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CONVEX POLYTOPES AS MATRIX INVARIANTS

Gerard Sierksma and Klaas de Vos

Abstract

For a convex polytope $P$ which is the convex hull of a finite number of points, the set $\pi(P)$ consists of all real square matrices $A$ such that $AP \subseteq P$, i.e. that leave $P$ invariant. In this paper the extremals of $\pi(P)$ are characterized for $P$ being a convex simplex, and the number of its extremes is determined.

1. Introduction

In Berman and Plemmons [2] the first chapter deals with matrices that leave a cone invariant, i.e. $\pi(K) = \{ A \in \mathbb{R}^{d \times d} \mid AK \subseteq K \}$ with $K$ a cone in $\mathbb{R}^d$. An extensive bibliography on properties of $\pi(K)$ can be found in this book. In e.g. Tam [8] it is shown that $\pi(K)$ is a polyhedral cone if $K$ is a polyhedral cone. One of the main problems is to characterize the extremals of such a polyhedral cone $\pi(K)$; see e.g. Adin [1]. Instead of taking a cone as matrix invariant we consider in this paper convex polytopes, with a convex polytope being the convex hull of a finite nonempty set of points in $\mathbb{R}^d$; see e.g. Eggleston [4] and Sierksma [6]. In Valentine [9] the term convex polyhedron is used. In a recent paper by Elsner [3] is also deviated from the idea of using cones; here so-called nontrivial convex sets are used as matrix invariants. In this paper we restrict ourselves mainly to convex simplices $S_0$ with one vertex at the origin. By a convex simplex $P$ in $\mathbb{R}^d$ we mean the convex hull of $d+1$ points in $\mathbb{R}^d$ with nonempty interior. We shall characterize the extremes of $\pi(S_0)$ and calculate its number. In general, we define for $X \subseteq \mathbb{R}^d$ the set of matrices

$$\pi(X) = \{ A \in \mathbb{R}^{d \times d} \mid AX \subseteq X \}. $$

Note that if $X = \{0\}$, then $\pi(X) = \mathbb{R}^{d \times d}$. If $X = \{x\}$ with $x \neq 0$, then $\pi(X)$ consists of all $(d, d)$-matrices with eigenvector $x$ and eigenvalue 1. Before restricting ourselves to convex polytopes we give the following result for arbitrary sets. Note that if $X$ is convex then so is $\pi(X)$. By cone $X$ we mean the convex cone generated by $X$, i.e. all nonnegative linear combinations of $X$. The set cone $X$ is also denoted by $X^G$; see [2].
**THEOREM 1:** Let $X \subset \mathbb{R}^d$. Then the following holds

(a) $\text{cone } \pi(X) \subset \pi(\text{cone } X)$;

(b) $\text{cone } \pi(X) = \pi(\text{cone } X)$ if $X$ is compact, convex and contains 0.

**PROOF:** (a) Take any $A \in \text{cone } \pi(X)$. Then there are matrices $A_1, \ldots, A_n \in \pi(X)$ such that $A = \sum_{i=1}^n \lambda_i A_i$ for all $i$. Furthermore let $x = \sum_{i=1}^n \mu_i x_i \in \text{cone } X$ with $\mu_i \geq 0$ and $x_i \in X$ for all $i$. Then it follows that $Ax = A(\sum_{i=1}^n \mu_i x_i) = \sum_{i=1}^n \mu_i A x_i = \sum_{i=1}^n \mu_i (\sum_{j=1}^{\lambda_j} A_j x_i) = \sum_{i,j} \lambda_i \mu_i A_j x_i \subset \text{cone } X$. Hence, $A \in \pi(\text{cone } X)$.

(b) Take any $A \in \pi(\text{cone } X)$ and let $X = \{0\}$. Hence $A(\text{cone } X) \subset \text{cone } X$. As $0 \in X$ and $X$ convex it follows that for each $x \in X$ there are $\lambda, \mu > 0$ and $y \in X$ such that

$$A(\lambda x) = \mu y,$$

or that $(\lambda/\mu)Ax \in X$. Let $\lambda^*$ be the infimum of all $\lambda/\mu$ over $x$. Then $\lambda^* \neq 0$, and $(\lambda^* A)X \subset X$. This means that $\lambda^* A \in \pi(X)$. As $\pi(X)$ is convex and contains 0, it follows that $A \in \text{cone } \pi(X)$.

In Theorem 1(b) we have a sufficient condition in order to obtain equality in (a). Note that we also have equality if $X = \text{cone } X$, because in that case both $X$ and $\pi(X)$ are convex cones; see e.g. Berman and Plemmons [2]. The following example shows that equality does not hold in general in Theorem 1. Take $X = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1, x_2 \geq 0, x_1 + x_2 = 1\}$. Then $\text{cone } X = \mathbb{R}^2_+$, and $\pi(\text{cone } X)$ consists of all nonnegative $(2,2)$-matrices. On the other hand $\pi(X)$ consists of all matrices

$$
\begin{pmatrix}
a & b \\
1-a & 1-b
\end{pmatrix}
$$

with $0 \leq a, b \leq 1$, so $\pi(X)$ consists of all nonnegative multiples of these matrices. Hence $\pi(\text{cone } X) \neq \text{cone } \pi(X)$.

In the following chapters we replace "cone" by "conv" and "extr", so we consider $\pi(\text{conv } X)$, $\text{conv } \pi(X)$ and $\pi(\text{extr } X)$, $\text{extr } \pi(X)$.

2. Polytopes and simplices as matrix invariants

The main purpose of this chapter is to study the commutativity of $\pi$ and conv, i.e. $\pi(\text{conv } X) = \text{conv } \pi(X)$ for $X$ a polytope. Clearly, if $X$ is convex so is $\pi(X)$, and in that case we have $\text{conv } \pi(X) = \pi(\text{conv } X)$.

**THEOREM 2:** For each $X$ in $\mathbb{R}^d$ the following holds

$$\text{conv } \pi(X) \subset \pi(\text{conv } X).$$
PROOF: Take any $X \subset \mathbb{R}^d$. Clearly $\pi(\text{conv } X)$ is convex in $\mathbb{R}^{d \times d}$. So we only have to show that $\pi(X) \subset \pi(\text{conv } X)$. Take any $A \in \pi(X)$. Then $AX \subset X$. Now let $x \in A(\text{conv } X)$. Then there are $x_1, \ldots, x_s \in X$ and $\lambda_1, \ldots, \lambda_s \geq 0$ with $\lambda_1 + \ldots + \lambda_s = 1$, such that $x = A(\sum_{i=1}^s \lambda_i x_i) = \sum_{i=1}^s \lambda_i Ax_i$. As $Ax_i \in X$ for each $i$ we have $x \in \text{conv } X$, and it follows that $A(\text{conv } X) \subset \text{conv } X$. Hence, $A \in \pi(\text{conv } X)$.

Equality does not hold in general in the above theorem as is shown by the following example. Take $X = \{(1,0), (0,1), (-1,0), (1,1), (0,0)\}$. Then $\text{conv } X = \{(x_1, x_2) | 0 \leq x_1, x_2 \leq 1\}$, and $\pi(\text{conv } X)$ consists of all non-negative matrices with row sums $\leq 1$. On the other hand the $(2,1)$-th element of each matrix in $\text{conv } \pi(X)$ is zero. Equality also does not hold, in general, in case $X$ consists of the extremals of a convex cone $X$, i.e. $X = \text{extr } K$.

It is well-known that $\text{conv } X = \text{conv}(\text{extr } K) = K$ (the Krein-Milman theorem). However, in general, $\pi(\text{conv } X) = \pi(K) \neq \text{conv } \pi(\text{extr } K)$. In order for $\pi(K)$ being equal to $\text{conv } \pi(\text{extr } K)$ it is therefore necessary that $\text{extr } \pi(K) \subset \pi(\text{extr } K)$ which is a conjecture of Loewy and Schneider [5].

In the following we shall write $0 \cup S$ instead of $\{0\} \cup S$. In the next theorem $S_0 = \text{conv}(0 \cup S)$ is a convex simplex i.e. a convex simplex with one vertex at the origin and $|S| = d$. The theorem shows the commutativity of $\pi$ and $\text{conv}$ for $0 \cup S$. The proof of it is after Theorem 6.

**Theorem 3**: Let $0 \cup S$ be the vertices of a convex simplex in $\mathbb{R}^d$. Then the following holds

$$\pi(\text{conv}(0 \cup S)) = \text{conv } \pi(0 \cup S).$$

In order to prove this theorem we need a nonsingular transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ such that

$$T(e_1, \ldots, e_d) = S$$

with $S$ as in the above theorem. We denote $E_d = \{e_1, \ldots, e_d\}$, i.e. the set of unit vectors in $\mathbb{R}^d$.

**Theorem 4**: The following assertions are equivalent:

(i) $A \in \pi(\text{conv}(0 \cup E_d))$;
(ii) $Ae_1, \ldots, Ae_d \in \text{conv}(0 \cup E_d)$;
(iii) $A \geq 0$, column sums of $A \leq 1$;
PROOF: We shall show that (i) \implies (ii) \implies (iii) \implies (i).

(i) \implies (ii): Let $A \in \pi(\text{conv}(0 \cup E_d))$, so $A(\text{conv}(0 \cup E_d)) \subset \text{conv}(0 \cup E_d)$. As $e_i \in \text{conv}(0 \cup E_d)$, it follows that $Ae_i \in \text{conv}(0 \cup E_d)$ for each $i = 1, \ldots, d$.

(ii) \implies (iii): As $Ae_j \in \text{conv}(0 \cup E_d)$ for each $j$, it follows that the $j$-th column of $A$ is equal to $\sum_{r=1}^{d} \lambda_{ij} e_r$ with $\lambda_{1j}, \ldots, \lambda_{dj} \geq 0$ and $\lambda_{1j} + \ldots + \lambda_{dj} \leq 1$, so the $j$-th column sum of $A$ is equal to $\sum_{r=1}^{d} \lambda_{ij}$ and is $\leq 1$. The matrix $A$ is nonnegative, because all $\lambda_{ij}$'s are nonnegative.

(iii) \implies (i): Take any $x \in \text{conv}(0 \cup E_d)$. Then there are scalars $\alpha_1, \ldots, \alpha_d \geq 0$ with $\alpha_1 + \ldots + \alpha_d \leq 1$ such that $x = \sum_{r=1}^{d} \alpha_r e_r$. Hence, $Ax = A(\sum_{r=1}^{d} \alpha_r e_r) = \sum_{r=1}^{d} \alpha_r A e_r$. As the column sums of $A$ are $\leq 1$, it follows directly that $Ae_i \in \text{conv}(0 \in E_d)$ for each $i = 1, \ldots, d$, and therefore we have $Ax \in \text{conv}(0 \cup E_d)$, and hence $A \in \pi(\text{conv}(0 \cup E_d))$.

Theorem 4 implies that all matrices in $\pi(0 \cup E_d)$ have Perron-Frobenius eigen-value $\leq 1$; this is the well-known Minkowski-theorem, see e.g. Sierksma [7].

LEMMA 5: Let $T : \mathbb{R}^d \to \mathbb{R}^d$ be a nonsingular transformation and let $X \subset \mathbb{R}^d$. Then the following assertions hold:

(a) $\text{conv}(TX) = T(\text{conv } X)$,
(b) $\pi(TX) = T\pi(X) T^{-1}$,
(c) $\text{conv } \pi(TX) = T[\text{conv } \pi(X)]T^{-1}$.

PROOF: (a) is left to the reader. In order to prove (b), take any $A \in \pi(TX)$. Hence, $A(TX) \subset TX$, and this implies that $(T^{-1}AT)(X) \subset X$, so that $T^{-1}AT \in \pi(X)$, or $A \in T[\pi(X)]T^{-1}$. The converse inclusion is shown similarly. To prove (c), take any $A \in \text{conv } \pi(TX) = \text{conv}[T\pi(X)T^{-1}]$. Then there are matrices $B_1, \ldots, B_k \in \pi(X)$, and scalars $\lambda_1, \ldots, \lambda_k \geq 0$ with $\lambda_1 + \ldots + \lambda_k = 1$ such that $A = \sum_{r=1}^{k} \lambda_r B_r$ (Convex combination of matrices from $\pi(X)$). As $\sum_{r=1}^{k} \lambda_r B_r \in \text{conv } \pi(X)$, it follows that $A \in T[\text{conv } \pi(X)]T^{-1}$. The other conclusion is also shown similarly.

The following theorem gives the commutativity of $\pi$ and $\text{conv}$ for $0 \cup E_d$.

THEOREM 6: $\pi(\text{conv}(0 \cup E_d)) = \text{conv } \pi(0 \cup E_d)$.

PROOF: According to Theorem 2, we only have to show that $\pi(\text{conv}(0 \cup E_d)) \subset \text{conv } \pi(0 \cup E_d)$. Take any $A \in \pi(\text{conv}(0 \cup E_d))$. Then $A(\text{conv}(0 \cup E_d)) \subset \text{conv}(0 \cup E_d)$. Theorem 4 then gives that $A \geq 0$ and that all column sums of $A$ are $\leq 1$. We must show now that $A$ can be written as a convex combination of matrices from $\pi(0 \cup E_d)$. To show this, we first define $A = A_i = \{a_{ij}^{(i)}\}$. Moreover, we define

$$
\lambda_i = \min_j \max_i a_{ij}^{(i)}.
$$
and \( I_1 \) is the matrix with precisely one 1 in the \( j \)-th column in the \((i,j)\)-th position if \( a_{ij} \) is the maximum in the \( j \)-th column of \( A_1 \) (if there are more maxima in the \( j \)-th column choose one!) and zeroes otherwise (\( j = 1, \ldots, d \)). Then consider the matrix

\[
A_2 = A_1 - \lambda_1 I_1,
\]

with \( A_2 = \{ a_{ij}^{(2)} \} \) and define

\[
\lambda_2 = \min \max a_{ij}^{(2)}.
\]

Also define \( A_3 = A_2 - \lambda_2 I_2 = A_1 - \lambda_1 I_1 - \lambda_2 I_2 \), where \( I_2 \) is defined for the matrix \( A_2 \) in the same way as \( I_1 \) for \( A_1 \). Continuing this process we obtain, after at most \( d^2 \) steps, the zero-matrix. So we obtain

\[
0 = A_{d^2+1} = A - \lambda_1 I_1 - \lambda_2 I_2 - \ldots - \lambda_{d^2} I_{d^2}.
\]

Hence, \( A = \sum_{i=1}^{d^2} \lambda_i I_i \). Clearly, \( A \geqslant 0 \), and each \( \lambda_i \geqslant 0 \). Note that a column of \( I_i \) becomes zero if the corresponding column of \( A_i \) is zero. If after \( d^2 - 1 \) steps there still is some nonzero element in \( A_{d^2-1} \) we have, say in the \( j \)-th column,

\[
a_{i,j} + \ldots + a_{d,j} - (\lambda_1 + \ldots + \lambda_{d^2}) = 0,
\]

or \( \sum_{i=1}^{d^2} \lambda_i = \sum_{i=1}^{d} a_{ij} \leqslant 1 \). And this means that in fact \( A \in \text{conv } \pi(0 \cup E_d) \).

The number of steps in the proof of the above theorem is, in general \( \leqslant d^2 \). Question: under what conditions is the number of steps equal to \( d^2 \)?

**Proof of Theorem 3:** First note that \( 0 \cup S \) is the set of vertices of a simplex, so \(|S| = d \). We must show that

\[
\pi(\text{conv}(0 \cup S)) \subset \text{conv } \pi(0 \cup S).
\]

Clearly, there is a nonsingular transformation \( T: \mathbb{R}^d \to \mathbb{R}^d \) such that \( S = T E_d = T(e_1, \ldots, e_d) \). According to Lemma 5 and Theorem 6 we have

\[
\pi(\text{conv}(0 \cup S)) = \pi(\text{conv}(0 \cup TE_d)) = \pi(T \text{conv}(0 \cup E_d)) = T(\pi(0 \cup E_d)) = T(\pi(0 \cup E_d)) T^{-1} = T(\pi(0 \cup E_d)) T^{-1} = \text{conv } \pi(T(0 \cup E_d)) = \text{conv } \pi(0 \cup E_d) = \text{conv } \pi(0 \cup S).
\]

**3. The extremes of \( \pi(S_0) \)**

In this chapter we shall characterize the extreme vertices of \( \pi(S_0) \) and determine their number. If \( P \) is a convex polytope then, in general, \( \pi(P) \) is not a polytope. For instance if we take the two points \( \begin{pmatrix} 0 \\ 0 \end{pmatrix} \) and \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \)
then \( P = \text{conv} \left( \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right) \) is convex but \( \pi(P) \) is not: in \( \pi(P) \) there are matrices \( A = (a_{ij}) \) with \( a_{11} = -a_{12} \), so \( a_{11} \) can be as large as possible which means that \( \pi(P) \) is not bounded, so certainly is not a convex polytope. The following theorem gives a sufficient condition in order to save the boundedness of \( \pi(P) \).

**Theorem 7:** If \( X \) is a bounded set in \( \mathbb{R}^d \) with \( \text{intconv}(0 \cup X) \neq \emptyset \) then \( \pi(X) \) is bounded.

**Proof:** Suppose, to the contrary, that \( \pi(X) \) is not bounded. Then there is a sequence of matrices \( A_k \) in \( \pi(X) \) such that one of the elements, say the \((i,j)\)-th element, of the \( A_k \)'s goes to infinity. As the interior of \( \text{conv}(0 \cup X) \) is nonempty, there is an element \( y \in \text{intconv}(0 \cup X) \) with \( y_i \neq 0 \). Then \( y = \sum_{i=1}^{s} \lambda_i x_i \) with \( x_i \in X \), \( \lambda_i \geq 0 \), and \( \lambda_1 + \ldots + \lambda_s \leq 1 \). Then \( (A_k y)_j \rightarrow \infty \) for \( k \rightarrow \infty \). As \( A_k y = \sum_{i=1}^{s} \lambda_i A_k x_i \) is a finite sum, we have \( A_k x_i \rightarrow \infty \) for \( k \rightarrow \infty \) and for some \( i \). As \( A_k x_i \in X \), it follows that \( X \) is not bounded which is a contradiction. Therefore we have in fact that \( \pi(X) \) is bounded.

It is open question whether \( \pi(X) \) is a convex polytope in case \( X \) is a convex polytope in \( \mathbb{R}^d \) with \( \text{intconv}(0 \cup X) \neq \emptyset \). Question: Is the number of extreme vertices of \( \pi(X) \) less then or equal to \((d + 1)^d\)? (see the following theorem)?

**Theorem 8:** Let \( 0 \cup S \) be the vertices of a convex simplex. Then the following holds:

\[
|\pi(0 \cup S)| = |\pi(0 \cup E_d)| = (d + 1)^d.
\]

**Proof:** There is a nonsingular transformation \( T: \mathbb{R}^d \rightarrow \mathbb{R}^d \) such that \( 0 \cup S = T(0 \cup E_d) \), so we have directly that \( |\pi(0 \cup S)| = |\pi(0 \cup E_d)| \). We only have to show that

\[
|\pi(0 \cup E_d)| = (d + 1)^d.
\]

First note that \( A0 = 0 \) for each \( A \in \pi(0 \cup E_d) \). The problem of determining the number of matrices in \( \pi(0 \cup E_d) \) is therefore equivalent to the problem of finding the number of bipartite graphs \((G, H)\) on \( 2(d + 1) \) vertices with \( |G| = |H| = d + 1 \), with one edge fixed, and such that the degree of each vertex in \( G \) is 1. Let there be a fixed edge between \( a \in G \) and \( b \in H \). Then there are for each vertex \( \neq a \) in \( G \) precisely \( d + 1 \) possibilities in \( H \). This holds for all of the vertices in \( G \) that are \( \neq a \). So the number of such bipartite graphs is equal to

\[
\frac{(d + 1) \times \ldots \times (d + 1)}{d \text{ times}}.
\]
Hence, \(|\pi(0 \cup E_d)| = (d + 1)^d\).

To illustrate the above theorem we consider the following example.
Let \(d = 2\) and \(S_0 = \text{conv}\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\} = \text{conv}(0 \cup E_2)\). Then we have
\[
\tau(0 \cup S) = \{\begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\}, \quad \text{and}
\]
\[
\tau(S_0) = \left\{\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \mid a, b, c, d \geq 0, a + c = 1, b + d = 1\right\}.
\]

Note that \(|\pi(0 \cup S)| = 3^2 = 9\). In order to characterize the extreme vertices of \(S_0\) we need the following two lemmas.

**Lemma 9:** Let \(X \subset \mathbb{R}^{d \times d}\) be convex and \(T: \mathbb{R}^d \to \mathbb{R}^d\) be nonsingular. Then the following holds:
\[
\text{extr}(TXT^{-1}) = T(\text{extr } X)T^{-1}
\]

**Proof:** Take any \(A \in \text{extr}(TXT^{-1})\). Hence, \(A \notin \text{conv}(TXT^{-1}) \setminus \{A\}\).
Suppose, to the contrary, that \(A \notin T(\text{extr } X)T^{-1}\). We shall show first that \(T^{-1}AT \in \text{extr } X\), or that \(T^{-1}AT \notin \text{conv } X \setminus \{T^{-1}AT\}\). Taking \(T^{-1}AT \in \text{conv } X \setminus \{T^{-1}AT\}\), there should exist matrices \(B_1, \ldots, B_k \in X\), all \(\neq T^{-1}AT\), and scalars \(\lambda_1, \ldots, \lambda_k \geq 0\) with \(\lambda_1 + \ldots + \lambda_k = 1\), such that
\[
T^{-1}AT = \sum_{i=1}^{k} \lambda_i B_i,
\]
or \(A = \sum_{i=1}^{k} \lambda_i TB_iT^{-1}\), and \(B_i \neq T^{-1}AT\) for all \(i\).
Because \(TB_iT^{-1} \in \text{conv}(TXT^{-1})\) for all \(i\), we have \(A \in \text{conv}(TXT^{-1}) \setminus \{A\}\), hence \(A \notin \text{extr}(TAT^{-1})\), and this is a contradiction. Therefore we have, \(T^{-1}AT \in \text{extr } X\), and this means that \(A \in T(\text{extr } X)T^{-1}\). The converse can be shown similarly.

**Lemma 10:** \(\text{extr conv } \pi(0 \cup E_d) = \pi(0 \cup E_d)\).

**Proof:** As all columns of the matrices in \(\pi(0 \cup E_d)\) consists of zero or unit vectors, no such a matrix can be written as a convex combination of the other ones in \(\pi(0 \cup E_d)\).

The next theorem characterizes the extreme vertices of \(\pi(S_0) = \pi(\text{conv}(0 \cup S))\); they are precisely the matrices that leave the vertices invariant.

**Theorem 11:** \(\text{extr } \pi(S_0) = \text{extr } \pi(\text{conv}(0 \cup S)) = \pi(0 \cup S)\).

**Proof:** Let \(T: \mathbb{R}^d \to \mathbb{R}^d\) be a nonsingular transformation such that
\[ 0 \cup S = T(0 \cup E_d). \] Then we have

\[
\text{extr} \pi(S_0) = \text{extr} \pi(\text{conv}(0 \cup S)) = \text{extr} \pi(\text{conv}(T(0 \cup E_d)))
\]

\[
= \text{extr\,conv} \pi(T(0 \cup E_d)) = \text{extr\,conv} [T \pi(0 \cup E_d) T^{-1}]
\]

\[
= \text{extr}(T \left[ \text{conv} \pi(0 \cup E_d) \right] T^{-1})
\]

\[
= T \left[ \text{extr\,conv} \pi(0 \cup E_d) \right] T^{-1} = T \left[ \pi(0 \cup E_d) \right] T^{-1}
\]

\[
= \pi(T(0 \cup E_d)) = \pi(0 \cup S).
\]

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References


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