbb{Q}_v) \mu_v Sp_1(\mathbb{Q}_v)$. Then we let the corresponding
element in $C_\infty^c(\text{Sp}_1(A))$ be
\[
\frac{1}{\text{vol}(\text{Sp}_1(\mathcal{O}_v)\mu_{\nu}\text{Sp}_1(\mathcal{O}_v))} g_v \otimes \prod_{w \neq \nu, \infty} \chi_w^* \otimes f_w^*
\]
where $\chi_w^*$ = characteristic function of the maximal compact subgroup $\text{Sp}_1(\mathcal{O}_w)$ and $f_w^*$ is a $C_\infty$ function of compact support such that $\sigma_{D_\nu}(f_w^*) = 0$ (for all $j$), $\sigma_{D_\nu}(f_w^*) = 0$ (for $j \neq 2t$), and $\sigma_{D_\nu}(f_w^*)$ = identity operator on the lowest weight space for $SO(2)$ and trivial on the remaining weight spaces (such a function $f_w^*$ is well known to exist by a result of Duflo and Langlands).

From the comments above, we have that
\[
\text{Tr}(\rho_Q(f)|_{F \cap R_1(Q)}) = \sum_i \frac{\langle \ell_{q, \psi_i} \ast f | \ell_{q, \psi_i} \rangle}{\langle \ell_{q, \psi_i} | \ell_{q, \psi_i} \rangle}.
\]
But we have that $\ell_{q, \psi_i} \ast f = \ell_{q_0(f)(q), \psi_i} = \ell_{q_0(\omega_Q(f)q), \psi_i} = \ell_{q, \psi_i} \ast \omega_Q(f) \ast \psi_i$. On the other hand,
\[
\langle \psi \ast \mathcal{L}(\varphi, \varphi) | \psi \rangle = \langle \ell_{q, \psi_i} | \ell_{q, \psi_i} \rangle
\]
for all $\psi$. With these facts we have
\[
\frac{\langle \ell_{q, \psi_i} \ast f | \ell_{q, \psi_i} \rangle}{\langle \ell_{q, \psi_i} | \ell_{q, \psi_i} \rangle} = \frac{\langle \psi_i \ast \omega_Q(f) \ast \mathcal{L}(\varphi, \varphi) | \psi_i \rangle}{\langle \psi_i \ast \mathcal{L}(\varphi, \varphi) | \psi_i \rangle} = \frac{\langle \psi_i \ast \omega_Q(f) \ast \psi_i \rangle}{\langle \psi_i | \psi_i \rangle}.
\]
(We note here that $\psi_i$ is an eigenfunction of $\mathcal{L}(\varphi, \varphi)$.)

Q.E.D.

**Remark 2.3:** We note that the operator $\mathcal{L}(\varphi, \varphi)$ operating on the eigenvector $\psi_i$ will have an eigenvalue (aside from a positive constant independent of $\psi_i$) of the form
\[
\frac{2}{|m|} \frac{L((m/2) - 1, \Pi_i, r_1)}{\xi((m/2) - 1, \xi(m/2)) \left( \prod_{q < \infty} (1 + q^{-(m/2 - 1)}) \right)}
\]
where $2/|m| = 1/(d(m))$, $d(m)$ = the formal degree of the discrete series representation $D_{m/2}$ and $L(\sigma, \Pi_i, r_1)$ is the Langlands L function given by
\[
\prod_{\rho < \infty} L_p(\sigma, (\Pi_i)_p, r_1)
\]
(see § 4 for the definition of $L_p(\ldots)$). Here $\Pi_i$ is the irreducible cuspidal representation of $\text{Sp}_1(\mathbb{A})$ which contains the vector $\psi_i$.

### § 3. Convergence of integrals in (1–2)

We show here the absolute convergence of the integrals appearing in (1–2). Indeed, we first note that $\text{Sp}_r(\mathbb{A}) = P(\mathbb{A}) \cdot K$ where $P$ is any parabolic of the following form: $P$ has a Levi factor of the type $\text{Sp}_j \times G_{\ell,-j}$. Then the integrals in (1–2) are of 2 types:

$$
\int_{\Delta_0(K) \backslash \text{Sp}_k(\mathbb{A}) \times \text{Sp}_l(\mathbb{A})} \psi_1(G) \psi_2(G') \pi_Q(\Omega_0(G, G'))(\varphi \otimes \overline{\varphi}')(0) \, dG \, dG'.
$$

(1)

We note that this integral can be expressed as a product

$$
\int_{\text{Sp}_k(\mathbb{A})} \left( \int_{\Delta_0(K) \backslash \Delta_0(\mathbb{A})} \psi_1(Gh) \psi_2(G) \, dG \right) \pi_Q(\Omega_0(h, 1))(\varphi \otimes \overline{\varphi}')(0) \, dh.
$$

However, by taking absolute values it suffices to show that

$$
\left| \int_{\text{Sp}_k(\mathbb{A})} \left( \int_{\Delta_0(K) \backslash \Delta_0(\mathbb{A})} |\psi_1(Gh)\psi_2(G)| \, dG \right) \pi_Q(\Omega_0(h, 1))(\varphi \otimes \overline{\varphi}')(0) \, dh \right|
\leq \|\psi_1\|^2 \|\psi_2\|^2
$$

$$
\int_{\text{Sp}_k(\mathbb{A})} |\pi_Q(\Omega_0(h, 1))(\varphi \otimes \overline{\varphi}')(0)| \, dh < \infty.
$$

This is equivalent to showing $h \mapsto \pi_Q(\Omega_0(h, 1))(\varphi \otimes \overline{\varphi}')(0)$ is a $L^1$ function on $\text{Sp}_k(\mathbb{A})$ (which we show in Proposition 1.1).

$$
\int_{\text{Sp}_k(\mathbb{A}) \times \text{Sp}_k(\mathbb{A}) \backslash \text{Sp}_k(\mathbb{A}) \times \text{Sp}_k(\mathbb{A})} \cdots
$$

$$
\int_{(\text{Sp}_k(\mathbb{A}) \times P(\mathbb{A})) \times K \times K'} \psi_1(\alpha \cdot k) \psi_2(\alpha' \cdot k') \pi_Q(\Omega_1(ak, \alpha'k'))(\varphi \otimes \overline{\varphi}')(0) \, d\alpha \, d\alpha' \, dk \, dk'.
$$

(2)

Thus assuming $\psi_1$ and $\psi_2$ and $\varphi \otimes \overline{\varphi}'$ are $K$ finite functions, it suffices to
prove that
\[ \int_{\text{Sp}_k \times \text{Sp}_{k-1}(\mathbb{A}) \setminus P \times P'(\mathbb{A})} |\psi_1(\alpha)\psi_2(\alpha')| \]
\[ \times \pi_Q(\Omega_1(\alpha, \alpha'))(\varphi \otimes \bar{\varphi}')(0)|d\alpha d\alpha' < \infty. \]

Here, \( P \) (resp. \( P' \)) is the parabolic subgroup having \( \text{Sp}_k \times G_{k-1} \) (resp. \( \text{Sp}_k \times G_{k-1}' \)) as its Levi factor. But then we have that the integral above can be decomposed as follows:

\[ \int_{\text{Sp}_k(\mathbb{A})} |\pi_Q(\Omega_1(h, 1))(\varphi \otimes \bar{\varphi}')(0)| \]
\[ \leq \left\{ \int_{(U_1 \times U_2)(K) \setminus U_1 \times U_2(\mathbb{A})} \left( \int_{G_1 \times G_2(\mathbb{A}) \setminus G_1 \times G_2(\mathbb{A})} \left( \int_{\text{Sp}_k(\mathbb{A}) \setminus \text{Sp}_k(\mathbb{A})} |\psi_1(n(x)g_1g_2h)| |\psi_2(n(y)g_1'g_2)| \right) dg_1 dg_2 \right) \right\} dh \]

where \( G_1 = G_{k-1}, \ G_2 = G_{k-1}' \), \( U_1 \) and \( U_2 \) are unipotent radicals in \( P \) and \( P' \). Then by using the same idea as in (1) above, we have that the inner integral is majorized by

\[ \int_{\text{Sp}_k(\mathbb{A}) \setminus \text{Sp}_k(\mathbb{A})} |\psi_1(n(x)g_1g_2)|^2 dg_2 \cdot \int_{\text{Sp}_k(\mathbb{A}) \setminus \text{Sp}_k(\mathbb{A})} |\psi_2(n(y)g_1'g_2)|^2 dg_2. \]

Hence, it suffices to show that

\[ \int_{U_1(K) \setminus U_1(\mathbb{A})} \int_{G_1(K) \setminus G_1(\mathbb{A})} |\det g_1|^{m/2} \]
\[ \times \left\{ \int_{\text{Sp}_k(\mathbb{A}) \setminus \text{Sp}_k(\mathbb{A})} |\psi_1(n(x)g_1g_2)|^2 dg_2 \right\} dg_1 dx < \infty \]

and that the function \( h \mapsto \pi_Q(\Omega_1(h, 1))(\varphi \otimes \bar{\varphi}')(0) \) is a \( L^1 \) function on \( \text{Sp}_k(\mathbb{A}) \).

For the first type of integral, we consider integrating \( |\psi_1(n(x)g_1g_2)|^2 \)
over the set \( \Sigma_1 \times D_t \times S_{t_1} \subset (U_1 \times G_1 \times \text{Sp}_k)(\mathbb{A}) \) where \( \Sigma_1 \) is any compact subset of \( U_1 \), \( D_t \), a fundamental domain in \( G_1(K) \setminus G_1(\mathbb{A}) \) of the form \( \Lambda \cdot \{ \text{diag}(x_{r+1}, \ldots, x_k) \mid x_i \geq t_1x_{i+1} \} \) (\( \Lambda \) a compact set) and \( S_{t_1} \), a fundamental domain in \( \text{Sp}_k(K) \setminus \text{Sp}_k(\mathbb{A}) \) of the form \( \Lambda' \cdot \{ \text{diag}(x_1, \ldots, x_r) \mid x_i \geq t_2x_{i+1}, x_r \geq t_2 \} \) (\( \Lambda' \) a compact set). We note that for the construction of fundamental domains it suffices to let \( t_1 = t_2 = t \). Then it suffices to show
that
\[ \int |\psi_1(D(x_1, \ldots, x_r)D(x_{r+1}, \ldots, x_k))|^2 \]
\[ \left( \prod_{i=1}^{i=r} x_i^{2i+1-r+m/2} \right) \left( \prod_{i=1}^{i=k} x_i^{-2k+2i-2+m/2} \right) \prod_{i=1}^{i=n} \frac{dx_i}{x_i} < \infty. \]

where the range of integration is
\[ \{(x_1, \ldots, x_k) \in (\mathbb{R}^+)^k | x_1 \]
\[ \geq tx_2, \ldots, x_{r-1} \geq tx_r, x_{r+1} \geq tx_{r+2}, \ldots, x_{k-1} \geq tx_k, x_k \geq t \} \]

We then show that such an integral can be decomposed in integration over certain sets. We note that the fundamental domain in the integration above can be decomposed into a union of the following sets:
\[ \bigcup_w \{(x_1, \ldots, x_k) | x_1 \geq tx_2, \ldots, x_{k-1} \geq tx_k, x_k \geq t \} \]

where \( w \) ranges over the cosets in Weyl group \((\text{Sp}_k) \mod(\text{Weyl group}(G_{e'})) \times \text{Weyl group}(\text{Sp}_{k-r})\). Then we use the rapid decrease properties of \( \psi_1 \) (a cusp form) on the various translates of the Siegel domain for \( \text{Sp}_k \) to deduce the finiteness of the above integrals.

On the other hand, we have by a similar computation to that given in Proposition 1.1 that
\[ \pi_Q(\Omega_j(h, 1))(\varphi \otimes \overline{\varphi})(\theta) \]
\[ = \int_{M_{mr}(A) \times M_{mr}(A)} \tau(\text{Tr}(\delta_1 Q \delta_2)) \pi_Q(h)(\varphi)(\delta_1) \overline{\varphi}(\delta_2) d(\delta_1) d(\delta_2). \]

Then using the same argument as in Proposition 1.1, we see that the above function is \( L^1 \) on \( \text{Sp}_k(A) \).

§ 4. Computation of local factors

(1) We first assume that \( v \) is finite in \( K \). Then we compute the series:
\[ Z(\omega_s, \lambda) = \sum_{n=0}^{\infty} \omega_s \left( \begin{array}{cc} \pi^n & 0 \\ 0 & \pi^{-n} \end{array} \right) \text{vol} \left[ K \cdot \left( \begin{array}{cc} \pi^n & 0 \\ 0 & \pi^{-n} \end{array} \right) \cdot K \right] q^{-n\lambda} \]

where the Haar measure on \( \text{Sp}_1(K_v) \) is so normalized that \( \text{vol}(K) = 1 \).
Then we know that
\[
\text{vol}\left[ K \left( \begin{array}{cc} \pi^n & 0 \\ 0 & \pi^{-n} \end{array} \right) K \right] = ((1 + q)/q)q^{2n}
\]
for \(n > 1\) and that
\[
\omega_s \left( \begin{array}{cc} \pi^n & 0 \\ 0 & \pi^{-n} \end{array} \right) = \left( \frac{q}{q + 1} \right) \left( \frac{q^{-n}}{1 - q^{-s}} \right)
\]
\[
\times \left[ q^{-sn}(q^{-1} - q^{-s}) + q^{sn}(1 - q^{1-s}) \right].
\]
Thus \(Z(\omega_s, \lambda)\) becomes the series
\[
1 + \left( \frac{1}{1 - q^{-s}} \right)
\]
\[
\times \sum_{n=1}^{n=+\infty} \left\{ q^{(-s-\lambda+1)n}(q^{-1} - q^{-s}) + q^{(s-\lambda+1)n}(1 - q^{1-s}) \right\}.
\]
Then using geometric series expansion we have that \(Z(\omega_s, \lambda)\) equals
\[
1 + \left( \frac{1}{1 - q^{-s}} \right) \left( \frac{q^{(-s-\lambda+1)}}{1 - q^{(-s-\lambda+1)}} (q^{-1} - q^{-s}) \right)
\]
\[
+ \left( \frac{q^{(s-\lambda+1)}}{1 - q^{s-\lambda+1}} \right) (1 - q^{1-s}) \right) \right].
\]
But then rationalizing with common denominators, we have that \(Z(\omega_s, \lambda)\) equals
\[
\frac{(1 + q^{-\lambda-1})(1 - q^{-\lambda})}{(1 - q^{-s-\lambda-1})(1 - q^{s-\lambda-1})}.
\]

**Remark 4.1:** We note that \(Z(\omega_s, \lambda)\) is related to the local Langlands \(L\)-function for the representation \(\pi_s\). We know that the \(L\)-group of \(\text{Sp}_s\) is \(\text{SO}(3, \mathbb{C})\) and that the \(L\)-factor associated to \(\pi_s\) and the standard 3-dimensional representation \(r_1\) of \(\text{SO}(3, \mathbb{C})\) is given by (see [B])
\[
L_p(w, \pi_s, r_1) = \frac{1}{(1 - q^s q^{-w})(1 - q^{-s} q^{-w})(1 - q^{-w})}
\]
Then we have that
\[
Z(\omega_s, \lambda) = L_p(\lambda - 1, \pi_s, r_1) \frac{1}{\xi_p(\lambda - 1) \xi_p(\lambda)} \frac{1}{\xi_p\left( \lambda - 1 + \frac{\pi \sqrt{1}}{\log q} \right)}
\]
where \( \zeta_p \) is the local zeta factor function \( \zeta_p(t) = \frac{1}{1 - q^{-t}} \). Thus \( Z(\omega_p, \lambda) \) is the special value (at \( w = \lambda - 1 \)) of the Langlands L-function and local \( GL_1 \) zeta factors.

(2) Let \( v \) be a real infinite prime. We then consider a matrix coefficient of the Weil representation \( \pi_{Q_v} \) of the form

\[
\tau = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}, \quad \sigma = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}
\]

Then we note that

\[
f(k(\theta)gk(\theta')) = e^{i \frac{1}{2} \pi \left( \frac{\theta - \theta'}{2} \right)} f(g)
\]

for all \( g \in SL_2(\mathbb{R}) \). Then we have that

\[
\text{trace}_{H}(\pi_{Q} \cdot f) = \int_{0}^{+\infty} \left( \int_{-\infty}^{+\infty} f\left( \begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right) dx \right) |r|^n dr
\]

(recall that \( f \) is a \( L^1 \)-function on \( Sp_4(\mathbb{R}) \)) for \( |\text{Re}(s)| < 1 \). We choose an orthogonal splitting of \( Q = Q_+ \oplus Q_- \) so that \( Q_+ (Q_- \text{ resp.)} \) is positive (negative) definite \( (a = \dim Q_+, b = \dim Q_-) \). Then, we choose \( \varphi_1(X) = P_+(X)P_-(X) \) \( e^{-\pi [X]} \) where \( P_+ \) is a harmonic polynomial in the \( + \) part of \( Q \) of degree \( r_1 \) and \( P_- \) is a harmonic polynomial in the \( - \) part of \( Q \) of degree \( r_2 \) (if \( b = 1 \) then \( P_- \) is the \( r_2 \)-th Hermite polynomial). Then we have that

\[
f\left( \begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right) = e^{2\pi \sqrt{-1} x r^2 Q(X)} e^{-\pi [r^2 + 1][X]}|P_+(X)|^2|P_-(X)|^2 dX
\]

Then we separate \( \mathbb{R}^m = \mathbb{R}^a \oplus \mathbb{R}^b \) and use polar coordinates on \( \mathbb{R}^a \) and \( \mathbb{R}^b \) to get that \( f \) above equals

\[
\|P_+\|_{L^2(S^a)} \|P_-\|_{L^2(S^b)} \Gamma\left(r_1 + \frac{a}{2} + 1\right) \Gamma\left(r_2 + \frac{b}{2} + 1\right)
\]

\[
(\text{if } b > 1)
\times r^{r_1 + r_2 + m/2} \left[ \pi (r^2 + 1) - 2\pi \sqrt{-1} x r^2 \right]^{-(r_1 + a/2)}
\times \left[ \pi (r^2 + 1) + 2\pi \sqrt{-1} x r^2 \right]^{-(r_2 + b/2)},
\]

or

\[
\|P_+\|_{L^2(S^a)} \Gamma\left(r_1 + \frac{a}{2} + 1\right) r^{r_1 + m/2}
\]
Then in both cases we integrate the resulting functions in the $x$ variable over $(-\infty, +\infty)$ and get that modulo the nonzero constants not involving $r$

\[
 (if \ b = 1) \quad \times \left[ \pi \left( r^2 + 1 \right) - 2\pi\sqrt{-1} \ x r^2 \right]^{-\frac{1}{2} \left( r_1 + \frac{a}{2} \right)} \\
\left\{ \sum_{\ell = 0}^{r_2} 2^{\ell+1} \binom{r_2}{\ell} \left( \frac{2r_2 - 2\ell}{(r_2 - \ell)!} \right) \right\} \\
\times \left( -1 \right)^i \left[ \pi \left( r^2 + 1 \right) + 2\pi\sqrt{-1} \ x r^2 \right]^{-\frac{1}{2} \left( r_2 + \ell \right)} 
\]

Then in both cases we integrate the resulting functions in the $x$ variable over $(-\infty, +\infty)$ and get that modulo the nonzero constants not involving $r$

\[
 r_{\ell_1} r_{\ell_2} + (\ell_1 + \ell_2 + m/2 - 2) \left( \frac{r - 1}{2} + 1 \right)^{-\frac{1}{2} \left( r_1 + r_2 \right) - m/2} \\
 (if \ b > 1) \quad \times \int_{-\pi/2}^{+\pi/2} e^{\left( -\frac{1}{2} \right) \left( r_1 - r_2 + ((a-b)/2) \right)} \cos \theta \ r_{\ell_1} r_{\ell_2} + (\ell_1 + \ell_2 + m/2 - 2) - \frac{1}{2} \ d\theta \\
\]

Then we observe that this latter integral will not vanish provided suitable choice of $r_1$ and $r_2$ is made. Indeed, if we choose $r_1$ and $r_2$ so that $r_1 - r_2 + (a-b)/2 = 0$ or 1 then clearly the above integral is nonvanishing. We note here if $r_1 - r_2 + (a-b)/2 = 0$ ($= 1$ resp.) then SO(2) type of $f$ is 0 ($= 1$ resp.). Then when we compute the above expression against $r^d r$ we get an integral of the form

\[
 \int_0^{+\infty} (1 + r^2)^{-\frac{1}{2} \left( r_1 + r_2 \right) - m/2} \ r_{\ell_1} r_{\ell_2} + (r_1 + r_2 + m/2 + s/2 - 2) \ d\theta 
\]

This integral by a simple substitution $r^2 = v$ becomes a beta integral of the form $B\left( \frac{1}{2} (r_1 + r_2) + (m/4) + (s/2) - \frac{1}{2}, \ \frac{1}{2} (r_1 + r_2) + (m/4) + (s/2) - \frac{1}{2} \right)$. But now if we assume that $\frac{1}{2} (r_1 + r_2) + m/4$ is an integer (equivalent to $b$ (even) iff $r_1 - r_2 + (a-b)/2 = 0$ resp.) then $B\left( \frac{1}{2} (r_1 + r_2) + (m/4) + (s/2) - \frac{1}{2}, \ \frac{1}{2} (r_1 + r_2) + (m/4) + (s/2) - \frac{1}{2} \right)$ is equal to

\[
\pi \ \text{sec} \left( \frac{\pi}{2} \left( \frac{r_1 + r_2}{m/4} + 1 \right) \right) \prod_{v=1}^{m/4 - 1} \left[ \left( v - \frac{1}{2} \right)^2 - \left( \frac{s}{2} \right)^2 \right]^{-1} \\
\times \frac{1}{\Gamma \left( r_1 + r_2 + \frac{m}{2} - 1 \right)} 
\]

Similarly if $\frac{1}{2} (r_1 + r_2) + m/4$ is half integral (equivalent to $b$ (odd) iff $r_1 - r_2 + (a-b)/2 = 0$ resp.) we have that $\left( r_1 + r_2 + (m/4) + \frac{m}{2} \right) \left( r_1 + r_2 + (m/4) + \frac{m}{2} - 1 \right)$ is equal to

\[
\pi \ \text{sec} \left( \frac{\pi}{2} \left( \frac{r_1 + r_2}{m/4} + \frac{1}{2} \right) \right) \prod_{v=1}^{m/4} \left[ \left( v - \frac{1}{2} \right)^2 - \left( \frac{s}{2} \right)^2 \right]^{-1} \\
\times \frac{1}{\Gamma \left( r_1 + r_2 + \frac{m}{2} - \frac{1}{2} \right)} 
\]
Now we consider the case $b = 1$ separately. For this we may assume that $r_2 = 0$. Then the above computations yield after integrating in the $x$ variable over $(-\infty, +\infty)$

\[ \pi \left( \frac{s}{2} \right) \csc \left( \pi \left( \frac{s}{2} \right) \right) \prod_{\nu = 1}^{v - \frac{1}{2}(r_1 + r_2) + \frac{m}{4} - 3/2} \left( \nu^2 - \left( \frac{s}{2} \right)^2 \right) \]

\[ \times \frac{1}{\Gamma \left( r_1 + r_2 + \frac{m}{2} - 1 \right)}. \tag{4-2} \]

Now we consider the case $b = 1$ separately. For this we may assume that $r_2 = 0$. Then the above computations yield after integrating in the $x$ variable over $(-\infty, +\infty)$

\[ r^{r_1 + m/2 - 2} (r^2 + 1)^{1 - (r_1 + m/2)} \times \int_{-\pi/2}^{+\pi/2} e^{i \theta (r_1 + (a - 1)/2)} (\cos \theta)^{r_1 + m/2 - 2} d\theta. \]

Then following the same reasoning as in the case above ($b > 1$) we deduce that after integrating against $r^d dr$ we get expressions for the form (4-1) with $r_1 + (a - 1)/2$ odd and of the form (4-2) with $r_1 + (a - 1)/2$ even (in these expressions we note that $r_2 = 0$ and $m = a + 1$). We note here that we have modified the $SO(2)$ type to be more general (i.e., either odd or even integral). The key observation as in the case above is that an integral of the form

\[ \int_{-\pi/2}^{+\pi/2} [\cos(\theta)]^{\mu - 1} e^{i \mu \theta} d\theta \]

(for $\mu$ a nonnegative integer) is nonvanishing. Indeed, we can write this integral as

\[ \sum_{j = 0}^{\mu - 1} \binom{\mu - 1}{j} \left( \int_{-\pi/2}^{+\pi/2} e^{i \theta (2j + 1)} d\theta \right). \]

Then the inner integral is up to a multiple of $\sqrt{-1}$

\[ \frac{1}{2j + 1} (-1)^j. \]

Then we know that

\[ \sum_{j = 0}^{\mu - 1} \binom{\mu - 1}{j} (-1)^j \frac{1}{2j + 1} = \frac{1}{2} \Gamma(1/2) \Gamma(\mu) = \frac{(\mu - 1)!2^{\mu - 1} \mu!}{(2\mu)!}. \]
References


(Oblatum 26-V-1982)

Department of Mathematics
The Ohio State University
231 West 18th Avenue
Columbus, OH 43210
USA