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Compositio Mathematica, tome 52, n° 2 (1984), p. 221-229

<http://www.numdam.org/item?id=CM_1984__52_2_221_0>
HOMOGENEOUS-RATIONAL MANIFOLDS AND UNIQUE FACTORIZATION

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Introduction. Statement of the results

All varieties occurring in this note are assumed to be defined over \( \mathbb{C} \). We call an affine (resp. projective) variety factorial if its affine (resp. homogeneous) coordinate ring is a unique factorization domain. It seems that the question whether a given affine or projective variety is factorial, goes back to Felix Klein and Max Noether in the late 19th century (see e.g. [20, p. 32]).

Some well-known examples of factorial projective varieties are, besides the trivial example \( \mathbb{P}_n \), the nonsingular quadric \( Q_n \subset \mathbb{P}_{n+1} \) for \( n \geq 3 \) (Klein), and the Grassmann variety \( G_{n,k} \) of \( k \)-planes in \( \mathbb{P}_n \) considered as a projective variety in \( \mathbb{P}_N \) via Plücker embedding, where \( N = \binom{n+1}{k+1} - 1 \) (the question whether \( G_{n,k} \) is factorial was raised by Severi around 1915 and answered in the affirmative by Samuel in the early 1960’s (cf. [1] and [20, pp. 37 ff.])).

Since the examples mentioned above are all homogeneous-rational manifolds, it is a quite natural question whether any homogeneous-rational manifold is factorial or, more realistically, to decide which of them are. Here, by a homogeneous-rational manifold we mean a compact homogeneous projective-rational complex manifold of positive dimension. Equivalently, a compact complex manifold \( X \) of positive dimension is homogeneous-rational if and only if either there is a connected semisimple complex Lie group \( G \) acting transitively on \( X \) such that \( X = G/H \), where \( H \) is a proper parabolic subgroup of \( G \), or \( X \) is homogeneous with vanishing first Betti number and nonvanishing Euler characteristic, or \( X \) is homogeneous and Kähler with \( H^1(X, \mathcal{O}) = 0 \) (see [2], [3], [8]).

Now, none of the above equivalent conditions for homogeneous-rationality involves an embedding of \( X \) into \( \mathbb{P}_N \). Hence the question for factoriality of homogeneous-rational manifolds should be stated more precisely in the following form: Given a holomorphic embedding \( f: X \rightarrow \mathbb{P}_N \) of a homogeneous-rational manifold \( X \), under which conditions (on \( X \) and \( f \)) is \( f(X) \) factorial?
We first define some rather special embeddings. For this purpose let $X$ be a homogeneous-rational manifold, $G$ a connected simply-connected semisimple complex Lie group acting transitively on $X$. A holomorphic embedding $f: X \to \mathbb{P}_N$ is called homogeneously normal if $f$ is $G$-equivariant, i.e. if there is a holomorphic representation $\phi_f: G \to \text{SL}(N+1, \mathbb{C})$ such that $\phi_f(g)(f(x)) = f(g(x))$ for all $g \in G$ and $x \in X$. It is not difficult to see that this definition is independent of $G$, i.e. if $G^*$ is another connected simply-connected semisimple complex Lie group acting likewise transitively on $X$, then a holomorphic embedding $f: X \to \mathbb{P}_N$ is $G$-equivariant if and only if $f$ is $G^*$-equivariant (cf. [22, Kap. II, Sect. 2.3]). A holomorphic embedding $f: X \to \mathbb{P}_N$ is called homogeneously minimal if it is homogeneously normal and if $N$ is minimal, i.e. $N \leq M$ for any homogeneously normal embedding $f^*: X \to \mathbb{P}_M$. Then we have the following result which is a special case of a theorem of Tits ([24, III.D]):

**Theorem T:** There exists a homogeneously minimal embedding of $X$, and this is unique up to an automorphism of the ambient projective space.

Note that it is necessary to assume the group $G$ to be simply-connected in order to obtain homogeneously minimal embeddings which everybody would expect (for instance, there is no $\text{PGL}(2, \mathbb{C})$-equivariant embedding of $\mathbb{P}_1$ in $\mathbb{P}_1$). In the case $X = G_{n,k}$, the homogeneously minimal embedding of $X$ is just the Plücker embedding. It should be pointed out that, in contrast to the name “minimal”, this $N$ is not necessarily so small: For instance, if $X = G/B$, where $B$ is a Borel subgroup of $G$, then $N = 2^{\dim X} - 1$. On the other hand, it is well-known that any projective-algebraic manifold of dimension $d$ admits an embedding into $\mathbb{P}_{2d+1}$ (cf. [9, p. 173]).

Let $X$ be a homogeneous-rational manifold and $G$ a connected semisimple complex Lie group acting transitively on $X$. Write $X = G/H$ with $H$ a proper parabolic subgroup of $G$, and denote by $H'$ the commutator group of $H$. We define the rank $^{(1)}$ of $X$, written $\text{rk}(X)$, as the dimension of the complex Lie group $H/H'$. Equivalently, $\text{rk}(X)$ is the number of maximal parabolic subgroups of $G$ which contain $H$. Using this description of the rank and a theorem of Remmert-van de Ven ([19, Satz (2.2)]), it is easy to see that the rank of $X$ depends only on $X$, but not on the group $G$. One can also show that $\text{rk}(X) = b_2(X)$, where $b_2(X)$ denotes the $2^{nd}$ Betti number of $X$ (cf. [5, p. 245] and [23, Remark in § 3]). Obviously, $\text{rk}(X) = 1$ if and only if $H$ is a maximal parabolic subgroup of $G$ and, by [18], this is equivalent to the condition that each holomorphic map $h: X \to Y$ of $X$ into a complex space $Y$ of dimension $< \dim X$ be

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$^{(1)}$ This definition of the rank has nothing whatsoever to do with the rank of a symmetric (in particular hermitian symmetric) space in differential geometry. However, our notation seems to be rather familiar, see e.g. [25, p. 114].
constant. In particular, every rank 1-homogeneous-rational manifold is irreducible. (A homogeneous-rational manifold $X$ is called irreducible if $\text{Aut}(X)$ is a simple complex Lie group, reducible otherwise. Evidently, a homogeneous-rational manifold $X$ is reducible if and only if there are homogeneous-rational manifolds $X_1, X_2$ such that $X \cong X_1 \times X_2$.)

Let us look at some examples: The homogeneous-rational manifolds of rank 1 are projective spaces, quadrics of dimension $\geq 3$, Grassmannians, “Grassmannians” of linear subspaces of $\mathbb{P}_{n+1}$ which lie on the quadric $Q_n \subset \mathbb{P}_{n+1}$ (cf. [24, II.C.7, II.C.11]), “Grassmannians” of linear subspaces of $\mathbb{P}_n$, $n \geq 5$ odd, which are totally isotropic with respect to a nullcorrelation (cf. [24, II.C.7, II.C.11]), and finally 24 pairwise non-isomorphic rank 1-homogeneous-rational manifolds, whose automorphism groups are exceptional simple complex Lie groups (cf. [23]). In higher rank, the probably best-known examples are, besides direct products of rank 1-homogeneous-rational manifolds, the flag manifolds of $\mathbb{P}_n$, $n \geq 2$, of rank $n$, the simplest being the rank 2-homogeneous-rational manifold $F_2 = \{(x, L) \in \mathbb{P}_2 \times \mathbb{P}_2^*; x \in L\}$ of dimension 3.

Now we are in position to state our main results.

**Theorem 1 (Factoriality Criterion):** The following statements about a holomorphic embedding $f: X \to \mathbb{P}_N$ of a homogeneous-rational manifold $X$ are equivalent:

(i) $f(X)$ is factorial;

(ii) (a) $\text{rk}(X) = 1$ and

(b) there is a linear $k$-plane $\mathbb{P}_k \subset \mathbb{P}_N$ such that $f(X) \subset \mathbb{P}_k$ and $f: X \to \mathbb{P}_k$ is homogeneously minimal.

The proof of this theorem is carried out in Section 2 by inspecting the divisor class group $\text{Cl}(X)$ of $X$ and using a criterion of factoriality which is due to Samuel. Fundamental for the proof is the following Normality Criterion which is proved in Section 1.

**Theorem 2 (Normality Criterion):** The following statements about a holomorphic embedding $f: X \to \mathbb{P}_N$ of a homogeneous-rational manifold $X$ are equivalent:

(i) $f(X)$ is projectively normal;

(ii) $f$ is homogeneously normal.

Recall that a projective variety is called projectively normal if its homogeneous coordinate ring is a normal domain. We suspect that (at least) part of the Normality Criterion is known, but we do not know any adequate reference (except for the case $X = G_{n,k}, f = \text{Plücker embedding}:$ Severi showed in 1915 that $f(X)$ is projectively normal (cf. [21, p. 100], see also [13]).

Let $X$ be a homogeneous-rational manifold homogeneously minimally embedded in $\mathbb{P}_N$. We define an affine kernel $X_a$ of $X$ to be the comple-
ment of a general hyperplane section in $X$. Thus $X_a$ is an affine variety, and we ask for the divisor class group of $X_a$. This question was raised by Remmert around 1965. We give a complete answer to this question:

**Theorem 3:** If $X$ is a homogeneous-rational manifold, then the divisor class group $\text{Cl}(X_a)$ of an affine kernel $X_a$ of $X$ is isomorphic to $\mathbb{Z}^{\text{rk}(X)-1}$. In particular, $X_a$ is factorial if and only if $\text{rk}(X) = 1$.

The proof of this theorem is similar to that of Theorem 1 and is also given in Section 2. It is done by investigating the canonical surjective mapping $\text{Cl}(X) \to \text{Cl}(X_a)$ between the divisor class groups of $X$ and $X_a$.

In Section 3, we give two applications of Theorem 1. First, the homogeneous coordinate ring $S$ of a homogeneously minimally embedded rank 1-homogeneous manifold $X$ is Gorenstein (for Grassmannians, this has been shown by Hochster in [11]). This is proved by first showing that $S$ is Cohen-Macaulay and then, of course, applying Murthy's Theorem ([17]). For the second application, let $X$ be a rank 1-homogeneous-rational manifold, homogeneously minimally embedded in $\mathbb{P}_N$, and let $R$ be the local ring of the vertex of the affine cone over $X$ in affine $(N+1)$-space. Using a result of Danilov ([6]), we prove that, unless $X$ is isomorphic to a projective space (in which case $R$ is regular), $R$ is a non-regular local unique factorization domain whose completion $\hat{R}$ is again factorial.

It should be noted that the proofs of the theorems as well as the applications, though not being very complicated, depend on an interplay of several mathematical fields: from representation theory of semisimple complex Lie algebras and Lie groups, we use Tits' embedding theorem and the Borel-Weil Theorem; from complex analysis, we use Bott's Theorem and results of Remmert-van de Ven; from algebraic geometry and commutative algebra, we use Samuel's Criterion of Factoriality, Murthy's Theorem, results of Danilov on the divisor class group of a complete local ring, etc.

It seems that most of our results carry over to varieties $G/H$ over more general algebraically closed ground fields $K$, at least for $\text{char } K = 0$.

It is the author's pleasure to thank Prof. R. Remmert for bringing the above mentioned problems to his attention as well as for many helpful conversations during the preparation of this paper.

**1. Proof of Theorem 2**

We first discuss the Borel-Weil Theorem, which will turn out to be crucial for the proof of our Normality Criterion. Let always $X = G/H$ be a homogeneous-rational manifold, where $G$ is a connected simply-connected semisimple complex Lie group acting transitively on $X$ and $H$ is a proper parabolic subgroup of $G$. We further denote by $\text{Cl}(X)$ the divisor class group of $X$. We begin with the following simple
**LEMMA 1:** $\text{Cl}(X) \cong H^1(X, \mathcal{O}^*) \cong H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{\text{rk}(X)}$.

**PROOF:** The first isomorphism follows from [10, p. 145]. Next, since $H^q(X, \mathcal{O}) = 0$ for $q \geq 1$ (cf. [5, Lemma 14.2]), from the exact cohomology sequence belonging to the short exact exponential sequence we obtain $H^1(X, \mathcal{O}^*) \cong H^2(X, \mathbb{Z})$. Finally, since $b_1(X) = 0$, $H^2(X, \mathbb{Z})$ is torsion-free, whence $H^2(X, \mathbb{Z}) \cong \mathbb{Z}^{\text{rk}(X)}$ because of $b_2(X) = \text{rk}(X)$. □

Now let $\mathcal{L}$ be a line bundle on $X$, $\mathcal{L} = \mathcal{O}(D)$ with $D$ a divisor on $X$. Let $|D| \cong \mathbb{P}(H^0(X, \mathcal{L}))$ be the corresponding linear system (which may be empty). Since $G$ is connected, $G$ acts trivially on $H^2(X, \mathbb{Z})$, hence on $\text{Cl}(X)$. Thus, for $D^* \in |D|$, $g \in G$, $g(D^*) \in |D|$, and hence $G$ acts on $|D| \cong \mathbb{P}(H^0(X, \mathcal{L}))$. Since $G$ is simply-connected, this action lifts to a linear action of $G$ on $H^0(X, \mathcal{L})$. In particular, $H^0(X, \mathcal{L})$ is a $G$-module.

Now we state the following special case of the Borel-Weil Theorem ([4]):

**THEOREM BW:** If $\mathcal{L}$ is a very ample line bundle on $X$, then the $G$-module $H^0(X, \mathcal{L})$ is irreducible.

**PROOF OF THEOREM 2:** Denote by $S = \sum_{n \geq 0} S_n$ the homogeneous coordinate ring of $f(X)$, and let $\bar{S}$ be the integral closure of $S$. Then we have (cf. [10, Ch. II, Ex. 5.14]): $\bar{S} = \sum_{n \geq 0} H^0(f(X), \mathcal{O}(n))$. Here, $\mathcal{O}(1)$ denotes the twisting sheaf of Serre (cf. [10, p. 117]), i.e., considered as a line bundle on $X$, $\mathcal{O}(1) = f^* \mathcal{H}$, where $\mathcal{H}$ is the hyperplane section bundle on $\mathbb{P}_N$, and $\mathcal{O}(n) = \mathcal{O}(1)^n$. Then, by Theorem BW, for all $n \geq 0$, $H^0(f(X), \mathcal{O}(n))$ is an irreducible $G$-module.

Now let $f$ be homogeneously normal. Then the linear action $\phi_f: G \to \text{SL}(N + 1, \mathbb{C})$ induces a natural action $\phi_f$ of $G$ on $S$. This action preserves the grading of $S$, i.e., $\phi_f(G)S_n \subseteq S_n$ for all $n \geq 0$, and, in fact, furnishes $S_n$ with the structure of a $G$-submodule of $H^0(f(X), \mathcal{O}(n))$. But $H^0(f(X), \mathcal{O}(n))$ is an irreducible $G$-module. Hence, for a fixed $n \geq 0$, we have either $S_n = 0$ which is clearly impossible, or $S_n = H^0(f(X), \mathcal{O}(n))$. Thus we obtain $S = \bar{S}$, and $f(X)$ is projectively normal.

Finally, if $f$ is not homogeneously normal, then evidently $S_1 \subsetneq H^0(f(X), \mathcal{O}(1))$, and hence $f(X)$ is not projectively normal. □

**EXAMPLE:** As a special case of Theorem 2, for $X = \mathbb{P}_1$, we obtain the following well-known facts:

1. The $d$-uple embedding of $\mathbb{P}_1$ in $\mathbb{P}_d$ (this embedding is given by the monomials in two variables of degree $d$) is projectively normal (cf. [10, Ch. IV, Ex. 3.4]).
2. The twisted quartic curve in $\mathbb{P}_3$ (this is given by the (non-homogeneously normal) embedding $[z_0 : z_1] \to [z_0^2 : z_0^2z_1 : z_0z_1 : z_1^3]$ of $\mathbb{P}_1$ in $\mathbb{P}_3$) is not projectively normal (cf. [10, Ch. I, Ex. 3.18]).
2. Proofs of Theorem 1 and Theorem 3

We begin with some useful remarks concerning very ample line bundles on homogeneous-rational manifolds. In this whole section, $X = G/H$ is a homogeneous-rational manifold, where, as usual, $G$ is a connected simply-connected semisimple complex Lie group acting transitively on $X$ and $H$ a proper parabolic subgroup of $G$. We further let $r = \text{rk}(X)$.

By Lemma 1, $H^1(X, \mathcal{O}^*) \cong \mathbb{Z}^r$, so it is reasonable to speak of line bundles of type $(n_1, \ldots, n_r) \in \mathbb{Z}^r$ on $X$. However, since a line bundle of type $(1, \ldots, 1)$ should be “positive”, this has to be rendered precise: Let $B$ be a Borel subgroup of $G$, and consider the $B$-action on $X$. Then there is an open $B$-orbit $U$ in $X$ (the open “Bruhat-cell”), and the complement $X - U$ is a divisor on $X$ consisting of $r$ irreducible components $D_1, \ldots, D_r$ (see [15] for details in the case $X = G/B$). Now, for $(n_1, \ldots, n_r) \in \mathbb{Z}^r$, define the line bundle of type $(n_1, \ldots, n_r)$ to be the line bundle belonging to the divisor $n_1D_1 + \ldots + n_rD_r$. Equivalently, line bundles of type $(n_1, \ldots, n_r)$ may be defined as follows: Let $R = T \cdot S$ be the reductive part of $H$, where $T \cong (\mathbb{C}^*)^r$ and $S$ is semisimple. Then the groups $T^*$ and $H^*$ of holomorphic characters of $T$ and $H$ are isomorphic, $T^* \cong H^*$. Obviously, $T^* \cong \mathbb{Z}^r$, the isomorphism being given by $\mathbb{Z}^r \ni (n_1, \ldots, n_r) \mapsto \chi_{(n_1, \ldots, n_r)}(z_1, \ldots, z_r) = z_1^{n_1} \cdot \ldots \cdot z_r^{n_r}$. Now the line bundle of type $(n_1, \ldots, n_r)$ on $X$ is the homogeneous line bundle given by the character $\chi_{(n_1, \ldots, n_r)} : T^* \to \mathbb{C}^*$, where $T^* \cong H^*$.

Furthermore, line bundles of type $(n_1, \ldots, n_r)$ on $X$ may be described in the following way: Let $P_1, \ldots, P_r$ the maximal parabolic subgroups of $G$ which contain $H$, let $X_i = G/P_i$, and let $\pi_i : X \to X_i$ the natural fibrations. Since $\text{rk}(X_i) = 1$, $H^1(X_i, \mathcal{O}^*) \cong \mathbb{Z}$. Now take positive generators $\mathcal{L}_i$ of $H^1(X_i, \mathcal{O}^*) \cong \mathbb{Z}$. Then the line bundle $\mathcal{L}$ of type $(n_1, \ldots, n_r)$ on $X$ is given by $\mathcal{L} = \pi_1^* (\mathcal{L}_1^{\otimes n_1}) \otimes \ldots \otimes \pi_r^* (\mathcal{L}_r^{\otimes n_r})$.

Now, very ample line bundles on $X$ can be easily characterised:

**Remark:** A line bundle $\mathcal{L}$ of type $(n_1, \ldots, n_r)$ on $X$ is very ample if and only if $n_i > 0$, $i = 1, \ldots, r$. In particular, a holomorphic embedding $f : X \to \mathbb{P}_N$ is homogeneously minimal if and only if $f$ is given by a base $s_0, \ldots, s_N$ of $H^0(X, \mathcal{L})$, where $\mathcal{L}$ is a line bundle of type $(1, \ldots, 1)$ on $X$.

**Proof:** The first part is contained in [4, §4], the second assertion follows from Tits’ Theorem (see [22, Korollar 2.2.2]).

We now come to the proof of Theorem 1. We employ Samuel’s

**Criterion of Factoriality** (cf. [10, Ch. II, Ex. 6.3]): A projective variety $V$ is factorial if and only if (1) $V$ is projectively normal, and (2) the divisor class group $\text{Cl}(V)$ of $V$ is isomorphic to $\mathbb{Z}$ and is generated by the class of a (suitable) hyperplane section.
PROOF OF THEOREM 1: By Samuel's Criterion and Lemma 1, clearly \( \text{rk}(X) = 1 \) if \( f(X) \) is factorial. So assume \( \text{rk}(X) = 1 \). Take a line bundle \( \mathcal{L}_0 \) of type (1) on \( X \). Hence \( \mathcal{L}_0 \) generates \( H^1(X, \mathcal{O}^*) \cong \mathbb{Z} \). By the Remark, \( f \) is given by sections \( s_0, \ldots, s_N \in H^0(X, \mathcal{L}) \), where \( \mathcal{L} \) is a line bundle of type \( (n) \) on \( X \), \( n \geq 1 \), i.e. \( \mathcal{L} = \mathcal{L}_0^n \) with \( n \geq 1 \). Let \( V^* \) be a hyperplane in \( \mathbb{P}_N \) not containing \( f(X) \) and \( V \) the corresponding hyperplane section. Then we have \( \text{Cl}(f(X))/\langle V \rangle \cong H^1(X, \mathcal{O}^*)/\langle \mathcal{L} \rangle \cong \mathbb{Z}/n\mathbb{Z} \). Now, by Samuel's Criterion, \( f(X) \) is factorial if and only if \( n = 1 \) and \( f(X) \) is projectively normal. Hence the assertion follows from Theorem 2 and the Remark.

PROOF OF THEOREM 3: Let \( f: X \to \mathbb{P}_N \) be a homogeneously minimal embedding of \( X \), and let an affine kernel \( X_a \) of \( X \) be given by \( X_a = f(X) - Z \), where \( Z \) is a general (i.e. smooth) hyperplane section of \( f(X) \). We consider the exact sequence \( \mathbb{Z} \to \text{Cl}(X) \to \text{Cl}(X_a) \to 0 \), where the map \( i \) is given by \( 1 \to 1 \cdot Z \) (cf. [10, Ch. II, Prop. 6.5]). Since the group \( \text{Cl}(X) \) is torsion-free, the map \( i \) is injective. We have to determine the image \( \text{Im}(i) \) of \( i \) in \( \text{Cl}(X) \). First, \( \text{Cl}(X) \cong H^1(X, \mathcal{O}^*) \cong \mathbb{Z}^r \). Next, by the Remark, \( f \) is given by a base \( s_0, \ldots, s_N \) of \( H^0(X, \mathcal{L}_0) \), where \( \mathcal{L}_0 \) is a line bundle of type \( (1, \ldots, 1) \) in \( H^1(X, \mathcal{O}^*) \). Hence, for the divisor class group of \( X_a \) we obtain: \( \text{Cl}(X_a) \cong \text{Cl}(X)/\text{Im}(i) \cong H^1(X, \mathcal{O}^*)/\langle \mathcal{L}_0 \rangle \cong \mathbb{Z}^r/\langle (1, \ldots, 1) \rangle \cong \mathbb{Z}^{r-1} \), whence the assertion. In particular, if \( r = 1 \), then \( \text{Cl}(X_a) = 0 \), and hence \( X_a \) is factorial (cf. [10, Ch. II, Prop. 6.2]).

3. Applications

In this section, we give two applications of Theorem 1 and the following special case of a theorem of Bott (cf. [5, Thm. IV']):

**THEOREM B:** If \( \mathcal{L} \) is a very ample line bundle on a homogeneous-rational manifold \( X \), then \( H^q(X, \mathcal{L}) = 0 \) for \( q \geq 1 \) and \( H^q(X, \mathcal{L}^{-1}) = 0 \) for \( q < \dim X \).

For the first application, recall that a noetherian ring \( A \) is called Cohen-Macaulay (Gorenstein, resp.) if, for every maximal ideal \( m \) of \( A \), the local ring \( A_m \) is Cohen-Macaulay (Gorenstein, resp.), i.e. \( \dim A_m = \text{depth } A_m \) (the injective dimension of \( A_m \) is finite, resp.). For generalities on Cohen-Macaulay and Gorenstein rings, see e.g. [14]. Now we can state:

**COROLLARY 1:** The homogeneous coordinate ring \( S \) of a homogeneously minimally embedded rank 1-homogeneous-rational manifold \( X \) is Gorenstein.
PROOF: According to a theorem of Murthy ([17], see also [7, Thm. 12.3]), a factorial Cohen-Macaulay factor ring of a Gorenstein ring is Gorenstein. Hence, by Theorem 1, it is sufficient to show that $S$ is Cohen-Macaulay. In fact, we have quite generally

PROPOSITION: If $f: X \to \mathbb{P}_N$ is a homogeneously normal embedding of a homogeneous-rational manifold $X$, then the homogeneous coordinate ring $S(f(X))$ of $f(X)$ is Cohen-Macaulay.

PROOF: It is well-known that the homogeneous coordinate ring $S(V)$ of a nonsingular projectively normal projective variety $V$ is Cohen-Macaulay provided $H^q(V, \mathcal{O}(n)) = 0$ for all $n \in \mathbb{Z}$ and $1 \leq q \leq \dim V - 1$ (this is e.g. a special case of Prop. B, p. 131, and Prop. 5.1 of [12]). Applying this theorem to our situation, it follows from Theorem 2 and Theorem B that $S(f(X))$ is Cohen-Macaulay. $\square$

For the second application, let $X$ be a homogeneous-rational manifold, and let $f: X \to \mathbb{P}_N$ be a homogeneously minimal embedding of $X$. Denote by $V(X)$ the affine cone over $f(X)$ in affine $(N + 1)$-space. Let $P$ be the vertex of $V(X)$ and $R = \mathcal{O}_{V(X), P}$ the local ring of $P$ on $V(X)$. Thus, by Theorem 2, $R$ is a normal domain, and, unless $X$ is isomorphic to $\mathbb{P}_n$ for some $n$, $R$ is not regular.

Now assume additionally $\text{rk}(X) = 1$. Then, by Theorem 1, $f(X)$ is factorial, and hence $R$ is a unique factorization domain (cf. [7, Cor. 10.3]). One may ask whether the completion $\hat{R}$ of $R$ with respect to its maximal ideal is again factorial. In general, if $A$ is a local noetherian Krull domain and $\hat{A}$ its completion, then, by Mori’s Theorem (cf. [7, Cor. 6.12]), $A$ is factorial if $\hat{A}$ is, but the converse is false in general (see [7, Example 19.9] for a counterexample). In our case, however, we have

COROLLARY 2: The ring $\hat{R}$ is a unique factorization domain.

PROOF: We use the following result of Danilov (cf. [6, Theorem in §2 and Prop. 8], see also [16, p. 532 f.]): If $V$ is a nonsingular projectively normal projective variety, $A$ the local ring of the vertex of the affine cone over $V$, and $\hat{A}$ the completion of $A$, then the divisor class groups of $A$ and $\hat{A}$ are isomorphic if and only if $H^1(V, \mathcal{O}(n)) = 0$ for all $n \geq 1$. Applying this theorem to our situation, by Theorem B we obtain $\text{Cl}(\hat{R}) = \text{Cl}(R) = 0$, whence the assertion. $\square$

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(Oblatum 22-VII-1982)

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