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# MODULI FOR KODAIRA SURFACES

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## 1. Introduction

Kodaira proved (see [2], Theorem 19) that a compact complex analytic surface with a trivial canonical bundle is a complex torus, a K3 surface or an elliptic surface of the form $\mathbb{C}^2/G$, where $\mathbb{C}^2$ denotes the space of two complex variables $(z, w)$ and $G$ is a properly discontinuous non-abelian group of affine transformations without fixed points of $\mathbb{C}^2$ which leave invariant the two-form $\mathrm{d}z \wedge \mathrm{d}w$.

Surfaces in the last class, referred to as Kodaira surfaces, are complex analytic fibre bundles of elliptic curves over an elliptic curve and, from the point of view of the differential (or even real-analytic) structure, parallelizable manifolds represented as non-trivial circle bundles over the real- three-dimensional torus: a denumerable family indexed by the single torsion coefficient of their first integral homology group. (Absence of torsion is taken as index one.)

If $X$ is a compact complex analytic surface with a trivial canonical bundle, a non-zero global holomorphic two-form $\eta$ will satisfy:

$$\mathrm{d}\eta = 0, \quad \eta \wedge \eta = 0 \quad \text{and} \quad \eta \wedge \bar{\eta} > 0 \text{ at every point of } X. \tag{1}$$

Conversely, any global complex-valued two-form $\eta$ on the underlying canonically oriented differential manifold $X_0$ of $X$, satisfying conditions (1), defines an integrable almost complex structure, that is, a complex structure on $X_0$, with respect to which $\eta$ is holomorphic (and nowhere null).

Thus, if we denote by $P$ the complex projective space associated to $H^2(X_0, \mathbb{C})$ and by $p \in P$ the point corresponding to the cohomology class of a global holomorphic non-zero two-form, for some complex structure on $X_0$ with a trivial canonical bundle, $p$ will necessarily lie in the open set $D$ determined on the quadric $p \cdot p = 0$ by the condition $p \cdot \bar{p} > 0$, where multiplication and conjugation respectively come from cup product and conjugation on $H^2(X_0, \mathbb{C})$.

The group of orientation preserving diffeomorphisms of $X_0$ acts naturally on $D$.

Starting with a Kodaira surface $X$ (compare [4]), we prove:

THEOREM 2: *Given any line $p \in D$ of $H^2(X_0, \mathbb{C})$, there exist representatives $\eta$ of $p$ satisfying conditions* (1).

THEOREM 3: *Any two such representatives define isomorphic complex analytic structures on $X_0$.*

From results of Kodaira and Yau (see [2]&[3]), it can be seen that any complex structure on $X_0$ appears in this manner, i.e. has a trivial canonical bundle.

Thus, a parametrising space for the isomorphism classes of complex structures on $X_0$ should be the quotient of $D$ by the action of the orientation preserving diffeomorphisms of $X_0$.

It turns out that this quotient may be identified to the product of the complex plane with a punctured disk (say, the unit disk without zero) and embodies the moduli space (Theorem 4).

Our approach involves related results on discrete co-compact subgroups of a certain four-dimensional nilpotent real Lie group (Theorem 1) and makes the treatment of this case quite similar to that of tori.

## 2. A description of Kodaira surfaces

Let $X$ be a Kodaira surface. Then (see [2]), the fundamental group of $X$ may be generated by four elements $g_i$, $i = 1, 2, 3, 4$, satisfying the following relations:

$$g_i g_j g_i^{-1} g_j^{-1} = id \text{ for all couples } (i, j),\ i < j,\ (i, j) \neq (3, 4) \tag{2}$$

and $g_3 g_4 g_3^{-1} g_4^{-1} = g_2^m$ for some positive integer $m$.

The universal covering of $X$ is complex analytically isomorphic to $\mathbb{C}^2$, the space of two complex variables $(z, w)$, is such a manner that $g_j$, regarded as covering transformations, take the form:

$$g_j(z, w) = (z + \alpha_j, w + \bar{\alpha}_j \cdot z + \beta_j), \qquad j = 1, 2, 3, 4 \tag{3}$$

where $\alpha_1 = \alpha_2 = 0$ and $\bar{\alpha}_3 \alpha_4 - \bar{\alpha}_4 \alpha_3 = \mathrm{m} \cdot \beta_2$.

We shall identify $\mathbb{C}^2$ with $R^4$, the space of four real variables $(x, y, u, v)$, by $z = x + iy$, $w = u + iv$.

Now, considering the group $A$ of all real-affine transformations of $\mathbb{C}^2 = R^4$, commuting with any transformation of the form

$$g(z, w) = (z + \alpha, w + \bar{\alpha} \cdot z + \beta), \tag{4}$$

one discovers that $A$ may be identified to $\mathbb{C}^2$ endowed with the following multiplication:

$$(z, w) * (\alpha, \beta) = (z + \alpha, w + \bar{\alpha} \cdot z + \beta). \tag{4'}$$

Thus, the complex structure of $\mathbb{C}^2$ determines a right-invariant complex structure on $A$ and any Kodaira surface appears as the compact quotient of $A$ by some discrete subgroup $\Gamma$, generated, according to (3), by $g_j(0, 0) = (\alpha_j, \beta_j)$, $j = 1, 2, 3, 4$.

Conversely, given a discrete subgroup $\Gamma$ of $A$, such that the space of left cosets $A/\Gamma = \{a \cdot \Gamma;\ a \in A\}$ is compact, the complex structure induced on this quotient will give a Kodaira surface, since the canonical bundle is trivial and $A/\Gamma$ can be neither a torus (see details below), nor a K3 surface.

## 3. Proof of the theorems

We identify the Lie algebra $a$ of right-invariant tangent vector fields on $A$ with $\mathbb{R}^4$, the tangent space of $A = \mathbb{C}^2 = \mathbb{R}^4$ at the origin, and denote by $X_i$, $i = 1, 2, 3, 4$, the canonical base.

Simple computations give:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x} + x\frac{\partial}{\partial u} - y\frac{\partial}{\partial v} \\ X_2 &= \frac{\partial}{\partial y} + y\frac{\partial}{\partial u} + x\frac{\partial}{\partial v} \\ X_3 &= \frac{\partial}{\partial u} \\ X_4 &= \frac{\partial}{\partial v}. \end{aligned} \tag{5}$$

Consequently:

$$\begin{aligned} \left[X_i, X_j\right] = 0 \quad & \text{for all couples } (i,j),\ i<j,\ (i,j) \neq (1, 2) \\ & \text{and } [X_1, X_2] = 2.X_4. \end{aligned} \tag{6}$$

It follows that $a$ is nilpotent and its center is spanned by $X_3$ and $X_4$. The exponential mapping will be a diffeomorfism and multiplication, in logarithmic coordinates, will have the form: (Campbell-Baker-Hausdorff formula)

$$X * Y = X + Y - \tfrac{1}{2}[X, Y] \qquad \text{for} \quad X, Y \in a. \tag{7}$$

These formulae make plain that the logarithmic image (i.e. the image by the inverse of the exponential mapping) of any discrete co-compact subgroup of $A$ has to span $a$ as a vector space. Therefore $\Gamma$ is non-abelian and must be isomorphic to the fundamental group of a Kodaira surface.

Given two such groups $\Gamma$ and $\Gamma'$, corresponding to the same torsion coefficient $m$, we may choose generators $g_j$ and $g'_j$ respectively, $j = 1, 2, 3, 4$, according to (2). We identify $a$ and $A$ by exp. If we look at $g_j$ and $g'_j$ as elements of $a$, then $\{g_j\}$ as well as $\{g'_j\}$ have to be bases of $a$ ($g_1$, $g'_1$ lie in the center of $a$ and $g_2$, $g'_2$ are proportional to $X_4$) and the linear map defined by $g_j \to g'_j$ is an automorphism of $a$ which carries $\Gamma$ onto $\Gamma'$.

We have proved:

THEOREM 1: *Discrete co-compact subgroups $\Gamma$ of $A$ are classified, up to automorphisms of $A$, by the single torsion coefficient of $H_1(A/\Gamma, \mathbb{Z})$.*

We consider now, dually, right-invariant forms on $A$ and set:

$$\begin{aligned}
\omega_1 &= \mathrm{d}x \\
\omega_2 &= \mathrm{d}y \\
\omega_3 &= \mathrm{d}u - x\,\mathrm{d}x - y\,\mathrm{d}y \\
\omega_4 &= \mathrm{d}v - x\,\mathrm{d}y + y\,\mathrm{d}x.
\end{aligned} \tag{8}$$

We have (Maurer-Cartan equations):

$$\mathrm{d}\omega_i = 0 \quad \text{for} \quad i = 1, 2, 3 \quad \text{and} \quad \mathrm{d}\omega_4 = -2\omega_1 \wedge \omega_2. \tag{9}$$

Let us fix the topological (differential) type $X_0$ of a Kodaira surface $X$.

Theorem 1 shows that, instead of obtaining the various complex structures on $X_0$ by different representations of the fundamental group, we may fix some $\Gamma_0$ and vary the right-invariant complex structure of $A$. Such a complex structure corresponds to a right-invariant complex two-form $\eta$ (determined up to multiplication by non-zero constants) with:

$$\mathrm{d}\eta = 0, \qquad \eta \wedge \eta = 0 \qquad \text{and} \qquad \eta \wedge \bar{\eta} > 0 \text{ at every point of } A. \tag{10}$$

A basis of right-invariant closed two-forms on $A$ is given by:

$$\theta_{ij} = \omega_i \wedge \omega_j, \qquad 1 \leqslant i < j \leqslant 4, \qquad (i, j) \neq (3, 4). \tag{11}$$

Moreover, $\theta_{12}$ is exact by (9) and any wedge product $\theta_{ij} \wedge \theta_{kl}$ of forms in (11) vanishes, except the case $\{i, j, k, l\} = \{1, 2, 3, 4\}$, when $\theta_{ij} \wedge \theta_{kl} = \pm \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}u \wedge \mathrm{d}v$.

Since the second Betti number of a Kodaira surface is four, we see that

$$\theta_{ij}, \qquad 1 \leqslant i < j \leqslant 4, \qquad (i,j) \neq (1,2), \qquad (3,4) \tag{12}$$

considered as forms on $X_0 = A/\Gamma_0$, determine a basis of $H^2(X_0, \mathbb{C})$.

If, with respect to this basis, a point $p \in D$ has homogeneous coordinates $(p_{ij})$, $1 \leqslant i < j \leqslant 4$, $(i,j) \neq (1, 2)$, $(3, 4)$, then clearly $\eta = p_{13}\theta_{13} + p_{14}\theta_{14} + p_{23}\theta_{23} + p_{24}\theta_{24}$ defines a right-invariant complex structure of $A$ and, considered on $X_0$, a representative for the line $p \in D$ of $H^2(X_0, \mathbb{C})$. This proves:

THEOREM 2: *Let $P$ be the projective space associated to $H^2(X_0, \mathbb{C})$ and $D := \{ p \in P | p \cdot p = 0, p \cdot \bar{p} > 0\}$, where multiplication comes from cup product.*

*For any line $p \in D$ of $H^2(X_0, \mathbb{C})$, there exist two-forms $\eta$, representing $p$ and such that $\mathrm{d}\eta = 0$, $\eta \wedge \eta = 0$ and $\eta \wedge \bar{\eta} > 0$ at every point of $X_0$.*

*Thus, $\eta$ is a nowhere null holomorphic two-form corresponding to some complex structure on $X_0$.* ■

Left translation of a closed right-invariant two form remains right-invariant and alters at most the coefficient of $\theta_{12}$, since the induced diffeomorphism on $X_0$ is homotopic to the identity.

So far as we are interested only in isomorphism classes of complex structures on $X_0$, it is readily seen that, by convenient left translations, we may altogether dispense with $\theta_{12}$ and restrict to forms

$$\eta = p_{13}\theta_{13} + p_{23}\theta_{23} + p_{14}\theta_{14} + p_{24}\theta_{24} \tag{13}$$

which are parametrised (up to multiplication by non-zero constants precisely by $D$.

REMARK 1: At this point, it is possible to perform a construction similar to that encountered in the case of tori (see [1]).

For $p = (p_{ij}) \in D$, we conceive the right-invariant complex structure on $A$ given by $\eta$ in (13) as a complex structure on $a$, that is, a certain point in the Grassmann manifold of complex two-planes in $a \otimes \mathbb{C}$.

The pull-back $K$ over $D$ of the canonical quotient vector bundle of the Grassmann manifold is, differentiably, identical to the trivial (real) vector bundle $a \times D$, so that $\Gamma_0$ acts naturally and holomorphically on $K$ (to the right). This action is properly discontinuous, without fixed points and $K/\Gamma_0$ projects onto $D$ as a complex analytic family of Kodaira surfaces. The fibre over $p \in D$ is $X_0$ endowed with the complex structure determined by $\eta$ in (13).

Furthermore, this family is complete.

Let now $\nu$ be a two-form on $X_0$, cohomologous to some fixed $\eta$

(considered on $X_0$) determined, according to (13), by a point $p \in D$. Suppose that $\nu \wedge \nu = 0$ and $\nu \wedge \bar{\nu} > 0$ at every point of $X_0$. We want to prove that $\eta$ and $\nu$ define isomorphic complex structures.

Since $X_0$ with the structure induced by $\nu$ has to be isomorphic to some fibre in the family above, our problem may be reformulated as follows:

Let $p$ and $p'$ be points in $D$, $\eta$ and $\eta'$ the corresponding two-forms given by (13).

Suppose that $f\colon X_0 \to X_0$ is an orientation preserving diffeomorphism such that $f^*p' = p$.

Then, one should prove that $\eta$ and $\eta'$ define isomorphic complex structures.

Let $F$ be a lifting of $f$ to the universal covering $A$ of $X_0$.

The map $F$ induces an automorphism $\varphi$ of the covering transformation group $\Gamma_0$ of $X_0$ and the arguments for Theorem 1 show that $\varphi$ extends uniquely to an automorphism $\Phi$ of $A$.

If $p'' \in D$ satisfies $\Phi^*p'' = p$ and $\eta''$ corresponds to $p''$ by (13), then $\eta$ and $\eta''$ define isomorphic complex structures because $\Phi^*\eta''$ and $\eta$ are both right-invariant and (conveniently proportioned) differ at most by multiples of $\theta_{12}$ which disappear by appropriate left translations.

This means that we reduced the problem to the case of a diffeomorphism $F$ commuting with the covering transformations. Consequently, $f_*$ is the identity on $H_1(X_0, \mathbb{Z})$ and, by de Rham and Poincaré duality, $f^*$ is the identity on $H^1(X_0, \mathbb{C})$ and $H^3(X_0, \mathbb{C})$.

Let $t_{ij} \in H^2(X_0, \mathbb{C})$ denote the cohomology class of $\theta_{ij}$ and let $t'_{ij} = f^*t_{ij}$, $1 \leqslant i < j \leqslant 4$, $(i, j) \neq (1, 2), (3, 4)$.

Since $f^*$ commutes with cup (wedge) products and the bilinear form on $H^2(X_0, \mathbb{C})$ is nondegenerate, one readily finds that:

$$\begin{aligned} &t'_{13} = t_{13}, \qquad t'_{23} = t_{23}, \\ &t'_{14} = t_{14} + \lambda \cdot t_{13}, \qquad t'_{24} = t_{24} + \lambda \cdot t_{23}. \end{aligned} \tag{14}$$

If we let $\Gamma_0$ be generated by (the exponential image of) $X_1$, $X_2$, $X_3$ and $-2/mX_4$, then $m/2t_{14}$ is an integral class and $m/2 \cdot \lambda$ has to be an integer $k$. (Integrate, e.g. on the two-cycles corresponding to the subalgebras spanned by $\{X_1, X_3\}$, $\{X_2, X_3\}$, $\{X_1, X_4\}$, $\{X_2, X_4\}$.)

But the same effect (14) on $H^2(X_0, \mathbb{C})$ will be obtained if we consider the automorphism of $A$ determined by the automorphism of $\Gamma_0$:

$$\begin{aligned} &X_1 \to X_1 \\ &X_2 \to X_2 \\ &X_3 \to X_3 + k \cdot \left(\frac{2}{m} X_4\right) = X_3 + \lambda \cdot X_4 \\ &\frac{-2}{m} X_4 \to \frac{-2}{m} \cdot X_4. \end{aligned}$$

According to previous discussions, this suffices to establish:

THEOREM 3: *The complex structure defined on $X_0$ by a two-form $\eta$ satisfying*

$$d\eta = 0, \qquad \eta \wedge \eta = 0 \qquad \text{and} \quad \eta \wedge \bar{\eta} > 0 \text{ at every point of } X_0,$$

*depends (up to isomorphism) exclusively on the line of $H^2(X_0, \mathbb{C})$ determined by the cohomology class of $\eta$.* □

It remains to describe the quotient of $D$ by the action of orientation preserving diffeomorphisms of $X_0$ and we saw already that for this purpose we may restrict our considerations to diffeomorphisms induced by orientation preserving automorphisms of $A$ sending $\Gamma_0$ onto $\Gamma_0$.

These automorphisms are entirely determined by their restriction to $\Gamma_0$ which (on generators) has to be of the following form:

$$\begin{aligned} &X_1 \to a \cdot X_1 + b \cdot X_2 + r \cdot X_3 + s \cdot V \\ &X_2 \to c \cdot X_1 + d \cdot X_2 + p \cdot X_3 + q \cdot V \\ &X_3 \to X_3 + k \cdot V \\ &V \to e \cdot V \end{aligned} \tag{15}$$

where $V = -2/mX_4$, all coefficients are integers and $ad - bc = e = \pm 1$.

Now, in homogeneous coordinates $(p_{13}, p_{23}, p_{14}, p_{24})$, $D$ is given by:

$$\begin{aligned} &p_{13} \cdot p_{24} - p_{23} \cdot p_{14} = 0 \\ &-p_{13} \cdot \bar{p}_{24} + p_{23} \cdot \bar{p}_{14} + p_{14} \cdot \bar{p}_{23} - p_{24} \cdot \bar{p}_{13} > 0 \end{aligned} \tag{16}$$

and the effect of (15) on $D$ is:

$$\begin{pmatrix} p_{13} \\ p_{23} \\ p_{14} \\ p_{24} \end{pmatrix} \to \begin{pmatrix} M & \dfrac{-2k}{m} \cdot M \\ 0 & e \cdot M \end{pmatrix} \cdot \begin{pmatrix} p_{13} \\ p_{23} \\ p_{14} \\ p_{24} \end{pmatrix} \tag{17}$$

where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det M = e = \pm 1.$$

Using (16), it follows that (up to sign) $p_{14}/p_{24}$ undergoes a modular transformation, while $p_{13}/p_{14}$ is translated by $-2k/m$.

But $(p_{13}, p_{23}, p_{14}, p_{24}) \to (p_{14}/p_{24}, p_{13}/p_{14})$ defines an isomorphism of $D$ onto $H_+ \times H_+ \cup H_- \times H_-$, where $H_+$ and $H_-$ denote the upper and lower half plane respectively.

The quotient of $H_+$ by the modular group is isomorphic to the complex plane and the quotient of $H_+$ by an infinite cyclic group of (real) translations is isomorphic to a punctured disk.

If we let $a=e=-d=-1$ and $b=c=p=q=r=s=k=0$ in (15), then the induced automorphism of $D$ clearly interchanges the two connected components of $D$.

We also notice that in our considerations we made use of a fixed orientation on $X_0=A/\Gamma_0$. But the case of the reversed orientation is the pull-back of the initial case by means of the orientation-reversing diffeomorphism of $X_0$ induced by: $X_i \to X_i$, $i=1, 2, 4$ and $X_3 \to -X_3$.

This concludes the arguments for:

THEOREM 4: *The moduli space corresponding to isomorphism classes of complex structures on $X_0$ may be identified to the product of the complex plane with a punctured disk.*

REMARK 2: Recall that a Kodaira surface $X$ is a complex analytic fibre bundle of elliptic curves over an elliptic curve. The typical fibre is isomorphic to the identity component of the automorphism group of $X$.

It can be seen that the first coordinate in the parametrising space above comes from the modulus of the base (i.e. of the meromorphic function-field on $X$), while the second one can be (holomorphically) mapped to the modulus of the fibre.

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