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R. BENEDETTI
M. DEDÒ

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COUNTEREXAMPLES TO REPRESENTING HOMOLOGY CLASSES BY REAL ALGEBRAIC SUBVARIETIES UP TO HOMEOMORPHISM

R. Benedetti and M. Dedò

1. Introduction and statements of the main results

By a *manifold* we shall mean a compact connected smooth manifold without boundary; by an *algebraic variety* an affine real algebraic variety. An *algebraic manifold* will be a compact connected non-singular algebraic variety. Of course we may consider any algebraic variety endowed with the fine (Euclidean) topology and an algebraic manifold simply as a manifold.

For any topological space X let us denote by $H^*_{ch}(X)$ the graded subring of $H^*(V)$ corresponding to $H^{(a)}_*(V)$ by the Poincaré duality D: $H_*(V) \to H^*(V)$.

For any manifold M let us denote by $T^*(M)$ the graded subring of $H^*(M)$ generated by the union of $H^*_{ch}(M)$ and the set of classes representable by submanifolds of M.

Finally, for any algebraic manifold V let us denote by $H_k^{(a)}(V)$ the subgroup of $H_k(V) = H_k(V; \mathbb{Z}_2)$ generated by the set of algebraic subvarieties of V of dimension k (see [BH]) and by $H^*_{(a)}(V)$ the graded subring of $H^*(V)$ corresponding to $H^{(a)}_*(V)$ by the Poincaré duality D: $H_*(V) \to H^*(V)$.

The main result of this note is the following:

THEOREM 1: *For each $d \geqslant 11$, there exists a manifold V of dimension d and a class $\alpha \in H^2(V)$ such that for every homeomorphism $h: V' \to V$ between V and an algebraic manifold V', the class $h^*(\alpha)$ does not belong to $H^2_{(a)}(V')$.*

By Tognoli's proof of the Nash conjecture ([T1]), every manifold M is diffeomorphic to an algebraic manifold M'. It is then a natural problem to study which smooth (or continuous) extra-structures over M can be realized algebraically with respect to a suitable choice of M'. There are several positive results in this direction (see [T2] for general references about these topics); for example:

(a) It is true a covariant version of the Nash conjecture with respect to any smooth action of a compact Lie group on M (Palais).

(b) There exists M' such that every continuous vector bundle over M' is isomorphic to a *strongly algebraic* one (that is, an algebraic vector bundle with A-coherent sections sheaf) (see the number 3 below).

(c) There exists M' such that $T^*(M')$ is in fact a subring of $H^*_{(a)}(M')$.

Notice, in particular, that this implies $H^*_{(a)}(M') = H^*(M')$ if dim $M' \leqslant 6$ (see [Tm]) and, in general, $H^1_{(a)}(M') = H^1(M')$. On the other hand, there are examples of algebraic manifolds V such that $H^1_{(a)}(V) \neq H^1(V)$ ([BT]). Theorem 1 says that, in general, *there does not exist an M' homeomorphic to a given M such that $H^*_{(a)}(M') = H^*(M')$*. Actually, a direct proof of this result, which is less precise than theorem 1 but in fact motivated this note, is slightly simpler (see Remark 3.3).

The following Theorem 2 is an analogue of Theorem 1 in terms of smooth unoriented bordism:

THEOREM 2: *For each $d \geqslant 11$, there exist a manifold V of dimension d, a manifold N and a map f: $N \to V$ such that for every homeomorphism g: $V \to V'$ between V and an algebraic manifold V', $g \circ f$: $N \to V'$ does not represent the same smooth unoriented bordism class of a regular rational map r: $P \to V'$, P being a non singular algebraic variety.*

Recall that, on the other hand, every manifold is cobordant to a non-singular algebraic variety. Then, a fortiori, we are able to produce counterexamples to the following problem: Let $N \xrightarrow[f]{} M$ be a (smooth) map between manifolds. Does there exist a diagram

$$\begin{array}{ccc} N & \xrightarrow[f]{} & M \\ h\downarrow & & \downarrow g \\ N' & \xrightarrow[f']{} & M' \end{array}$$

such that: (i) f' is a regular rational map between algebraic manifolds; (ii) h and g are homeomorphisms; (iii) f' approximates $g \circ f \circ h^{-1}$?

Notice that a positive answer to this last question would have been crucial to solve completely (in some sense) the problem of giving a topological characterization of real algebraic sets (see [AK1], [AK2]). In fact, the strongest result about this problem at the moment (see [AK2]) asserts roughly speaking that a (compact) polyhedron of dimension n is homeomorphic to a real algebraic set if it admits a "good" topological resolution of singularities and, moreover, the above approximation problem has positive answer at least for all manifolds of dimension $\leqslant n$. Unfortunately it is false for $n \geqslant 11$.

Section 3 is largely inspired to the results and the methods of [BS], [BH] and [G]; for this reason, whenever the proof of any statement of this section is simply obtained by adapting one of these quoted papers, we shall only sketch it and give precise reference.

The result of this paper has been announced in [BD].

2. A topological result

It is rather easy to see that, in general, for a given manifold M, $H^*_{ch}(M) \neq H^*(M)$ (see, for example, $M = S^3$). Actually, for our purposes, we need such examples for the second cohomology group.

2.1. Proposition: *For each $d \geqslant 11$ there exist a manifold V of dimension d and a class $\alpha \in H^2(V)$ which does not belong to $H^2_{ch}(V)$.*

Let $K = K(Z_2, 2)$ be an Eilenberg-McLane space, A its 5-skeleton, $u \in H^2(K)$ the fundamental class, $v = i^*(u)$, where i: $A \to K$ is the inclusion.

2.2. Lemma: *v does not belong to $H^2_{ch}(A)$.*

Proof: First of all recall that i^*: $H^i(K) \to H^i(A)$ is an isomorphism for $i \leqslant 4$ and is injective for $i = 5$. Assume that $v \in H^2_{ch}(A)$. Since $H^1(A) = 0$, it is necessarily of the form $v = f^*(w_2)$, where f is a continuous map f: $A \to B0(k)$ and w_j denotes the j-th Stiefel-Whitney class of the universal bundle. By an iterated application of the Wu formulas, one has

$$Sq^2Sq^1w_2 + w_2Sq^1w_2 = Sq^1(w_4) + w_1^2w_3 + w_1^3w_2.$$

By applying f^* to both sides of this equality, one obtains

$$Sq^2Sq^1v + vSq^1v = Sq^1(f^*(w_4)).$$

Note that $Sq^1(f^*(w_4)) = 0$; in fact, $H^4(A) = Z_2$ is generated by v^2; hence, if $f^*(w_4) = v^2$, then $Sq^1v^2 = 0$ (by the Cartan's formula for Steenrod squares). At last, by using the injectivity of i^*: $H^5(K) \to H^5(A)$, we have the relation

$$Sq^2Sq^1u + uSq^1u = 0.$$

This is impossible, because we know after Serre's computation ([S]) that $H^*(K)$ is the polynomial ring over Z_2 generated by all the $Sq^I(u)$, where $I = (i_1, \dots, i_k)$ is an admissible sequence of excess $e(I)$ less than 2 (that is $i_j \geqslant 2i_{j+1}$ for $j < k$ and $e(I) = 2i_1 - \Sigma_j i_j < 2$). Note that in particular Sq^2Sq^1u, Sq^1u and u belong to that set of generators. The lemma is proved. □

PROOF OF 2.1: We shall use a standard trick (see, for example, [Tm]). Realize the 5-skeleton A of K as a subpolyhedron of $\mathbb{R}^d$ (this is possible since $d \geq 11$). Let Q be any closed neighbourhood of A in $\mathbb{R}^d$ such that Q is a smooth manifold with boundary and A is a retract of Q (note that A can be assumed to be connected). Take the smooth double V of Q; a retraction r: $Q \to A$ induces naturally a map p: $V \to A$, such that $j^* \circ p^*$ is the identity on $H^*(A)$ (j is the inclusion of A in V); V and $\alpha = p^*(v)$ satisfy the statements of 2.1. □

3. A theorem of Grothendieck

Let X be a complex nonsingular connected quasi projective variety. Let $K(X)$ be the Grothendieck group constructed with the coherent (algebraic) sheaves over X and $K_l(X)$ be that constructed with the locally free sheaves over X (that is, the "algebraic vector bundles over X") (see [BS], pag. 105). The natural homomorphism ξ: $K_l(X) \to K(X)$ is in fact an isomorphism (Theorem 2 of [BS]), so that it is possible to extend the definition of the Chern classes to any coherent sheaf $\mathscr{F}$ (naturally considered as an element of $K(X)$) and, moreover, every such a class $c_i(\mathscr{F})$ belongs to the Chow ring $A(X)$ of algebraic cycles of X up to rational equivalence (see [BS] and [G]).

Associate to every subvariety Y of X: (i) $\theta_Y \in K(X)$; (ii) the class $Cl_X(Y)$ in $A(X)$. A very useful theorem of Grothendieck asserts: *If p is the codimension of Y in X, then $c_p(\theta_Y) = (-1)^{p-1}(p-1)!Cl_X(Y)$.* (For the proof see [G] p. 151 or [H] p. 53.) We want to outline a similar theory in the real case.

A good category of sheaves for this purpose is that of *A-coherent* sheaves. These are defined in [T3] (Section 4) and [BrT], to which we refer for their basic properties.

By definition, An A-coherent sheaf $\mathscr{F}$ over an algebraic manifold V is a coherent algebraic sheaf such that there exists an exact sequence of sheaves

$$\theta_V^m \to \theta_V^r \to \mathscr{F} \to 0.$$

In the complex case the category of A-coherent sheaves coincides with that of coherent sheaves (by the theorem A of Cartan-Serre). On the contrary, in the real case, there exist even locally free sheaves (over $\mathbb{R}^n$, $n \geq 2$) which are not A-coherent (see [T3], pp. 40–41).

Let us denote by $\mathscr{A}$ the set of all A-coherent sheaves over V (up to isomorphism) and by $\mathscr{L}$ the set of all locally free A-coherent sheaves over V. By the usual construction (see [BS], p. 105) we can define the Grothendieck groups $K_{\mathscr{A}}(V)$ of V with basis $\mathscr{A}$ and $K_{\mathscr{L}}(V)$ with basis $\mathscr{L}$. The inclusion $\mathscr{L} \subset \mathscr{A}$ induces a natural homomorphism ϵ: $K_{\mathscr{L}}(V) \to K_{\mathscr{A}}(V)$. Our first claim is the following:

Claim: ϵ is an isomorphism

One can prove this claim by following step by step that of [BS] in the complex case (pp. 105–108). For the sake of clarity we shall use the convention that, in the following, lemma n' will be the analogue of lemma n in [BS].

LEMMA 8′: *Let $0 \to \mathscr{G} \to L' \to L \to 0$ be an exact sequence, where $\mathscr{G} \in \mathscr{A}$ and L', $L \in \mathscr{L}$. Then $\mathscr{G} \in \mathscr{L}$.*

LEMMA 9′: *Let $0 \to \mathscr{G} \to L_p \to \dots \to L_0 \to \mathscr{F} \to 0$ be an exact sequence where $\mathscr{G}, \mathscr{F} \in \mathscr{A}$ and $L_j \in \mathscr{L}$. If $p \geqslant \dim V - 1$, then $\mathscr{G} \in \mathscr{L}$.*

The proofs of these lemmas, that is of the fact that $\mathscr{G}$ is in both cases locally free, can be achieved by repeating "verbatim" the proofs of the corresponding Lemmas 8 and 9 of [BS].

LEMMA 10′: *Every $\mathscr{F} \in \mathscr{A}$ is a quotient of a sheaf $L \in \mathscr{L}$.*

PROOF: Since any θ_V^p is clearly A-coherent, the lemma follows trivially from the definition of A-coherent sheaves.

LEMMA 10′ bis: *Every $\mathscr{F} \in \mathscr{A}$ has a exact resolution (a priori not finite) $\dots \to L_q \to L_{q-1} \to \dots \to L_0 \to \mathscr{F} \to 0$ where $L_j \in \mathscr{L}$.*

PROOF: It follows from Lemma 10′ and the fact that, if $\mathscr{F} \xrightarrow{\varphi} \mathscr{G}$ is any morphism of A-coherent sheaves, then $\ker \varphi$ (and also $\varphi(\mathscr{F})$ and coker φ) belongs to $\mathscr{A}$ (see [T3], p. 44).

As an immediate consequence of the previous lemma we get:

COROLLARY: *Every $\mathscr{F} \in \mathscr{A}$ admits an exact sequence L: $0 \to L_n \to \dots \to L_0 \to \mathscr{F} \to 0$ where $L_j \in \mathscr{L}$.*

For every sequence L as before, define $\gamma(L) = \Sigma(-1)^p L_p \in K_{\mathscr{L}}(V)$. In order to prove that γ actually induces the inverse of ϵ we must prove:

LEMMA 11′: *$\gamma(L)$ depends only on $\mathscr{F}$.*

LEMMA 12′: *$\gamma(L)$ is an additive function of $\mathscr{F}$.*

The proof of these lemmas works formally as in [BS]. It is enough to notice that the direct sum of sheaves in $\mathscr{A}$ (resp. in $\mathscr{L}$) clearly belongs to $\mathscr{A}$ (resp. to $\mathscr{L}$) and to apply several times the properties of morphisms between A-coherent sheaves recalled in the proof of Lemma 10′bis. In particular this implies that the sheaf (a, c) of the technical Lemma 13 of [BS] belongs to $\mathscr{A}$ (here we assume that a, b, $c \in \mathscr{A}$ and we look for $L \in \mathscr{L}$).

Thus the claim is proved and we are able (by the Whitney sum formula) to extend the definition of the Stiefel-Whitney classes to any A-coherent sheaf over V.

Recall that, if Y is a subvariety of V, then θ_Y belongs to $\mathscr{A}$ (see [T3] p. 44, Corollary 4 and [BrT] Theorem 1 and Lemma 2 of Section 2); then in particular we can define the Stiefel Whitney classes $w_j(\theta_Y)$.

We recall now some further properties of $\mathscr{A}$, concerning the good behaviour of $\mathscr{A}$ with respect to the operation of complexification.

(a) Assume $V \subset \mathbb{R}^n$ and let $\tilde{V} \subset \mathbb{C}^n$ be the complexification of V (that is the smallest complex affine subvariety of $\mathbb{C}^n$ containing V).

CONVENTION: By an open neighbourhood U of V in $\tilde{V}$ we mean an open non singular neighbourhood of the form $U = \tilde{V} - S$, where S is a closed set of $\tilde{V}$ (in the Zariski topology) defined over $\mathbb{R}$.

Let $\mathscr{I} \in \mathscr{A}$; then for any exact sequence $\theta_V^m \xrightarrow{\alpha} \theta_V^r \to \mathscr{I} \to 0$ there exist an open neighbourhood U of V in $\tilde{V}$ and an exact sequence of coherent algebraic sheaves over U $\theta_U^m \xrightarrow{\tilde{\alpha}} \theta_U^r \to \tilde{\mathscr{I}} \to 0$, where $\tilde{\alpha}$ extends α. Any such $\tilde{\mathscr{I}}$ is called a complexification of $\mathscr{I}$. Two such complexifications of $\mathscr{I}$ coincide over an open neighbourhood of V in $\tilde{V}$ and for each $x \in V$ we have $\tilde{\mathscr{I}}_X \cong \mathscr{I}_X \otimes_{\mathbb{R}} \mathbb{C}$ (see [T3] Section 4b, or [BrT] Section 3).

(b) For every exact sequence of A-coherent sheaves over V $\mathscr{I}_1 \xrightarrow{\alpha_1} \mathscr{I}_2 \to \dots \xrightarrow{\alpha_{p-1}} \mathscr{I}_p$ (eventually $p = 2$) there exist an open neighbourhood U of V in $\tilde{V}$ and an exact sequence $\tilde{\mathscr{I}}_1 \xrightarrow{\tilde{\alpha}_1} \tilde{\mathscr{I}}_2 \to \dots \xrightarrow{\tilde{\alpha}_{p-1}} \tilde{\mathscr{I}}_p$, where $\tilde{\mathscr{I}}_j$ is a complexification of $\mathscr{I}_j$ defined over U and $\tilde{\alpha}_j$ extends α_j (the references are the same as the previous point (a)).

We can state now the main result of this section.

Let Y be a subvariety of V; let us associate to Y $\theta_Y \in K_{\mathscr{A}}(V)$ and the homology class $[Y] \in H_*^{(a)}(V)$.

3.1. PROPOSITION: *If the codimension of Y in V is* 2, *then* $D([Y]) = w_2(\theta_Y)$.

As an immediate corollary one gets

3.2. COROLLARY: *For every algebraic manifold V, if $z \in H^2_{(a)}(V)$, then $z \in H^2_{ch}(V)$.*

PROOF OF 3.1: Apply the proof of the Grothendieck theorem as developed in [H] p. 53 to a suitable open neighbourhood U of V in $\tilde{V}$ (in the sense of the previous remark (a)) and to the complexification $\tilde{Y}$ of Y in

U. By using the results of [H] it is not hard to see that in our hypothesis we can construct a Koszul resolution defined over $\mathbb{R}$ (in particular we can choose the hypersurfaces H_j involved in the proof to be defined over $\mathbb{R}$). This means that we can find a non singular open neighbourhood U of V in $\tilde{V}$ and an A-coherent locally free resolution of θ_Y:

$$(E)\ O \to L^n \to \dots \to L^1 \to \theta_Y \to 0$$

which extends (in the sense of the properties of A-coherent sheaves recalled above) to a locally free resolution of sheaves defined over U:

$$(E)\ O \to \tilde{L}^n \to \dots \to \tilde{L}^1 \to \theta_{\tilde{Y}} \to 0$$

such that Grothendieck's formula $Cl_U(\tilde{Y}) = -c_2(\theta_{\tilde{Y}})$, if computed by means of (E), actually holds in $A_{\mathbb{R}}(U)$, that is in the Chow ring of the cycles of U defined over $\mathbb{R}$, up to rational equivalence over $\mathbb{R}$. Recall that every Chern class $c_i(\tilde{L}^j)$ lies in $A_{\mathbb{R}}(U)$ and in particular also $c_2(\theta_{\tilde{Y}})$ (see [BH] pp. 495, 498). On the other hand, let $\rho(c_2(\theta_{\tilde{Y}}))$ and $\rho(\tilde{Y}) = [Y]$ be the classes of $H_{d-2}(V)$ defined by the real parts of $c_2(\theta_{\tilde{Y}})$ and $\tilde{Y}$ respectively ($d = \dim V$). Since the Grothendieck's formula above holds in $A_{\mathbb{R}}(U)$, we have $\rho(c_2(\theta_{\tilde{Y}})) = [Y]$ (see, for instance, Proposition 5.13 of [BH]). Finally, it follows immediately from the proposition of p. 498 in [BH] that $w_2(\theta_Y) = D\rho(c_2(\theta_{\tilde{Y}}))$. The proposition is proved.

3.3. Remark: If we assume that $H_2(V) = H_2^{(a)}(V)$, the proof of the above proposition becomes slightly simpler, because we can avoid to consider the details of the proof of Grothendieck's theorem; let (E) be any resolution of θ_Y and $(\tilde{E})$ any extension of (E) in the sense of the above remarks (a) and (b). $Z = c_2(\theta_{\tilde{Y}}) + Cl_U(\tilde{Y})$ can be considered as an algebraic cycle of U defined over $\mathbb{R}$ which represents the zero of $A(U)$. Let z be the class defined by the real part of Z, $z = \rho(Z) = D^{-1}w_2(\theta_Y) + [Y] \in H_{d-2}(V)$; in order to prove that z is zero, it is enough to show that, for every $c \in H^{d-2}(V)$, $Dz \cup c = 0$; moreover, we can assume that $c = D^{-1}([C])$, where C is an algebraic subvariety of V of dimension 2. Let $\tilde{C}$ be the complexification of C in U. By the *Chow's moving lemma* (see for instance [R]) there exists C' representing the same class as $\tilde{C}$ in $A_{\mathbb{R}}(U)$ such that the intersection $Z \cdot C'$ (in the sense of algebraic cyles) is defined. Since Z is zero in $A(U)$, $Z \cdot C' = 0$; hence $Dz \cup c = D\rho(Z) \cup D\rho(C') = D(\rho(Z \cdot C')) = 0$ (see the proposition of p. 494 in [BH]; actually, this remark is a particular case of Proposition 5.14 (2) of [BH]).

4. Proofs of the main results and final remarks

Proof of Theorem 1: Take V and α as in the proposition 2.1 and apply 2.1 and 3.2.

PROOF OF THEOREM 2: Take V and α as before and let $\beta = D^{-1}\alpha$; by [Tm] $\beta = \beta_1 + \ldots + \beta_k$, where each β_j is represented by a (smooth) map f_j: $N_j \to V$ and N_j is a manifold. Recall that, for every algebraic manifold M, a homology class of M is represented by an algebraic subvariety if and only if it is represented by a regular rational map r: $P \to M$, where P is a (non singular) algebraic variety (see [BT]). If, for every j, there exists a homeomorphism h_j: $V \to V_j'$ such that $h_j \circ f_j$: $N_j \to V_j'$ represents the same smooth unoriented bordism class of a regular rational map r_j: $P_j \to V_j'$, then α would be in $H^2_{ch}(V)$. We end this section with two conjectures:

CONJECTURE 1: *Theorem* 1 *holds for* $d \geqslant 7$.

CONJECTURE 2: *For every algebraic manifold* V, $H^*_{(a)}(V) = T^*(V) \cap H^*_{(a)}(V)$.

Note that this last conjecture would imply the following:

for each $k \geqslant 2$, *there exist a manifold* V *of dimension* $d = d(k)$ *and a class* $\alpha \in H^k(V)$ *such that for every homeomorphism* h: $V' \to V$ *between* V *and an algebraic manifold* V', *the class* $h^*(\alpha)$ *does not belong to* $H^k_{(a)}(V')$.

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Riccardo Benedetti and Maria Dedò
Istituto Matematico "Leonida Tonelli" dell'Università di Pisa
Via F. Buonarroti 2
56100 Pisa
Italy