COMPOSITIO MATHEMATICA

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Compositio Mathematica, tome 52, nº 3 (1984), p. 315-324
<http://www.numdam.org/item?id=CM_1984__52_3_315_0>

Compositio Mathematica 52 (1984) 315–324

ON UNIFORMLY REGULAR MEASURES

A. Sapounakis

1. Introduction

The concept of uniform regularity of measures on a compact space, a property which allows uniform approximation of the measure on all closed sets, was introduced and discussed in [2] and [3]. There, it is shown that these measures, share a lot of properties with measures on metric spaces. For instance, they have separable supports, separable L^p spaces, $1 \leqslant p < +\infty$, and their associated topological measure spaces can be represented in a certain way by metric measure spaces.

Some extensions of the notion of uniform regularity in case when the underlying space X is not compact are given in [5] and [6]. In this case, there is no unique uniformity inducing the topology of X and thus the notion of uniform regularity depends on a given uniformity $\mathscr{U}$.

In this paper we examine the notion of uniform regularity in conjunction with the various uniformities which induce the completely regular topology of X. In §3 it is shown that a τ-additive measure μ, which is uniformly regular with respect to some admissible uniformity $\mathscr{U}$, must be universally uniformly regular (i.e. uniformly regular with respect to any admissible uniformity). On the other hand it is proved that a universally uniformly regular measure is necessarily τ-additive, and thus extending a result of Babiker [5] for measures on locally compact spaces, to arbitrary completely regular spaces. Also here it is proved that the notions $\mathscr{C}^*$-uniform regularity and universally uniform regularity are equivalent for two-valued measures, which is not true in general for real-valued measures (see example 4.5 in [14]).

In §4 the concept of uniform regularity is examined for measures on product spaces. It is shown that a τ-additive measure μ on $X_1 \times X_2$ is uniformly regular if and only if its projections on X_1, X_2 are uniformly regular, thus generalizing a result of [4]. Finally it is shown by an example that this result is not true in general for σ-additive measures.

2. Definitions and preliminary results

Let X be a completely regular (Hausdorff) space. By $C(X)$ we mean the ring of all real-valued continuous functions on X and by $C^*(X)$ the subring of bounded functions. A zero set in X is a set of the form $f^{-1}(0)$ for some $f \in C(X)$. A cozero set in X is a complement of a zero set. We denote by $\mathscr{B}^*$ (resp. $\mathscr{B}_0^*$) the minimal algebra generated by the zero (resp. closed) sets of X and by $\mathscr{B}$ (resp. $\mathscr{B}_0$) the family of Baire (resp. Borel) sets, which is the σ-algebra generated by $\mathscr{B}^*$ (resp. $\mathscr{B}_0^*$).

Baire measures and Borel measures are finite, non-negative, finitely additive set functions defined on $\mathscr{B}^*$ and $\mathscr{B}_0^*$ respectively. Further we assume that all measures are regular in the sense of inner approximation by zero sets in the Baire case and by closed sets in the Borel case. Finally by a measure we shall mean either a Baire or a Borel measure.

A Baire measure μ is said to be

(i) σ-additive, if for any decreasing sequence $\{Z_n\}$ of zero subsets, with $\cap_n Z_n = \emptyset$, we have $\mu(Z_n) \to 0$.

(ii) u-additive, if for every partition of unity $\{f_a\}_{a \in A}$ we have $\Sigma_a \int_X f_a \,\mathrm{d}\mu = \mu(X)$.

(iii) τ-additive, if for any decreasing net $\{Z_a\}$ of zero subsets of X with $\cap_a Z_a = \emptyset$, we have $\mu(Z_a) \to 0$.

For Borel measures, σ-additivity and τ-additivity are defined similarly by replacing zero sets by closed sets, while the definition of u-additivity remains unchanged. A Borel measure is τ-additive iff its Baire restriction is τ-additive. Clearly the τ-additivity implies u-additivity which in turn implies σ-additivity. Further a σ-additive measure μ is u-additive if and only if for every continuous pseudometric d on X there exists a d-closed, d-separable subset of X with full μ-measure [15].

Now let μ be a Baire measure on X and $\tilde{\nu}$ the corresponding Borel measure on βX defined by

$$\int_X f \mathrm{d}\mu = \int_{\beta X} \bar{f} \mathrm{d}\tilde{\nu}$$

where $f \in C^*(X)$ and $\bar{f}$ is the continuous extension of f to βX. Then the additivity properties of μ are characterized in terms of the measure theoretic properties of $\tilde{\nu}$ as follows.

THEOREM 2.1. (KNOWLES [9]):

(*a*) *μ is σ-additive if and only if $\tilde{\nu}(Z) = 0$ for every zero set Z in βX, disjoint from X.*

(*b*) *μ is τ-additive if and only if $\tilde{\nu}(K) = 0$ for every compact subset K of βX, disjoint from X.*

A similar characterization of u-additivity is given in [[11], Theorem 3.2]. For more information about the additivity properties of measures we refer to [15], [9] and [16].

Now let $\mathscr{U}$ be an admissible uniformity of X. A Baire measure μ is called $\mathscr{U}$-uniformly regular if there exists a sequence $\{V_n\} \subset \mathscr{U}$ such that for any zero set $Z \subset X$, $V_n(Z) \in \mathscr{B}^*$ for all n and $\mu(Z) =$

$\lim_{n\to\infty}\mu(V_n(Z))$, where $V_n(Z)=\{x\in X:(x,y)\in V_n$ for some $y\in Z\}$. If we replace zero sets by closed we obtain the definition of uniform regularity for Borel measures [5].

We note that for every sequence $\{U_n\}\subset\mathscr{U}$ there exists a sequence $\{V_n\}\subset\mathscr{U}$ such that $V_n\subset U_n$ and $V_n(E)$ is a cozero subset of X, for every $E\subset X$. Indeed let $\mathscr{V}$ be the uniformity generated by $\{U_n\}$. Clearly $\mathscr{V}\subset\mathscr{U}$ and there is a continuous pseudometric ρ such that the family of all sets of the form $W_\epsilon=\{(x,y):\rho(x,y)<\epsilon)$ is a base of $\mathscr{V}$. Further for each $E\subset X$, the set $W_\epsilon(E)$ is open in the pseudometric topology induced by ρ and so must be a cozero subset of X. If follows that there is a sequence $\{V_n\}\subset\{W_\epsilon\}_{\epsilon>0}$ with the desired properties.

This discussion suggests the following.

PROPOSITION 2.2: *Let ν be a Borel measure on X and μ its Baire restriction. Then if ν is $\mathscr{U}$-uniformly regular so is μ. Moreover if ν is τ-additive the converse is also true.*

The last part of Proposition 2.2 follows from the simple observation that for a τ-additive measure ν and a closed subset F of X there exists a zero set $Z\subset X$ such that $F\subset Z$ and $\nu(F)=\nu(Z)$ (see Proposition 3.2 in [5]).

We need one more definition. A measure μ is called universally uniformly regular if it is $\mathscr{U}$-uniformly regular for every admissible uniformity $\mathscr{U}$ of X. In [5] it is shown that every universally uniformly regular measure on a locally compact space is τ-additive.

3. Uniform regularity and additivity

Let $\mathscr{U}_0$ be the uniformity of X generated by all continuous pseudometrics on X and C^* the uniformity generated by the bounded continuous functions of X. For a measure μ on X we consider the following regularity conditions.

(i) μ is universally uniformly regular.

(ii) μ is $\mathscr{C}^*$-uniformly regular.

(iii) μ is $\mathscr{U}_0$-uniformly regular

Clearly since $\mathscr{U}_0$ is the largest admissible uniformity of X we have that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). The next theorem shows that for τ-additive measures these conditions are equivalent.

THEOREM 3.1: *Let X be a completely regular space and μ a $\mathscr{U}_0$-uniformly regular τ-additive measure on X. Then μ is universally uniformly regular.*

PROOF: By Proposition (2.2) is is enough to prove the theorem in the case when μ is a Baire measure.

Let $\mathscr{U}$ be any admissible uniformity of X. We first note that for every zero set $Z\subset X$, there exists a sequence $\{U_k\}_{k=1,2\ldots}\subset\mathscr{U}$ such that $U_k(Z)$

$\in \mathscr{B}$ and

$$\mu(Z) = \lim_{k\to\infty} \mu(U_k(Z)) \tag{$*$}$$

This follows from the fact that $Z = \cap_{U \in \mathscr{U}} U(Z) = \cap_{U\in\mathscr{U}} \mathrm{Cl}_X(U(Z))$ and the τ-additivity of μ.

Now by the hypothesis there exists a sequence $\{V_n\}_{n=1,2\ldots} \subset \mathscr{U}_0$ such that $V_{n+1}^2 \subset V_n = V_n^{-1}$ for all n, $V_n(A)$ is cozero for every $A \subset X$ and $\mu(Z) = \lim_{n\to\infty}\mu(V_n(Z))$, for every zero set $Z \subset X$. For every n the family $\{V_n(x)\}_{x\in X}$ is a cover of X by cozero sets and using the τ-additivity property of μ we can find a countable set $Q_n \subset X$ such that $\mu(V_n(Q_n)) = \mu(X)$. Set $Q = \cup_{n\in N} Q_n$, $Y = \cap_{n=1}^{\infty} V_n(Q)$ and $\mathscr{Z}$ the family of all zero subsets of X which are complements of finite unions of sets of the form $V_n(x)$, where $x \in Q$. Clearly $\mathscr{Z}$ is countable and using $(*)$ we may find a decreasing sequence $\{U_k\}_{k=1,2,\ldots} \subset \mathscr{U}$ such that $U_k(C) \in \mathscr{B}$ and $\mu(C) = \lim_{k\to\infty}\mu(U_k(C))$, for every $C \in \mathscr{Z}$.

We show that $\mu(Z) = \lim_{k\to\infty}\mu(U_k(Z))$, for every zero set $Z \subset X$. Indeed set $K = \cap_{n=1}^{\infty} V_n(Z)$. Then for each $x \in Y \cap (X \setminus K)$ there is a $n_x \in N$ and $y_x \in Q$ such that $x \in V_{n_x}(y_x)$ and $V_{n_x}(y_x) \cap K = \emptyset$. Using this we can find a non-increasing sequence $\{C_n\} \subset \mathscr{Z}$, such that $K \subset \cap_n C_n \subset K \cup (X \setminus Y)$. Then since $\mu(Y) = \mu(X)$ we have

$$\mu(Z) = \mu(K) = \lim_{n\to\infty} \mu(C_n) = \lim_{n\to\infty}\lim_{k\to\infty} \mu(U_k(C_n))$$

$$= \mu\Big(\bigcap_{k,n} U_k(C_n)\Big) \geqslant \mu\Big(\bigcap_{k=1}^{\infty} U_k(Z)\Big) = \lim_{k\to\infty}\mu(U_k(Z)) \geqslant \mu(Z).$$

Thus the proof of the theorem is complete.

We note that in the proof of the above theorem the τ-additivity property is used twice. The second time it is used for sets of the form $V_n(x)$, which can be chosen to be open in the pseudometric topology generated by $\{V_n\}$, so that what it is really used here is the u-additivity property of μ. Using this idea and the fact that for every zero set $Z \subset X$ there is a sequence $\{U_k\}_{k=1,2,\ldots} \subset \mathscr{C}^*$ such that $Z = \cap_{k=1}^{\infty} U_k(Z)$ we may easily conclude the following.

COROLLARY 3.2: *A u-additive measure μ is $\mathscr{U}_0$-uniformly regular if and only if it is $\mathscr{C}^*$-uniformly regular.*

We remark that a $\mathscr{C}^*$-uniformly regular measure μ is u-additive if and only if it is σ-additive. Indeed by the $\mathscr{C}^*$-uniform regularity of μ there exists a sequence $\{U_n\} \subset \mathscr{C}^*$ such that $U_n(Z) \in \mathscr{B}(X)$ and $\mu(Z) = \lim_{n\to\infty}\mu(U_n(Z))$, for every zero $Z \subset X$. Then there exists a countable set $Q \subset X$ such that $U_n(Q) = X$ for all n. Now if d is a continuous pseudometric on X and S is the d-closure of Q we have that S is a zero set of X and $\mu(S) = \lim_{n\to\infty}\mu(U_n(S)) = \mu(X)$.

It is natural to ask whether the assumption of τ-additivity is important in Theorem 3.1. The answer is given by the next theorem.

THEOREM 3.3: *A universally uniformly regular, finitely additive measure μ on a completely regular space X, is τ-additive.*

PROOF: Without loss of generality we may assume that μ is a Baire measure.

We assume (for the purpose of contradiction) that μ is not τ-additive. Then by Theorem 2.1 there exists a compact subset K of $\beta X\setminus X$ such that $\tilde{\nu}(K) > 0$. We claim that without loss of generality we may assume that $\tilde{\nu}(K) = \tilde{\nu}(\beta X)$. To see this define a linear functional I on $\mathscr{C}^*(X)$ by

$$I(f) = \int_K \bar{f}\mathrm{d}\tilde{\nu}$$

where $\bar{f}$ denotes the Stone extension of f to βX. Then by the integral representation theorem there exists a Baire measure μ_1 on X such that $I(f) = \int_X f\mathrm{d}\mu_1$. It follows that μ_1 is universally uniformly regular (since $\mu_1 \leqslant \mu$) and $\tilde{\nu}_1(K) = \tilde{\nu}_1(\beta X)$, where $\tilde{\nu}_1$ denotes the Borel measure on βX associated with μ_1. Thus the proof of the claim is complete.

Now for each $x \in X$ there is a function $\bar{f}_x \in C(\beta X)$ such that $0 \leqslant \bar{f}_x \leqslant 1$, $\bar{f}_x(x) = 1$ and $\bar{f}_x(z) = 0$ for every $z \in K$. It follows that $\int_X f_x \mathrm{d}\mu = 0$, where f_x is the restriction of $\bar{f}_x$ to X. Let $\mathscr{F} = \{f \in C^*(X): 0 \leqslant f \leqslant 1$ and $\int_X d\mu = 0\}$ and $\mathscr{U}$ the uniformity of X generated by $\mathscr{F}$. Then $\mathscr{U}$ generates the topology of X. Indeed let $U = \{\xi \in X: h(\xi) > 0\}$ such that $h \in C^*(X)$ and $0 \leqslant h \leqslant 1$. For $x \in U$ let $g = f_x h$ and $\epsilon = h(x)/2$. Then $g \in \mathscr{F}$ and $\{\xi \in X: |g(\xi) - g(x)| < \epsilon\} \subset U$.

It follows by the hypothesis that μ is $\mathscr{U}$-uniformly regular and so there exists a sequence $\{U_n\} \subset \mathscr{U}$ such that $U_n(Z) \in \mathscr{B}(X)$ for all n and $\mu(Z) = \lim_{n\to\infty}\mu(U_n(Z))$, for every zero set $Z \subset X$. Moreover without loss of generality we may assume that $U_n = \{(x, y): |f_i(x) - f_i(y)| < \epsilon_n$, $i = 1, 2, \ldots n\}$, where $f_i \in \mathscr{F}$ and ϵ_n is a decreasing to zero, sequence of positive numbers.

For every n let $Z_n = \{x \in X: \sup_{1\leqslant i\leqslant n} f_i(x) \geqslant \epsilon_n\}$ and $W_n = \{x \in X: \sup_{1\leqslant i\leqslant n+1} f_i(x) > \epsilon_{n+1}\}$. Clearly Z_n is a zero set, W_n a cozero set and $Z_n \subset W_n \subset Z_{n+1}$. Moreover we show that for every zero set $Z \subset X$ we have

$$\mu(Z) = \begin{cases} 0 & \text{if} \quad Z \subset Z_n \quad \text{for some } n \\ \mu(X) & \text{Otherwise} \end{cases}$$

Indeed if $Z \cap (X\setminus Z_n) \neq \emptyset$ for all n we can easily show that $U_n(Z) \supset X\setminus Z_n$

for all n. It follows that $\mu(U_n(Z)) \geqslant \mu(X \setminus Z_n) = \mu(X)$ and $\mu(Z) = \lim_{n\to\infty}\mu(U_n(Z)) = \mu(X)$.

This in particular shows that $X = \cup_{n=1}^{\infty} Z_n$ and so $\{Z_n\}$ is a regular sequence in the sense of [17]. Thus by Theorem 13 Part I of [16] there exists a function g in $C^*(X)$ such that $0 \leqslant g \leqslant 1$ and $Z_n = \{x \in X: g(x) \leqslant 1 - 1/n\}$. Let $\bar{g}$ be the Stone extension of g and $Z_0 = \{x \in \beta X: \bar{g}(x) = 1\}$. Then it is not hard to see that Z_0 is the support of $\tilde{\nu}$ (i.e. $\tilde{\nu}(V) > 0$ for every open subset V of βX with $Z_0 \cap V \neq \emptyset$) and since $\tilde{\nu}$ is a two-valued measure we deduce that Z_0 is a singleton. It follows that $\beta X \setminus X$ contains a G_δ point which is a contradiction [Cor. 9.6 in [8]].

In [14] assuming the continuum hypothesis it is given an example of a σ-additive, $\mathscr{C}^*$-uniformly regular measure which is not τ-additive. This example can be used to show that the conditions (i), (ii) (in the beginning of this section) are not in general equivalent even for σ-additive measures. The next result shows that (i), (ii) are always equivalent for two-valued measures.

PROPOSITION 3.4: *Every two-valued $\mathscr{C}^*$-uniformly regular measure on X is τ-additive.*

PROOF. Without loss of generality we may assume that μ is a probability Baire measure.

Assume that μ is not τ-additive. Then there exists a point $x_0 \in \beta X - X$ such that $\tilde{\nu} = \delta_{x_0}$ (the Dirac measure at x_0). Now let $\{U_n\}$ be a sequence in $\mathscr{C}^*$ such that $U_n(Z) \in \mathscr{B}(X)$ and $\mu(Z) = \lim_{n\to\infty}\mu(U_n(Z))$ for every zero subset Z of X. Without loss of generality we may assume that $U_n = \{(x, y): |f_i(x) - f_i(y)| < \epsilon_n,\ i = 1, 2, \ldots n\}$ where $f_i \in \mathscr{C}^*(X)$ and $\{\epsilon_n\}$ is a decreasing sequence of positive numbers.

Define a sequence $\bar{g}_n \in C(X)$ by $\bar{g}_n(x) = \sup_{1\leqslant i\leqslant n}|\bar{f}_i(x) - \bar{f}_i(x_0)|$ where $\bar{f}_i$ is the Stone extension of f_i to X. Let $Z_0 = \cap_{n=1}^{\infty}\{x \in \beta X: \bar{g}_n(x) = 0\}$. Then Z_0 is a zero subset of βX and $x_0 \in Z_0$. Moreover we show that if Z is a zero subset of X with $\mu(Z) = 0$ then $Z_0 \cap Cl_{\beta X}(Z) = \emptyset$. Indeed if $\mu(Z) = 0$ there exists an n such that $\mu(U_n(Z)) = 0$. Then for each $z \in Z$ we have

$$\epsilon_n \leqslant \int_X \sup_{1\leqslant i\leqslant n} |f_i - f_i(z)| \mathrm{d}\mu = \int_{\beta X} \sup_{1\leqslant i\leqslant n} |\bar{f}_i - f_i(z)| \mathrm{d}\tilde{\nu} = \bar{g}_n(z)$$

and so $Z_0 \cap Cl_{\beta X}Z = \emptyset$.

Now let $x_1 \in Z_0$ such that $x_1 \neq x_0$ (such x_1 exists since x_0 is not a G_δ point of βX). Then we find a zero subset Z_1 of βX such that $x_1 \in Cl_{\beta X}(Z_1 \cap X)$ and $x_0 \notin Z_1$. It follows that $\mu(Z_1 \cap X) = 1$ though $\tilde{\nu}(Z_1) = 0$ which is a contradiction.

In [5] it is given an example of a purely finitely additive, two-valued measure μ which is $\mathscr{C}_0$-uniformly regular. Clearly by the above proposition, μ is not $\mathscr{C}^*$-uniformly regular and so we get an example showing that (ii) and (iii) are not in general equivalent for two valued measures. Moreover if we assume the existence of real-valued measurable cardinals we can produce examples of σ-additive, $\mathscr{U}_0$-uniformly regular measures which are not $\mathscr{C}^*$-uniformly regular.

We now state two immediate corollaries of proposition 3.4.

COROLLARY 3.5: *Let μ be a two-valued finitely additive measure on X. Then μ is $\mathscr{C}^*$-uniformly regular if and only if it is universally uniformly regular.*

COROLLARY 3.6: *Every continuous (i.e. $\mu^*(\{x\})=0$ for every $x \in X$), $\mathscr{C}^*$-uniformly regular finitely additive measure μ is non-atomic.*

In [[1], Proposition 4.2] it is proved that every σ-additive non-atomic measure is $\mathscr{C}^*$-uniformly regular on points (i.e. $\exists\{V_n\}_{n=1,2,\dots} \subset \mathscr{C}^*$ such that $\mu^*(\{x\}) = \lim_{n\to\infty}\mu(V_n(x))$ for every $x \in X$). On the other hand in [[10], Theorem 3.19] it is shown that a continuous, σ-additive, $\mathscr{C}^*$-uniformly regular on points measure must be non-atomic. Corollary 3.6 gives an analogous result for finitely additive measures, assuming a stronger regularity condition. The following example shows that the assumption of $\mathscr{C}^*$-uniform regularity of μ cannot be replaced in Corollary 3.6 by the weaker assumption of $\mathscr{C}^*$-uniform regularity on points.

EXAMPLE 3.7: Let N be the set of all positive integers with the discrete topology. For a fixed $x \in \beta N - N$ we define a continuous measure μ on N by

$$\int_N f \mathrm{d}\mu = \bar{f}(x)$$

where $f \in C^*(N)$ and $\bar{f}$ is the unique continuous extension of f on βN. Clearly μ is atomic, though it is $\mathscr{C}^*$-uniformly regular to points. To see this for $n \in N$ define $f_n \in C^*(N)$ by $f_n(x) = \min\{n, x\}$ and $V_n = \{(x, y) \in N \times N : |f_n(x) - f_n(y)| < 1\}$. It follows that $\{V_n\}_{n=1,2,\dots} \subset \mathscr{C}^*$ and $\lim_{n\to\infty}\mu(V_n(x)) = 0$ for each $x \in N$.

4. Measures on product spaces

In view of Theorem 3.1 the notion of uniform regularity of a τ-additive measure μ is not dependent upon a particular uniformity and so we will just say that μ is uniformly regular.

We start this section with a lemma, the proof of which is easy and will be omitted.

LEMMA 4.1: *Let f be a continuous, open function from X onto Y and μ a*

τ-additive, uniformly regular measure on X. Then the image measure $v = f(\mu)$ is also uniformly regular.

We note that the hypothesis "f is open" cannot be dropped in the above lemma, as Example 2.1 in [4] shows.

We now give the main result of this section.

THEOREM 4.2: *Let X_1, X_2 be two completely regular spaces and $X = X_1 \times X_2$. Denote by p_i the projection of X onto X_i, $i = 1, 2$. For a τ-additive measure μ on X let μ_i be the image measure of μ with respect to p_i, $i = 1, 2$. Then μ is uniformly regular on X iff both μ_1, μ_2 are uniformly regular on X_1, X_2 respectively.*

PROOF: By Proposition 2.2 it is enough to show the theorem in the case when μ is a Borel measure. Clearly the previous lemma shows that the uniform regularity of μ implies that of μ_1, μ_2.

Conversely, by the uniform regularity of μ_1, μ_2 we find two decreasing sequences, say $\{U_m\}$ and $\{V_n\}$ in $\mathscr{U}_0(X_1)$, $\mathscr{U}_0(X_2)$ respectively, such that for every closed $F_1 \subset X_1$ and closed $F_2 \subset X_2$ we have

$$\mu_1(F_1) = \lim_{m \to \infty} \mu_1(U_m(F_1)) \quad \text{and} \quad \mu_2(F_2) = \lim_{n \to \infty} \mu_2(V_n(F_2)).$$

For each m, n we define a sequence in the product uniformity $\mathscr{U}_0(X_1) \times \mathscr{U}_0(X_2)$ of X by

$$W_{mn} = \{((x_1, x_2), (y_1, y_2)) : (x_1, y_1) \in U_m \quad \text{and} \quad (x_2, y_2) \in V_n\}$$

We show that

$$\mu(F) = \lim_{m,n} \mu(W_{mn}(F)) \tag{$**$}$$

for every closed subset $F \subset X$.

Let $\mathscr{F}$ be the family of all closed subsets of X of the form $p_1^{-1}(F_1) \cup p_2^{-1}(F_2)$, where F_i is a closed subset of X_i, $i = 1, 2$.

We first show that $(**)$ is valid for every $F \in \mathscr{F}$. Indeed let $F = p_1^{-1}(F_1) \cup p_2^{-1}(F_2)$ then we have

$$W_{mn}(F) = p_1^{-1}(U_m(F_1)) \cup p_2^{-1}(V_n(F_2))$$

and

$$\lim_{m,n} \mu(W_{mn}(F)) = \mu\Big(\bigcap_{m,n} W_{mn}(F)\Big)$$

$$= \mu\left(p_1^{-1}\Big(\bigcap_{m=1}^{\infty} U_m(F_1)\Big) \cup p_2^{-1}\Big(\bigcap_{n=1}^{\infty} V_n(F_2)\Big)\right)$$

$$= \mu\left(p_1^{-1}(F_1) \cup p_2^{-1}(F_2)\right) = \mu(F).$$

Now let F be any closed subset F of X. Then using the τ-additivity property of μ we may find a sequence $\{F_k\} \subset \mathscr{F}$ such $F \subset F_k$ for all k and $\mu(F) = \mu(\bigcap_{k=1}^{\infty} F_k)$. It follows that

$$\lim_{m,n} \mu(W_{mn}(F)) = \mu\Big(\bigcap_{m,n} W_{mn}(F)\Big) \leqslant \mu\Big(\bigcap_{k} \bigcap_{m,n} W_{mn}(F_k)\Big)$$

$$= \mu\left(\bigcap_{k=1}^{\infty} F_k\right) = \mu(F) \leqslant \lim_{m,n} \mu((W_{mn}(F)).$$

Thus $(**)$ is valid for every closed $F \subset X$ and μ is uniformly regular.

The following result is a consequence of Theorem 4.2 and Corollary 3.10 of [14].

COROLLARY 4.3: *Let μ be a τ-additive measure on $X_1 \times X_2$, where X_1, X_2 are totally ordered, first countable spaces. Then μ is uniformly regular on $X_1 \times X_2$.*

REMARKS 4.4:

(i) We note that Theorem 4.2 and Corollary 4.3 can be extended to uncountable products but are not valid for uncountable products as Example 5.5 in [3] shows.

(ii) Theorem 4.2 is proved in [4] in the special case when X_1, X_2 are compact spaces, μ_1, μ_2 are Radon measures on X_1, X_2 respectively and μ the unique Radon extension of the product measure $\mu_1 \times \mu_2$.

We finally give an example showing that Theorem 4.2. is not valid in general for non-τ-additive measures.

EXAMPLE 4.5: Let S be the real line with the left-closed right-open interval topology and $X = S^2$. Let D be the set $\{(x, y) \in S^2 : x + y = 0\}$ and μ a probability, continuous, σ-additive Baire measure on X with $\mu(D) = 1$ (see Example 4 in [13] for the existence of such measures). We will first show that μ is not $\mathscr{U}_0$-uniformly regular.

We assume (for the purpose of contradiction) that μ is $\mathscr{U}_0$-uniformly regular. We note that since X satisfies the countable chain condition, μ must be u-additive (see Theorem 4.1) in [12]) and so by Corollary 3.2 μ is $\mathscr{C}^*$-uniformly regular. Let $\{U_n\}$ be a sequence of elements in $\mathscr{C}^*$ such that $U_{n+1}^2 \subset U_n = U_n^{-1}$ for all n and $\mu(Z) = \lim_{n\to\infty} \mu(U_n(Z))$ for every zero $Z \subset X$. Then it is easy to see that there exists a countable set $Y \subset X$ such that $U_n(Y) = X$ for all n. For $n \in N$ and $y \in Y$ we define a sequence $\{p_{n,y}\} \subset X$ such that

$$p_{n,y} \in U_n(Y) \cap D \qquad \text{if} \quad U_n(y) \cap D \neq \emptyset$$

Now let K be a compact (in the original topology) subset of $\mathbb{R}^2$ such that $K \subset D \setminus \{p_{n,y}\}$ and $\mu(K) > 0$. Set

$$Z = \bigcap_{(x,y)\in K} X \setminus ([x, x+1) \times [y, y+1))$$

Then by Lemma 1.2 of [7] Z is a clopen set and $\mu(Z) < 1$. Further it is easy to see that $U_n(Z) \supset D$ for all n and so $\lim_{n\to\infty} \mu(U_n(Z)) = 1$ which is a contradiction. Thus μ is not $\mathscr{U}_0$-uniformly regular though since S is a closed subset of a totally ordered space (see Example 2.2 (b) and Theorem 2.9 in [12]) every measure on S is uniformly regular (see Corollary 3.10 in [14]).

Acknowledgement

I wish to thank George Koumoullis for pointing out the above example.

References

[1] A.G.A.G. BABIKER: Uniform continuity of measures on completely regular spaces. *J. London Math. Soc. (2)* 5 (1972) 451–458.

[2] A.G.A.G. BABIKER: Uniform regularity of measures on compact spaces. *J. Reine Angew, Math.*, 289 (1977) 188–198.

[3] A.G.A.G. BABIKER: On uniformly regular topological measure spaces. *Duke Math. J.* 43 (1976) 775–789.

[4] A.G.A.G. BABIKER: Structural properties of uniformly regular measures on compact spaces. *Ann. Soc. Sci. Bruxelles Sec I 90* (1976) 289–296.

[5] A.G.A.G. BABIKER: Uniform regularity of measures on locally compact spaces. *J. Reine Angew. Math. 298* (1978) 65–73.

[6] A.G.A.G. BABIKER and A. SAPOUNAKIS: Some regularity conditions on topological measure spaces. *J. London Math. Soc. (2)23* (1981) 495–502.

[7] W. BADE: Two properties of the Sorgenfrey plane. *Pacific J. Math. 51* (1974) 349–354.

[8] L. GILLMAN and M. JERISON: *Rings of continuous functions.* New York,: Van Nostrand (1960).

[9] J.D. KNOWLES: Measures on topological spaces. *Proc. London Math. Soc. (3) 17* (1967) 139–156.

[10] G. KOUMOULLIS: Measures on topological spaces. Ph.D. thesis, Athens University 1978 (in Greek).

[11] G. KOUMOULLIS: Perfect, u-additive measures and strict topologies. *Illinois J. Math. (3) 26* (1982) 466–478.

[12] D.T. LUTZER: On generalized ordered spaces. *Diss. Math. 89* (1971) 5–36.

[13] W. MORAN: The additivity of measures on completely regular spaces. *J. London Math. Soc. 43* (1968) 633–639.

[14] A. SAPOUNAKIS: Additivity properties and uniform regularity of measures on topological spaces, to appear.

[15] F.D. SENTILLES and R.F. WHEELER: Linear functionals and partitions of unity in $C_b(X)$. *Duke Math. J. 41* (1974) 483–396.

[16] V.S. VARADARAJAN: Measures on topological spaces. *Math. Sb. 55 (1961) 33–100 (Russian); Amer. Math. Soc. Transl. (2) 48* (165) 141–228 (English).

(Oblatum 17-VIII-1982)

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