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q -PSEUDOCONVEX and q -COMPLETE DOMAINS

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Introduction

The Levi problem was originally posed in the following terms: if D is a domain in \mathbb{C}^n with C^2 boundary which is pseudoconvex is D a domain of holomorphy?

It was then realised that the hypothesis on the boundary can be removed if pseudoconvexity is replaced by completeness, which is a concept that makes sense in any analytic manifold, and the final solution of the Levi problem due to Grauert says that a complete analytic manifold is necessarily Stein [3].

The original spirit of the problem has not been betrayed: domains with C^2 boundary in \mathbb{C}^n are pseudoconvex if and only if they are complete ([4] p. 50).

The same is not true any more if \mathbb{C}^n is replaced by any analytic manifold: a well known example of Grauert provides a subset with C^2 boundary of a complex torus which is pseudoconvex but all holomorphic functions thereon are constant.

In this paper we prove that a q -pseudoconvex open subset of a Stein manifold is necessarily q -complete (the converse is also true, see [2]). This seems to be one of those facts that every complex analyst believes, perhaps for psychological reasons, but no precise reference is, to my knowledge, available and all mathematicians whom I have asked so far don't seem to know how a precise proof should go; the modest aim of this paper is to fill this gap and provide a definite reference.

Most of the ideas in the proof are due to Mike Eastwood to whom I am, once more, deeply grateful.

We briefly recall the basic definitions:

DEFINITION 1: Let D be an open subset of an analytic manifold M of dimension n ; we say that D has C^2 boundary if for all $x \in \partial D$ there exists an open neighbourhood U of x and a C^2 function $\varphi: U \rightarrow \mathbb{R}$, called *defining function* of D at x s.t. $D \cap U = \{y \in U \text{ s.t. } \varphi(y) < 0\}$, and

$d\varphi(x) \neq 0$; in these conditions we can consider the *complex Hessian*

$$\mathcal{H}(\varphi)(x) = \left(\frac{\partial^2 \varphi}{\partial z_i \partial \bar{z}_j}(x) \right)_{i,j=1}^n$$

where z_1, z_2, \dots, z_n are local holomorphic coordinates at x . The signature of this Hermitian matrix does not depend on the choice of the local holomorphic coordinates but it does depend on φ . However the *Levi form*

$$\mathcal{L}(\varphi)(x) =: \mathcal{H}(\varphi)(x)|_{T_x \partial D},$$

where $T_x \partial D = \{v = \sum_1^n v_i \partial / \partial z_i \in T_x M \text{ s.t. } \sum_1^n v_i \partial \varphi / \partial z_i(x) = 0\}$ is the holomorphic tangent space of ∂D at x , has a signature that depends only on D and x .

If $n(x)$ denotes the number of negative eigenvalues of $\mathcal{L}(\varphi)(x)$ we say that D is *q-pseudoconvex* if $n(x) \leq q$ for all $x \in \partial D$.

DEFINITION 2: A complex n -dimensional manifold D is said to be *q-complete* if we can find a *q-plurisubharmonic exhaustion function* on D i.e. a C^2 function $\Psi: D \rightarrow \mathbb{R}$ s.t.

(1) for all $c \in \mathbb{R}$ the set $B_c = \{x \in D \text{ s.t. } \Psi(x) < c\}$ is relatively compact in D and

(2) The complex Hessian $\mathcal{H}(\Psi)(x)$ has at least $n - q$ positive eigenvalues for all x in D .

0-pseudoconvex and 0-complete domains are simply called pseudoconvex and complete.

THEOREM: *If D is a domain with C^2 boundary in a Stein manifold M and D is q-pseudoconvex then it is also q-complete.*

PROOF: We shall divide the proof into several steps.

Step 1: As there is always an analytic embedding of M into \mathbb{C}^N , for some large N (see [5] p. 359) we can suppose at once that M is an analytic submanifold of \mathbb{C}^N . Choose a holomorphic tubular neighbourhood $p: V \rightarrow M$ and set $\tilde{D} = p^{-1}(D)$ (cfr. [1] proof of Lemma 1, p. 131). We claim that, after shrinking V if necessary,

(a) $\forall x \in \partial \tilde{D} \cap V$, $\partial \tilde{D}$ is C^2 at x ,

(b) If we consider \tilde{D} as an open subset of \mathbb{C}^N then $n(x, \tilde{D}) = n(p(x), D)$, for all $x \in \partial \tilde{D} \cap V$.

Indeed, since the problem is local we can suppose that local coordinates z_1, z_2, \dots, z_N have been chosen s.t., near x , $M = \{z \text{ s.t. } z_{N-n+1} = z_{N-n+2} = \dots = z_N = 0\}$, z_1, z_2, \dots, z_n are local coordinates of M at x and $p(z_1, z_2, \dots, z_N) = (z_1, z_2, \dots, z_n, 0, \dots, 0)$.

Let \tilde{U} be a neighbourhood of x in \mathbb{C}^N so small that z_1, z_2, \dots, z_N are defined in \tilde{U} and that there exists a C^2 defining function $\Phi: U = \tilde{U} \cap M \rightarrow \mathbb{R}$ for D with $d\Phi(x) \neq 0$ and $\tilde{U} \subseteq V$. By shrinking \tilde{U} if necessary we can also suppose that $\tilde{U} \subseteq p^{-1}(U)$.

Define $\tilde{\Phi}: \tilde{U} \rightarrow \mathbb{R}$ by $\tilde{\Phi} = \Phi \circ p$ i.e. $\tilde{\Phi}(z_1, z_2, \dots, z_N) = \Phi(z_1, z_2, \dots, z_n, 0, \dots, 0)$. Then $\tilde{\Phi}$ is a defining function for \tilde{D} at x . Moreover

$$\begin{aligned} T_x \partial \tilde{D} &= \left\{ v \in T_x \mathbb{C}^N \text{ s.t. } \sum_{i=1}^N \frac{\partial \tilde{\Phi}}{\partial z_i}(x) v_i = 0 \right\} \\ &= \left\{ v \in T_x \mathbb{C}^N \text{ s.t. } \sum_{i=1}^n \frac{\partial \Phi}{\partial z_i}(p(x)) v_i = 0 \right\} \simeq T_{p(x)} \partial D \times \mathbb{C}^{N-n} \end{aligned}$$

where as usual $v = \sum_{i=1}^N v_i \partial / \partial z_i$, and

$$\frac{\partial^2 \tilde{\Phi}(x)}{\partial z_i \partial \bar{z}_j} = \begin{cases} \frac{\partial^2 \Phi(p(x))}{\partial z_i \partial \bar{z}_j} & \text{if } i, j \leq n \\ 0 & \text{otherwise.} \end{cases}$$

This proves the claim.

Step 2: So, if we suppose that D is q -pseudoconvex we have that, $\forall x \in V \cap \partial \tilde{D}$, $\partial \tilde{D}$ is C^2 at x and $n(x, \tilde{D}) \leq q$.

Consider the function $\rho: \mathbb{C}^N \rightarrow \mathbb{R}$ given by

$$\rho(y) = \begin{cases} \text{dist}(y, \partial \tilde{D}) & \text{if } y \in \bar{\tilde{D}} \\ -\text{dist}(y, \partial \tilde{D}) & \text{if } y \in \mathbb{C}^N - \tilde{D}, \end{cases}$$

Where dist denotes the Euclidean distance. Since $\forall x \in \partial \tilde{D}$, $\partial \tilde{D}$ is C^2 at x , we can conclude that there exists a neighbourhood \tilde{U}' of ∂D in \mathbb{C}^N on which ρ is C^2 (by the inverse function theorem).

By shrinking \tilde{U}' if necessary, we can also suppose that $\forall y$ in \tilde{U}' there exists exactly one point $c(y) \in \partial \tilde{D} \cap \tilde{U}'$ which is the closest point to y under the Euclidean distance, that $d\rho(c(y)) \neq 0$ and that $n(c(y), \tilde{D}) \leq q$.

Let $\varphi: \tilde{U}' \cap \tilde{D} \rightarrow \mathbb{R}$ be the function $\varphi = \log \rho$; we claim that the Hessian $(\mathcal{H}\varphi)(y)$ has at most q positive eigenvalues $\forall y$.

Indeed suppose that this is false, i.e. there exists a point y in $\tilde{U}' \cap \tilde{D}$ s.t. $(\mathcal{H}\varphi)(y)$ has (at least) $q+1$ positive eigenvalues; the geometric interpretation is: there are linear coordinates (t_1, t_2, \dots, t_N) of \mathbb{C}^N s.t. the Hermitian form given by the matrix

$$(C_{jk})_{j,k=1}^{q+1} = \left(\frac{\partial^2 \varphi(y)}{\partial t_j \partial \bar{t}_k} \right)_{j,k=1}^{q+1}$$

is positive definite on the linear subspace V of $T_1\mathbb{C}^N = \mathbb{C}^N$ spanned by $(\partial/\partial t_1, \partial/\partial t_2, \dots, \partial/\partial t_{q+1})$.

By Taylor's theorem we have

$$\begin{aligned} \varphi\left(y + \sum_{j=1}^{q+1} t_j \frac{\partial}{\partial t_j}\right) &= \log \rho\left(y + \sum_{j=1}^{q+1} T_j \frac{\partial}{\partial t_j}\right) \\ &= \log \rho(y) + \operatorname{Re}\left(\sum_{i=1}^{q+1} a_i t_i + \sum_{j,k=1}^{q+1} b_{jk} t_j \bar{t}_k\right) \\ &\quad + \sum_{j,k=1}^{q+1} C_{jk} t_j \bar{t}_k + o(|t|^2), \end{aligned}$$

where $a_i = \frac{1}{2}\partial\varphi/\partial t_i(y)$ and $b_{jk} = \partial^2\varphi(y)/\partial t_j\partial t_k$ are constants, and $o(|t|^2)$ has the property that $\lim_{t \rightarrow 0} o(|t|^2)/|t|^2 = 0$ and so also

$$\lim_{t \rightarrow 0} \frac{o(|t|^2)}{\sum_{j,k} C_{jk} t_j \bar{t}_k} = 0.$$

In order to simplify notation omit the limits of the summands and write $A(t) = y + \sum t_j \partial/\partial t_j$, $B(t) = \exp(\sum a_i t_i + \sum b_{jk} t_j \bar{t}_k)$.

Then the above equality can be written as $|\rho(A(t)) - \rho(y)|B(t)| = \{\exp(\sum C_{jk} t_j \bar{t}_k + o(|t|^2)) - 1\}\rho(y)|B(t)| = \{\sum C_{jk} t_j \bar{t}_k + o'(|t|^2)\}\rho(y)|B(t)|$, where the last equality is obtained by expanding in Taylor series the function \exp and $o'(|t|^2)$ has the same properties as $o(|t|^2)$. Then one has

$$\lim_{t \rightarrow 0} \frac{\rho(A(t)) - \rho(y)|B(t)|}{\sum C_{jk} t_j \bar{t}_k} = \rho(y),$$

so we can choose $\epsilon > 0$ small enough s.t. $\forall t, |t| < \epsilon$, one has

- (a) $A(t) \in \tilde{D} \cap \tilde{U}'$ and
- (b) $|\rho(A(t)) - \rho(y)|B(t)| > \rho(y)/2 \cdot \sum C_{jk} t_j \bar{t}_k$.

Set $u = c(y) - y$ and define an analytic function T on the open ball $B_\epsilon = \{t \in \mathbb{C}^{q+1} \text{ s.t. } |t| < \epsilon\}$:

$$T: B_\epsilon \rightarrow \mathbb{C}^N \text{ is given by } T(t) = A(t) + uB(t).$$

We can also suppose that ϵ is so small that $T(t) \in \tilde{U}'$ if $t \in B_\epsilon$. Then it is easy to check, and a picture shows how, that if $t \in B_\epsilon$ one has

- (c) $\rho(T(t)) \geq \rho(A(t)) - |u||B(t)| \geq |u|/2 \sum C_{jk} t_j \bar{t}_k \geq 0$

This in particular proves that $T(t) \in \tilde{D}$ for all $t \in B_\epsilon - \{0\}$, and, since

$\rho(T(0)) = \rho(c(y)) = 0$, 0 is a minimum for the function $\rho \circ T: B_\epsilon \rightarrow \mathbb{R}$, and so, taking partial derivatives,

$$\frac{\partial \rho \circ T(0)}{\partial t_j} = 0 \quad \text{for all } j = 1, 2, \dots, q + 1.$$

Using the chain rule and the fact that T is analytic we have:

(d) $\sum_{h=1}^N \partial \rho / \partial z_h(c(y)) \partial T_h / \partial t_j(0) = 0$ for $j = 1, 2, \dots, q + 1$.

In other words the vectors $\partial T / \partial t_j(0)$, $j = 1, 2, \dots, q + 1$, are in $T_{c(y)} \partial \bar{D}$.

Moreover, $\forall t$ in \mathbb{C}^{q+1} , we have

(e) $\sum_{j,k=1}^{q+1} \partial^2 \rho \circ T(0) / \partial t_j \partial \bar{t}_k t_j \bar{t}_k \geq |u| / 4 \sum_{j,k=1}^{q+1} C_{jk} t_j \bar{t}_k$.

To prove this we first observe that it is clearly enough to check it for small $|t|$.

From the above inequality (c), using Taylor series, we deduce

$$\operatorname{Re} \left(\sum d_{jk} t_j \bar{t}_k \right) + \sum \frac{\partial^2 \rho \circ T(0)}{\partial t_j \partial \bar{t}_k} t_j \bar{t}_k + O''(|t|^2) \geq \frac{|u|}{2} \sum C_{jk} t_j \bar{t}_k,$$

for all $t \in B_\epsilon$, where $d_{jk} = \partial^2 \rho \circ T(0) / \partial t_j \partial \bar{t}_k$ are constants and $O''(|t|^2)$ has the same properties as $O(|t|^2)$.

Then, after reducing ϵ if necessary, we have, $\forall t \in B_\epsilon$,

$$\operatorname{Re} \left(\sum d_{jk} t_j \bar{t}_k \right) + \sum \frac{\partial^2 \rho \circ T(0)}{\partial t_j \partial \bar{t}_k} T_j \bar{t}_k \geq \frac{|u|}{4} \sum C_{jk} t_j \bar{t}_k.$$

Let $t'_j = e^{i\theta} t_j$ for $0 \leq \theta \leq 2\pi$; writing t' in the above inequality and observing that the second and third term are unchanged under the substitution $t \rightarrow t'$, we deduce, $\forall \theta$,

$$\operatorname{Re} \left(e^{i2\theta} \sum d_{jk} t_j \bar{t}_k \right) + \sum \frac{\partial^2 \rho \circ T(0)}{\partial t_j \partial \bar{t}_k} t_j \bar{t}_k \geq \frac{|u|}{4} \sum C_{jk} t_j \bar{t}_k,$$

and by choosing θ so that the first term is negative we prove the inequality (e).

Using again the chain rule and the fact that T is analytic we have that the Hermitian form

$$\left(\sum_{h,m=1}^N \frac{\partial^2 \rho(c(y))}{\partial z_h \partial \bar{z}_m} \cdot \frac{\partial T_h}{\partial t_j}(0) \cdot \overline{\left(\frac{\partial T_m}{\partial t_k}(0) \right)} \right)_{j,k=1}^{q+1}$$

is positive definite.

It follows easily that the Hermitian form $(\partial^2 \rho(c(y)) / \partial z_h \partial \bar{z}_m)_{h,m=1}^N$ is positive definite on the linear subspace V of $T_{c(y)} \partial \bar{D}$ spanned by the

vectors $\partial T/\partial T_j(0)$, $j = 1, 2, \dots, q+1$; in particular it follows automatically that these vectors are linearly independent, so that $\dim_{\mathbb{C}} V = q+1$; but since $-\rho$ is a defining function for \bar{D} at $c(y)$, we have that $n(c(y), \bar{D}) \geq q+1$ and this contradicts our hypothesis, so that the claim is proved.

Step 3: By restricting φ to $\tilde{U}' \cap D$ we find a C^2 function, called again $\varphi: W = \tilde{U}' \cap D \rightarrow \mathbb{R}$ s.t.

- (a) $\lim_{y \rightarrow \partial D} \varphi(y) = -\infty$,
- (b) $(\mathcal{H}\varphi)(y)$ has at most q positive eigenvalues $\forall y$ in W .

Let F be a closed subset of M s.t. $D - W \subseteq \text{int } F \subseteq F \subseteq D$, and let $0 \leq \Psi \leq 1$ be a C^2 bump function s.t. $\Psi = 0$ on F , $\Psi = 1$ in a neighbourhood of $M - D$, and suppose that F is chosen so that $\varphi(y) \leq 0$ for $y \notin F$.

By considering the function $\varphi' = \varphi \cdot \Psi$, we have that

- (a) $\lim_{y \rightarrow \partial D} \varphi'(y) = -\infty$,
- (b) $(\mathcal{H}\varphi')(y)$ has at most q positive eigenvalues $\forall y \in D - F$,
- (c) $\varphi' \leq 0$.

Now we use the fact that M is Stein and so 0-complete (see [5], lemma p. 358) i.e. there exists a 0-plurisubharmonic exhaustion function $\lambda: M \rightarrow \mathbb{R}$.

$\forall n \in \mathbb{Z}$, the set $K_n = \{y \in M \text{ s.t. } \lambda(y) \leq n\}$ is compact, therefore so is $F \cap K_n$ and there exist constants C_n s.t.

$$C_n (\mathcal{H}\lambda)(y) - (\mathcal{H}\varphi')(y) > 0 \quad \forall y \in F \cap K_n.$$

Now choose a C^2 function $f: \mathbb{R} \rightarrow \mathbb{R}$ with the properties

- (a) $f' > 0, f'' > 0$ always,
- (b) $f'(r) > C_{E(r)+1}$, r , where $E(r)$ denotes the integral part of r .
- (c) $f'(r) > C_0 \quad \forall r$, and consider the C^2 function

$$\chi = f \circ \lambda - \Psi': D \rightarrow \mathbb{R}.$$

First we notice that, $\forall c \in \mathbb{R}$, $B_c = \{y \in D \text{ s.t. } \chi(y) \leq c\}$ is contained, by the property (c) of φ' in $\{y \in D \text{ s.t. } f \circ \lambda(y) \leq c\}$ which is compact by the assumptions on f and λ . Moreover B_c is closed in D and, since $\lim_{y \rightarrow \partial D} \varphi'(y) = -\infty$, it is also closed in M . Thus B_c is compact and χ is an exhaustion function.

For all $y \in D$ we have

$$(\mathcal{H}\chi)(y) = f''(\lambda(y)) \cdot A(y) + f'(\lambda(y)) \cdot (\mathcal{H}\lambda)(y) - (\mathcal{H}\varphi')(y),$$

where $A(y) = \left(\partial \lambda / \partial z_i(y) \cdot \overline{\partial \lambda / \partial z_j(y)} \right)_{i,j=1}^n$ is a semipositive Hermitian form. If $y \in D - F$ then there exists a linear subspace V of $T_y D$, of dimension $n - q$ where $-(\mathcal{H}\varphi')(y)$ is positive semidefinite. Therefore $(\mathcal{H}\chi)(y)$ is positive definite on V .

If $y \in F$ then either $y \in K_0 \cap F$ in which case

$$\begin{aligned} (\mathcal{H}\chi)(y) &\geq f'(\lambda(y))(\mathcal{H}\lambda)(y) - (\mathcal{H}\varphi')(y) \\ &\geq C_0(\mathcal{H}\lambda)(y) - (\mathcal{H}\varphi')(y) > 0, \end{aligned}$$

or $y \in (K_{n+1} - K_n) \cap F$ for some integer $n \geq 0$, in which case

$$\begin{aligned} f'(\lambda(y)) &> C_{n+1} \quad \text{and so} \\ (\mathcal{H}\chi)(y) &> C_{n+1}(\mathcal{H}\lambda)(y) - (\mathcal{H}\varphi')(y) > 0. \end{aligned}$$

Therefore χ is also q -plurisubharmonic and we can finally say that the theorem is proved. \square

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