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## REMARKS ON $p$ -TORSION OF ALGEBRAIC SURFACES

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This paper is divided into two independent parts. In the first part, we show that if  $Y$  is a nonsingular model of a weighted complete intersection surface with only rational double points as singularities, then  $\text{Pic}(Y)$  is torsion-free. In the second part, we give an example of a surface with torsion-free crystalline cohomology, but for which the Hodge-de Rham spectral sequence is non-degenerate. At present, this is the only known example of this phenomenon.

The first part was inspired by a question of P. Blass, and the second part by a question of L. Illusie. I should like to thank both of them, as well as N. Katz and M. Raynaud, for their encouragement.

Throughout the paper, we work over an algebraically closed field  $k$  of characteristic  $p$ . All surfaces considered will be reduced, irreducible, and complete, unless otherwise stated.

### 1. The Picard group of a weighted complete intersection surface

For the definition and basic properties of weighted complete intersections, see [9].

**THEOREM:** *Let  $X$  be a weighted complete intersection surface with only rational double points as singularities. Let  $Y$  be a nonsingular model of  $X$ . Then  $\text{Pic}(Y)$  and  $\text{Pic}(X)$  are torsion-free.*

**PROOF:** First, notice that it is enough to prove the theorem for one nonsingular model of  $X$ . Therefore, we may assume that  $Y$  is a minimal resolution of the singularities of  $X$ .

**LEMMA 1:** (*Artin*). *Let  $X$  be a surface with only rational double points as singularities, let  $g: Y \rightarrow X$  be a minimal resolution of the singularities of  $X$ , and let  $\mathcal{L} = \mathcal{O}_X(D)$  be an invertible sheaf on  $Y$ . If  $D \cdot E = 0$  for all components  $E$  of the exceptional divisors obtained by resolving the singularities of  $X$ , then there exists an invertible sheaf  $\mathcal{L}'$  on  $X$  such that  $\mathcal{L} = g^* \mathcal{L}'$ .*

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PROOF: See [1], Cor. 2.6.

COROLLARY: *If  $g: Y \rightarrow X$  is as in Lemma 1, then  $g^*: \text{Pic}(X) \rightarrow \text{Pic}(Y)$  is an isomorphism on torsion.*

By Lemma 1 and its corollary, it is enough to show that  $\text{Pic}(X)$  is torsion-free.

LEMMA 2: *Let  $\ell$  be a prime number,  $\ell \neq p$ . Then  $\text{Pic}(X)$  is  $\ell$ -torsion-free.*

PROOF: (This is presumably well-known, but a proof does not appear in the literature.) In the proof of Theorem 3.7 of [9], Mori constructs a finite morphism  $f: \tilde{X} \rightarrow X$ , where  $\tilde{X}$  is an ordinary complete intersection, and  $f^*: \text{Pic}(X) \rightarrow \text{Pic}(\tilde{X})$  is an injection. Therefore it is enough to show that if  $X$  is an ordinary complete intersection of dimension 2 (with arbitrary singularities) then  $\text{Pic}(X)$  is  $\ell$ -torsion-free.

For this, we follow the argument of Hartshorne [5, IV.3]. (See also [11].) Let  $P$  be the ambient projective space in which  $X$  is a complete intersection, let  $U$  be an open neighborhood of  $X$  in  $P$ , let  $\hat{P}$  be the completion of  $P$  along  $X$ , and let  $X_n$  be the  $(n-1)$ st infinitesimal neighborhood of  $X$  in  $P$ . Then by the argument of Hartshorne,  $\text{Pic}(P) \simeq \text{Pic}(U) \simeq \text{Pic}(\hat{P}) \simeq \varinjlim \text{Pic}(X_n)$ . We now show, by induction on  $n$ , that the  $\ell$ -torsion part of  $\text{Pic}(X_n)$  is isomorphic to the  $\ell$ -torsion part of  $\text{Pic}(X)$ , and since  $\text{Pic}(P) \simeq \mathbb{Z}$ , this will conclude the proof of Lemma 2. We use the exact sequence [5, op. cit.]

$$0 \rightarrow I^n/I^{n+1} \rightarrow \mathcal{O}_{X_{n+1}}^* \rightarrow \mathcal{O}_{X_n}^* \rightarrow 0$$

where  $I$  is the ideal sheaf defining  $X$  as a subscheme of  $P$ . Taking cohomology, we get an exact sequence

$$\begin{aligned} H^1(X, I^n/I^{n+1}) &\rightarrow H^1(X_{n+1}, \mathcal{O}_{X_{n+1}}^*) \rightarrow H^1(X_n, \mathcal{O}_{X_n}^*) \\ &\rightarrow H^2(X, I^n/I^{n+1}). \end{aligned}$$

But  $H^1(X, I^n/I^{n+1}) = 0$  [5, op. cit.], and  $H^2(X, I^n/I^{n+1})$  is a  $p$ -torsion group (or torsion-free, if  $p = 0$ ). Thus the map  $\text{Pic}(X_{n+1}) \rightarrow \text{Pic}(X_n)$  is an isomorphism on  $\ell$ -torsion. This completes the proof of Lemma 2.

We again assume that  $X$  is a weighted complete intersection surface with only rational double points as singularities.

LEMMA 3:  *$\text{Pic}(X)$  is  $p$ -torsion free.*

PROOF: (Compare [SGA7], Exp. XI, Th. 1.8.) Suppose  $\mathcal{L}$  is a nontrivial line bundle on  $X$  which is killed by  $p$ . We construct a global Kahler

1-form on  $X$ , using the  $d \log$  map. The best description of this map for our purposes is found in [4], p. 220. Let  $\{f_{i,j}\}$  be a 1-cocycle representing the class of  $\mathcal{L}$  in  $H^1(X, \mathcal{O}_X^*)$ , then  $\{f_{i,j}^p\}$  is a coboundary, so we may write  $f_{i,j}^p = g_i/g_j$ ,  $g_i \in H^0(U_i, \mathcal{O}_X^*)$ . Then  $dg_i/g_i$  is a global section of the sheaf of Kahler differentials  $\Omega_X^1$  on  $X$ . Since  $d \log$  is an injective map from  ${}^p\text{Pic}(Y)$  to  $H^0(Y, \Omega_Y^1)$  ( ${}^p\text{Pic}(Y)$  denotes the kernel of multiplication by  $p$  on  $\text{Pic}(Y)$ ), and since the map is functorial, we see that  $d \log: {}^p\text{Pic}(X) \rightarrow H^0(X, \Omega_X^1)$  is injective also.

Therefore it is enough to show that  $H^0(X, \Omega_X^1) = 0$ . For this, we use the exact sequence of locally free sheaves on  $P$

$$0 \rightarrow \mathcal{O}_P \rightarrow \bigoplus_{i=0}^n \mathcal{O}_P(e_i) \rightarrow T_P \rightarrow 0$$

[9, Remark 2.4], where  $P$  is the ambient weak projective space of dimension  $n$  in which  $X$  is a weighted complete intersection,  $(e_0, \dots, e_n)$  are the weights of the variables, and  $T_P$  denotes the tangent bundle of  $P$ . Dualizing, and restricting to  $X$ , we get

$$0 \rightarrow \Omega_P^1|_X \rightarrow \bigoplus_{i=0}^n \mathcal{O}_X(-e_i) \rightarrow \mathcal{O}_X \rightarrow 0.$$

From this, and [9, Remark 2.2 and Prop. 3.3], it is easy to see that

$$H^0(X, \Omega_X^1|_P) = 0.$$

Next, we use the exact sequence

$$0 \rightarrow \bigoplus_{j=1}^s \mathcal{O}_X(-f_j) \rightarrow \Omega_P^1|_X \rightarrow \Omega_X^1 \rightarrow 0.$$

where  $f_j$ ,  $1 \leq j \leq s = n - 2$ , are the degrees of the defining equations of  $X$  as a weighted complete intersection in  $P$ . Taking cohomology, we find an exact sequence

$$H^0(X, \Omega_P^1|_X) \rightarrow H^0(X, \Omega_X^1) \rightarrow \bigoplus_{j=1}^s H^1(X, \mathcal{O}_X(-f_j)).$$

We know from above that  $H^0(X, \Omega_P^1|_X) = 0$ , and  $H^1(X, \mathcal{O}_X(-f_j)) = 0$  by [9, Prop 3.3]. Therefore  $H^0(X, \Omega_X^1) = 0$  and the proof of Lemma 3 is complete.

Since  $\text{Pic}(X)_{\text{tors}} \cong \text{Pic}(Y)_{\text{tors}}$ , and these groups are finite, Lemmas 2 and 3 imply the theorem.

**REMARK:** The motivating examples to which our theorem applies are the generic Zariski surfaces introduced by P. Blass in [2] and [3]. Blass uses the phrase “generic Zariski surface” in two different senses in these two

papers, but in both cases, it refers to the nonsingular model of a weighted hypersurface with only rational double points, to which our theorem applies. Incidentally, it is not known if  $H^0(Y, \Omega_Y^1) = 0$  where  $Y$  is a generic Zariski surface.

**2. Raynaud surfaces with torsion-free crystalline cohomology**

We use freely the results and notations of [6]. Let  $X$  be a quasi-elliptic Raynaud surface over a algebraically closed field of characteristic 3 (or, in the notation of [7], a generalized Raynaud surface of type (3,1,d)). Then there is a map  $f: X \rightarrow C$ , where  $C$  is a smooth curve, and all fibres are curves of arithmetic genus 1 with a single cusp. Let  $\mathcal{L} = R^1f_*\mathcal{O}_X$ , a locally free sheaf of rank one on  $C$ . We know that  $\mathcal{L}^6 \simeq K_C$ .

**PROPOSITION:** *Let  $X$  be a Raynaud surface over a curve  $C$  of genus  $g > 1$ . Then  $h^0(Z^1) = h^0(K_C) + h^0(\mathcal{L}^3)$ , where  $Z^1$  is the sheaf of closed 1-forms on  $X$ , and  $h^0(\Omega_X^1) = h^0(Z^1) + \dim \ker g: H^0(C, \mathcal{L}^5) \rightarrow H^1(C, \mathcal{L}^3)$ .*

**PROOF:** This is Theorem 4.5 of [6]. The description of the map  $g$  is not needed in the present paper.

**THEOREM:** *Suppose  $f: X \rightarrow C$  is a Raynaud surface,  $g(C) > 1, \mathcal{L} = R^1f_*\mathcal{O}_X$ . If  $H^0(C, \mathcal{L}^3) = 0$ , then the crystalline cohomology of  $X$  is torsion-free.*

**PROOF:** By Serre duality,  $H^1(C, \mathcal{L}^3) = 0$ . Then the dimension of the image of  $d: H^0(\Omega_X^1) \rightarrow H^0(\Omega_X^2)$  is  $h^0(\mathcal{L}^5)$ . By Serre duality,  $h^0(\mathcal{L}^5) = h^1(\mathcal{L})$ , and by the Leray spectral sequence  $h^1(C, \mathcal{L}) = h^2(X, \mathcal{O}_X)$ . Again by Serre duality,  $h^2(X, \mathcal{O}_X) = h^0(X, \Omega^2)$ , so  $d$  is surjective. Moreover, by the above proposition, the kernel of  $d$  (which is  $H^0(X, Z^1)$ ) consists of 1-forms pulled up from the base so  $\dim \ker d = g$ .

From the Leray spectral sequence, we get an exact sequence

$$0 \rightarrow H^1(C, \mathcal{O}_C) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow H^0(C, \mathcal{L}) \rightarrow 0.$$

Since  $H^0(C, \mathcal{L}^3) = 0, H^0(C, \mathcal{L}) = 0$  also. Hence,  $H^1(X, \mathcal{O}_X) \simeq H^1(C, \mathcal{O}_C)$  so all of  $H^1(X, \mathcal{O}_X)$  lives forever in the Hodge-de Rham spectral sequence. So we find that  $h^1_{DR}(X) = 2g$ . Since  $\text{Jac}(C) = \text{Alb}(X)$  (the fibres of  $f$  are rational curves), we see that  $h^1_{DR}(X) = B_1$ , where  $B_1$  is the first Betti number of  $X$ . It is now easy to see that  $H^1_{\text{cris}}(X/W)$  is torsion-free, using the universal coefficient theorem and Poincare duality for crystalline cohomology. In fact, one can show (using the methods of [8]) that  $d: H^2(W\mathcal{O}_X) \rightarrow H^2(W\Omega_X^1)$  is injective, hence  $H^2_{\text{cris}}(X/W)$  is isomorphic to  $W^2(-1)$  as an  $F$ -crystal.

Notice also that if  $X$  is as in the theorem, then the Hodge-de Rham spectral sequence is non-degenerate, since  $h^0(X, \Omega^2) = h^1(C, \mathcal{L})$  and

since  $h^0(\mathcal{L}) = 0$ , the Riemann-Roch theorem implies  $h^1(\mathcal{L}) = 2(g - 1)/3 > 0$ .

We now want to exhibit a surface satisfying the hypothesis of the theorem. We know that if  $(C, \mathcal{L})$  is a pair consisting of a smooth complete curve  $C$ , and a line bundle  $\mathcal{L}$  on  $C$ , then there is a Raynaud surface  $X$  together with a map  $f: X \rightarrow C$  such that  $R^1 f_* \mathcal{O}_X = \mathcal{L}$  if and only if there is a nowhere vanishing section  $dt$  of  $\Omega_C^1 \otimes \mathcal{L}^{-6}$  killed by the Cartier operator  $C: \Omega_C^1 \otimes \mathcal{L}^{-6} \rightarrow \Omega_C^1 \otimes \mathcal{L}^{-2}$ . We say that the triple  $(C, \mathcal{L}, dt)$  is a Tango curve, or, more precisely, a curve with Tango structure. (see [6], Section 1.)

LEMMA: *If  $(C, \mathcal{L}, dt)$  is a curve with tango structure, then  $(C, \mathcal{L} \otimes T, dt)$  is also a curve with Tango structure where  $T$  is a line bundle of order 2 in  $\text{Pic}(C)$ .*

PROOF: Obvious. □

Now observe that since  $\mathcal{L}^6 \simeq K_C$ ,  $\mathcal{L}^3$  is a theta characteristic in the sense of Mumford [10]. Replacing  $\mathcal{L}$  by  $\mathcal{L} \otimes T$ , and letting  $T$  run through all elements of  $\text{Pic}(C)$  of order 2, we see that all theta characteristics of  $C$  are of the form  $\mathcal{L}^3$ , where  $(C, \mathcal{L}, dt)$  is a Tango curve. Therefore, to get a Raynaud surface  $f: X \rightarrow C$  such that  $H^0(C, \mathcal{L}^3) = 0$ , it is enough to find a Tango curve  $C$  and a theta characteristic  $\mathcal{M}$  on  $C$  such that  $H^0(C, \mathcal{M}) = 0$ . However, any hyperelliptic Tango curve will do for this. (See [10], p. 191. Take  $\mathcal{M} = b_s^{(-1)}$ .) Examples of hyperelliptic Tango curves are given in [6], p. 481. An interesting open problem is to find an *ordinary* Tango curve satisfying the hypothesis of the theorem. (If  $(C, \mathcal{L}, dt)$  is a curve with Tango structure such that  $H^0(C, \mathcal{L}) \neq 0$ , then  $C$  cannot be ordinary, since it gives rise to a *holomorphic* differential on  $C$  killed by the Cartier operator. Thus, the examples in [6] are not ordinary.)

Notice that the observation that  $\mathcal{L}^3$  is a theta characteristic implies that the moduli space of Raynaud surfaces  $X$  (together with the map  $f: X \rightarrow C$ ) in characteristic three is disconnected, since a Raynaud surface such that  $\mathcal{L}^3$  is an even theta characteristic cannot be deformed into one such that  $\mathcal{L}^3$  is an odd theta characteristic by [10, Section 1].

## References

- [1] M. Artin: Some numerical criteria for contractability of curves on algebraic surfaces. *Amer. Journ. Math.* 84 (1962) 485–496.
- [2] P. BLASS: Zariski surfaces, preprint.
- [3] P. BLASS: Some geometric applications of a differential equation in characteristic  $p > 0$  to the theory of algebraic surfaces, preprint.
- [4] E. BOMBIERI and D. MUMFORD: Enriques' classification of surfaces in char. p. III. *Invent. Math.* 35 (1976) 197–232.
- [5] R. HARTSHORNE: Ample subvarieties of algebraic varieties. *Lecture Notes in Mathematics*, Vol. 156. Berlin, Heidelberg, New York: Springer (1970).

- [6] W.E. LANG: Quasi-elliptic surfaces in characteristic three. *Ann. Sci. Ec. Norm. Sup.* (4), 12 (1979) 473–500.
- [7] W.E. LANG: Examples of surfaces of general type with vector fields, preprint.
- [8] W.E. LANG and N.O. NYGAARD: A short proof of the Rydakov-Saferevic theorem. *Math. Ann.* 251 (1980) 171–173.
- [9] S. MORI: On a generalization of complete intersections, *J. Math. Kyoto Univ.* 15 (1975) 619–646.
- [10] D. MUMFORD: Theta characteristics of an algebraic curve. *Ann. Sci. Ec. Norm. Sup.* (4), 4 (1971) 181–192.
- [11] L. ROBBIANO: A problem of complete intersections, *Nagoya Math. J.* 52 (1973) 129–132.
- [SGA7] Seminaire de Geometrie Algebrique 7, Tome II, *Lecture Notes in Mathematics*, Vol. 340.

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