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MONODROMY OF FUNCTIONS DEFINED ON ISOLATED SINGULARITIES OF COMPLETE INTERSECTIONS

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A basic tool in the study of an analytic function germ \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) with an isolated singularity at the origin (or of the corresponding hypersurface germ \( Y = f^{-1}(0) \)) is the well-known local monodromy group ([4], [8], [12]).

This widely studied monodromy group can be defined in two equivalent ways:

(i) Using a morsification of the function \( f \).

(ii) Using a line in the base space \( B \) of a versal deformation for \( Y \), in general position with respect to the discriminant hypersurfaces \( \Delta \subset B \).

In this paper we extend the construction (i) above to function germs \( f : (X, 0) \to (\mathbb{C}, 0) \) defined on a complete intersection \( (X, 0) \subset (\mathbb{C}^{n+p}, 0) \) with an isolated singular point at the origin and such that \( X_0 = f^{-1}(0) \) is also a complete intersection with an isolated singularity at 0 (here \( n = \dim X > 0 \)).

In this way we obtain an action of a fundamental group \( \pi = \pi_1(\text{disc} \setminus \{s\} \text{ points}) \) on the exact sequence of the pair \( (\tilde{X}, \tilde{X}_0) \) in homology (with \( \mathbb{Z} \)-coefficients):

\[
0 \to H_n(\tilde{X}) \to H_n(\tilde{X}, \tilde{X}_0) \overset{\partial}{\to} H_{n-1}(\tilde{X}_0) \to 0
\]  

(\( * \))

where \( \tilde{X}, \tilde{X}_0 \) are the Milnor fibers of \( X \) and \( X_0 \) ([5]) chosen such that \( \tilde{X}_0 \subset \tilde{X} \) and \( s = \mu(X) + \mu(X_0) \) is the sum of their Milnor numbers.

More precisely, the action of \( \pi \) on \( H_n(\tilde{X}) \) is trivial, while the actions on the other two homology groups can be described in terms of Picard-Lefschetz formulas with respect to thimbles \( \Delta_k \in H_n(\tilde{X}, \tilde{X}_0) \) and corresponding vanishing cycles \( \delta_k = \partial \Delta_k \in H_{n-1}(\tilde{X}_0) \).

The \( \pi \)-exact sequence (\( * \)) is proved to be a contact invariant of the function \( f \) i.e. it depends only on the isomorphism class (in a natural sense) of the pair of complete intersections \( (X, X_0) \). This fact, as well as the independence of the sequence (\( * \)) on the choice of the morsification for \( f \) is obtained by a simple application of the Thom-Mather Second Isotopy Lemma.
To give some explicit examples, we compute next the \( \pi \)-sequence (\(*\)) for all the \( R \)-simple functions \( f \) defined on an isolated hypersurface singularity \( X \) of dimension \( n > 1 \), as listed in [1].

Note that the \( \pi \)-sequence (\(*\)) gives us in particular two monodromy groups

\[
G_0(f) = \operatorname{Im}\{ \pi \to \operatorname{Aut} H_{n-1}(\tilde{X}_0) \}
\]

\[
G(f) = \operatorname{Im}\{ \pi \to \operatorname{Aut} H_n(\tilde{X}, \tilde{X}_0) \}.
\]

We prove that \( G_0(f) \) is precisely the monodromy group of the complete intersection \( X_0 \) defined as in (ii). In fact the morsification process used above gives rise to a line in the base space \( B \) of a (suitable chosen) versal deformation of \( X_0 \), whose direction depends on the function \( f \) and is not generic with respect to the discriminant \( \Delta \subset B \).

That is why we need a slightly modified version of a result of Hamm-Lê on the fundamental group \( \pi_1(B \setminus \Delta) \) (see Lemma 3.5).

Then we show that the other monodromy group \( G(f) \) is a semidirect product of \( G_0(f) \) with a free abelian group \( \mathbb{Z}^a \) and we also give some estimates for the rank \( a \).

Finally we remark that constructions similar to some of ours (i.e. morsifications and connections with versal deformations) have been used many a time before (e.g. by Iomdin [7] and Lê [10]) but always with different aims in view, as far as we known.

We would like to express our deep gratitude to Professor V.I. Arnold for a very stimulating discussion.

§1. Morsifications and monodromy map of pairs

Let \( X: g_1 = \ldots = g_p = 0 \) be an analytic complete intersection in a neighbourhood of the origin of \( \mathbb{C}^{n+p} \), with an isolated singular point at 0. \( (n \geqslant 1, p \geqslant 0) \). Consider also an analytic function germ

\[
f: (\mathbb{C}^{n+p}, 0) \to (\mathbb{C}, 0)
\]

such that \( X_0 = f^{-1}(0) \cap X \) is again a complete intersection with an isolated singularity at 0.

For \( \epsilon \gg \delta > 0 \) chosen sufficiently small, it is known that the Milnor fiber of \( X \)

\[
X_\epsilon = \{ x \in B_\epsilon; \; g(x) = r \}
\]

is a compact \( C^\infty \)-manifold with boundary for any \( r \in \mathbb{C}^p \) sufficiently general with \( 0 < |r| \leqslant \delta \), where

\[
B_\epsilon = \{ x \in \mathbb{C}^{n+p}; \; |x| \leqslant \epsilon \}.
\]
The space $X_r$ (denoted in the introduction by $\tilde{X}$) has the homotopy type of a bouquet of $n$-spheres, the number of which is by definition the Milnor number $\mu(X)$ of the complete intersection $X$.

For $r$ small enough, it is easy to see that $f' = f|\text{Int} \ X_r$ has only a finite number of critical points $a_1, \ldots, a_k$ and moreover $a_i \to 0$ when $r \to 0$ for any $i = 1, \ldots, k$.

Let us denote by $\mu(f', a_i)$ the Milnor number of the function $f'$ at the critical point $a_i$.

One has the following property, in analogy with a result of Lê ([10], (3.6.4)).

**Proposition 1.1:**

$$\sum_{i=1}^{k} \mu(f', a_i) = \mu(X) + \mu(X_0).$$

**Proof:** Let $D_\delta$ denote the open disc $\{z \in \mathbb{C}; |z| < \delta\}$. For $\epsilon, \delta$ and $r$ suitable chosen, the inclusion

$$E = X_r \cap f^{-1}(D_\delta) \hookrightarrow X_r$$

is a homotopy equivalence (see for instance [10] (3.5)) and moreover the restriction

$$f|_{\partial E}: \partial E \to D_\delta$$

where $\partial E = \partial X_r \cap f^{-1}(D_\delta)$

is a submersion.

Let $b \in D_\delta$ be a regular value of $f = f|E$ and let $c_i = f(a_i) \in D_\delta$ be the (not necessarily distinct) critical values of $\tilde{f}$.

Then $F = \tilde{f}^{-1}(b)$ is the Milnor fiber of the complete intersection $X_0$ and the exact sequence of the pair $(E, F)$ shows that $H_n(E, F)$ is a free abelian group of rank $s = \mu(X) + \mu(X_0)$. ($\mathbb{Z}$-coefficients for homology are used throughout in this paper).

We compute now this group in a different way, following ([9], §5). Choose small disjoint closed discs $D_i$ centered at the critical values $c_i$ and fix some points $b_i \in \partial D_i$.

For each $i$, take a $C^\infty$-embedded interval $I_i$ from $b$ to $b_i$ such that $I_i = \bigcup I_i$ can be contracted within itself to $b$ and $D_\delta$ can be contracted to $C = \bigcup D_i \cup I$.

Since $\tilde{f}$ induces a (proper) locally trivial fibration

$$E \setminus f^{-1}\{c_i\} \to D_\delta \setminus \{c_i\},$$

these retractions can be lifted to the corresponding subsets of $E$ and we
get the following isomorphisms

\[ H_n(E, F) \xrightarrow{\sim} H_n(\tilde{f}^{-1}(C), F) \xrightarrow{\sim} H_n(\tilde{f}^{-1}(C), \tilde{f}^{-1}(I)). \]

By excision, the last group is equal to

\[ \bigoplus_i H_n(\tilde{f}^{-1}(D_i), \tilde{f}^{-1}(b_i)) \]

Assume that \( a_{i1}, \ldots, a_{im} \) are the critical points of \( \tilde{f} \) in the fiber over \( c_i \). Let \( B_j \) be the intersection of a small closed ball centered at \( a_{ij} \) with \( \tilde{f}^{-1}(D_i) \) and denote with \( F_i \) the fiber \( \tilde{f}^{-1}(b_i) \).

It follows that

\[ H_n(\tilde{f}^{-1}(D_i), F_i) = H_n\left( \bigcup_{j=1}^m B_j \cup F_i, F_i \right) = \bigoplus_{j=1}^m H_n(B_j, B_j \cap F_i). \]

Moreover

\[ H_n(B_j, B_j \cap F_i) \xrightarrow{\partial^3} H_{n-1}(B_j \cap F_i) \]

is a free abelian group of rank \( \mu(f', a_{i_j}) \) by the definition of the Milnor numbers of \( f' \), if the discs \( D_i \) and the balls \( B_j \) are chosen small enough.

We consider now the problem of the existence of morsifications of the function \( f' : X_r \to \mathbb{C} \), i.e. small deformations of \( f' \) having only nondegenerate critical points with distinct critical values.

If \( P \) denotes the vector space of polynomials in \( x_1, \ldots, x_{n+p} \) of degree \( \leq 3 \), it is easy to show by standard transversality arguments that there is a Zariski open subset \( U \subset P \) such that the function

\[ f_q = (f + q)|X_r \]

is a Morse function for any \( q \in U \).

Moreover, if we have chosen already \( \epsilon \gg \delta > 0 \) such that (1.2) and (1.3) hold true for any generic \( r \in \mathbb{C}^p \) with \( |r| \leq \delta \), then there is an \( \eta > 0 \) such that \( |q| < \eta \) implies similar properties for \( f_q \).

Suppose now we have two polynomials \( q_0, q_1 \in U \) such that \( |q_1| < \eta \). We can find a \( C^\infty \)-path \( q_t \) in \( U \) such that \( q_t = q_0 \) for \( 0 \leq t \leq a \), \( q_t = q_1 \) for \( 1 - a \leq t \leq 1 \) and \( |q_t| < \eta \) for any \( t \in [0, 1] \), where \( a \in (0, 1/3) \).

Consider the spaces

\[ D = D_\delta \times (0, 1) \quad \text{and} \quad E = \{(x, t) \in X_r \times (0, 1) ; f_{q_t}(x) \in D_\delta \} \]
and the proper map
\[ \varphi: \tilde{E} \to \tilde{D}, \quad \varphi(x, t) = (f_{q_i}(x), t). \]

If \( a_i(t) \) (resp. \( c_i(t) \)) denote the critical points (resp. critical values) of \( f_{q_i} \) for \( i = 1, \ldots, s \), then we can stratify the map \( \varphi \) as follows ([2], Chap. I). The strata in \( \tilde{D} \) are given by
\[ \tilde{D}_1 = \{(c_i(t), t); t \in (0, 1), i = 1, \ldots, s \} \quad \text{and} \quad \tilde{D}_3 = \tilde{D} \setminus \tilde{D}_1. \]

The strata in \( \tilde{E} \) are given by
\[ \tilde{E}_1 = \{(a_i(t), t); t \in (0, 1), i = 1, \ldots, s \} \]
\[ \tilde{E}_{2n-2} = \{(x, t); t \in (0, 1), x \in \left(f_{q_i}\right)^{-1}(c_i(t)) \cap \partial X_r, i = 1, \ldots, s \} \]
\[ \tilde{E}_{2n-1} = \{(x, t); t \in (0, 1), x \in \left(f_{q_i}\right)^{-1}(c_i(t)) \cap \text{Int } X_r, i = 1, \ldots, s \} \]
\[ \tilde{E}_{2n} = (\partial X_r \times (0, 1)) \cap (\tilde{E} \setminus \tilde{E}_{2n-2}) \]
\[ \tilde{E}_{2n+1} = \tilde{E} \setminus \text{the union of the other strata } \tilde{E}_k \text{ defined above.} \]

The lower index gives the real dimension of the stratum. (These definitions work for \( n \geq 2 \). The simpler case \( n = 1 \) is left to the reader.)

The Whitney-Thom regularity conditions are obviously satisfied for any pair of strata.

By Thom-Mather Second Isotopy Lemma ([2], II, (5.8)) we obtain a commutative diagram
\[ \begin{array}{ccc}
\varphi^{-1}(D_\delta \times \alpha) & \xrightarrow{H} & \varphi^{-1}(D_\delta \times (1 - \alpha)) \\
\downarrow f_{q_0} & & \downarrow f_{q_1} \\
D_\delta \times \alpha & \xrightarrow{h} & D_\delta \times (1 - \alpha)
\end{array} \]

where \( \alpha \in (0, a) \) and \( H, h \) are homomorphisms compatible with the induced stratifications.

In particular we get the following result.

**Lemma 1.4:** The topological type of the map of pairs
\[ f_q: \left( \varphi^{-1}(D_\delta), \varphi^{-1}(C) \right) \to (D_\delta, C) \]
where $C$ is the set of critical values of the function $f_0$ is independent of the polynomial $q \in U$, $|q| < \eta$.

It is also clear the independence of the topological type of the map above of the choice of (suitable) $\epsilon$, $\delta$ and $r$. Moreover, if we change the function $f$ to a function $f_1 = f + k$, where $k$ is a function in the ideal $(g_1, \ldots, g_p)$ of the complete intersection $X$, note that the distance $\|f_1 - f\|_X$ can be made as small as we want by taking $r$ small enough.

Using a stratification argument as above it follows that the topological type of the map of pairs in (1.4) depends only on the restriction $f|X$ i.e. on a function in $m_X = m/(g_1, \ldots, g_p)$, where $m \subset \mathcal{O}_{n+p}$ is the maximal ideal.

(We shall consider throughout in this paper only functions $f \in m_X$ such that $X_0 = f^{-1}(0)$ is a complete intersection with an isolated singularity at 0).

The discussion below will also imply independence from the defining equations $g_i = 0$ of $X$, and hence we can give the following.

**Definition 1.5:** The topological type of the map of pairs in (1.4) will be called the *monodromy map of pairs* of the function $f \in m_X$ and will be denoted simply by

$$f^*: (E^*, E^*_e) \rightarrow (D, C).$$

This topological object is constant in $\mu$-constant families in the following precise sense (compare to [12], §9).

Let $(X_t, 0) \subset (\mathbb{C}^{n+p}, 0)$ be a smooth family of complete intersections with isolated singular points at the origin such that $\dim X_t = n$ and $\mu(X_t) = \text{const.}$ for $t \in [0, 1]$. Assume that $f_t \in m_{X_t}$ is a smooth family of function germs such that $\mu(f_t^{-1}(0)) = \text{const}$.

Using the construction of morsifications and stratification arguments as above, one can then show that the monodromy map of pairs of the function $f_t$ is independent of $t$.

A special case of this situation is the following.

**Definition 1.6** [1]: We say that two function germs $f_1, f_2 \in m_X$ defined on the complete intersection $(X, 0)$ are $\mathcal{C}^\infty$-(contact)-equivalent if there is an automorphism $u$ of the local $\mathcal{C}$-algebra $\mathcal{O}_X$ such that $(u(f_1)) = (f_2)$, where $(a)$ means the ideal generated by $a$ in $\mathcal{O}_X$.

Since the complete intersections $X$ and $X_{0_i} = f_i^{-1}(0) \ i = 1, 2$ have isolated singularities at the origin, the question of $\mathcal{C}^\infty$-equivalence of $f_1$ and $f_2$ can be settled in a jet space $J^k(n + p, p + 1)$, via the action of a connected algebraic group $G_X^k$ (the particular case when $X$ is a hypersurface is treated in detail in [1]).

It follows that $(X, f_1)$ and $(X, f_2)$ can be connected by a $\mu$-constant family $(X_t, f_t)$ as above and we get thus the following.
COROLLARY 1.7: If two function germs \( f_1, f_2 \in m_X \) are \( \mathcal{K} \)-equivalent then their associated monodromy maps \( f_1^* \) and \( f_2^* \) are the same.

§2. Monodromy exact sequence. Examples

Let \( f^*: (E^*, E^*) \to (D, C) \) be the monodromy map of pairs of a function \( f \in m_X \) as in §1.

If \( b \in D \setminus C \) and \( F = (f^*)^{-1}(b) \), then the locally trivial fibration \( E^* \setminus E_c^* \to D \setminus C \) defines in the usual way an action of the fundamental group \( \pi = \pi_1(D \setminus C) \) on the middle homology group \( H_{n-1}(F) \) of the fiber.

Moreover, for any homotopy class \( w \in \pi \) there is a well defined homomorphism

\[
\tau_w: H_{n-1}(F) \to H_n(E^*, F)
\]
called the extension along the path \( w \). For a detailed construction and the main properties of \( \tau_w \) we send to ([9], (6.4)).

We can define an action of the fundamental group \( \pi \) on the homology group \( H_n(E^*, F) \) by the formula

\[
w \cdot x = x + (-1)^{n-1} \tau_w(\partial x)
\]
where \( \partial \) is the connecting homomorphism in the exact sequence of the pair \( (E^*, F) \)

\[
0 \to H_n(E^*) \xrightarrow{i} H_n(E^*, F) \xrightarrow{\partial} H_{n-1}(F) \to 0.
\] (2.2)

If we consider the trivial action of \( \pi \) on \( H_n(E^*) \), then this exact sequence is a \( \pi \)-exact sequence, i.e. the homomorphisms \( i \) and \( \partial \) are \( \pi \)-equivariant.

Let \( \tilde{X} \) (say equal to \( X_r \) in §1) and \( \tilde{X}_0 \) (say equal to \( X_r \cap f^{-1}(b) \)) denote the associated Milnor fibers of the complete intersections \( X \) and \( X_0 \).

The corresponding exact sequence

\[
0 \to H_n(\tilde{X}) \to H_n(\tilde{X}, \tilde{X}_0) \xrightarrow{\partial} H_{n-1}(\tilde{X}_0) \to 0
\] (2.3)
is isomorphic to the exact sequence (2.2) and via this isomorphism we can transfer the \( \pi \)-actions on the homology groups in (2.3).

DEFINITION 2.4: The \( \pi \)-exact sequence (2.3) constructed as above is called the monodromy exact sequence of the function \( f \).
EXAMPLE 2.5: If the complete intersection $X$ is smooth, then the sequence (2.3) becomes

$$0 \to 0 \to H_n(\tilde{X}, \tilde{X}_0) \xrightarrow{\partial} H_{n-1}(\tilde{X}_0) \to 0$$

and hence it contains the same information as the action of $\pi$ on $H_{n-1}(\tilde{X}_0)$ i.e. the classical monodromy action for the hypersurface $X_0$. □

Put again $s = \mu(X) + \mu(X_0) = rkH_n(\tilde{X}, \tilde{X}_0)$ and let $C = \{c_1, \ldots, c_s\}$. We denote by $w_k \in \pi$ the elementary path encircling $c_k$ ([9] (6.1)) and chose the order of these paths such that

$$w_s \cdots w_1 = w_0$$

where $w_0$ is the class of the path $w_0(t) = b \cdot e^{2\pi it}$, $0 \leq t \leq 1$ (we assume here $|b| > |c_k|$ for any $k = 1, \ldots, s$).

We recall from the proof of (1.1) the isomorphisms

$$H_n(\tilde{X}, \tilde{X}_0) = H_n(E^*, F) = \bigoplus_{k=1}^{s} H_n((f^*)^{-1}(D_k), (f^*)^{-1}(b_k))$$

Since $f^*$ is a morsification, each of the last homology groups is free abelian of rank one.

We shall denote by $\Delta_1, \ldots, \Delta_s$ the corresponding generators of the group $H_n(\tilde{X}, \tilde{X}_0)$, which are precisely the thimbles of Lefschetz ([9] (6.2)).

With these notations, the $\pi$-actions in the exact sequence (2.3) can be described in terms of Picard-Lefschetz formulas.

LEMMA 2.6: \hspace{1cm}

$$\langle, \rangle$$

where $\langle, \rangle$ denotes the intersection form on $H_{n-1}(\tilde{X}_0)$ and $k = 1, \ldots, s$.

PROOF. The second formula is the usual Picard-Lefschetz formula (see for instance ([8], §5)). The first one follows from (2.1) and the formula for $\tau_w$ given in ([9], (6.7.1)). □

It follows that in order to determine the monodromy exact sequence it is enough to fix a basis $\{\delta_k\}$ of the group $H_{n-1}(\tilde{X}_0)$ and to compute with respect to it the vanishing cycles $\partial \Delta$, and the intersection form.
As examples of this method, we give the description of the monodromy exact sequences of the $\mathcal{R}$-simple functions defined on an isolated hypersurface singularity $X$ with $\dim X > 1$ which were classified in ([1], §3).

In all these cases $X_0$ is an isolated hypersurface singularity of type $A_k$ for some $k$ and we can chose a distinguished basis of vanishing cycles $(\delta_i)$ for $H_{n-1}(\tilde{X}_0)$ corresponding to a Dynkin diagram of type $A_k$ ([4], (2.4)).

Moreover, using the stabilization of singularities (i.e. addition of a sum of squares to the given equation of $X_0$ as described in [4] (2.3)), we can assume $n = 1$ when we compute $\partial\Delta_i$.

The results are given below, without these tedious computations.

**Proposition 2.7:** For the simple function of type $B_m$ $(m \geq 2)$ given by $X$: $x_1^m + x_2^2 + \ldots + x_{n+1}^2 = 0$ and $f = x_1$ there is a basis of thimbles $\Delta_1, \ldots, \Delta_m$ of $H_n(\tilde{X}, \tilde{X}_0)$ and a vanishing cycle $\delta$ which generates $H_{n-1}(\tilde{X}_0)$ such that $\partial\Delta_k = \delta$ for any $k = 1, \ldots, m$.

**Proposition 2.8:** For the simple function of type $C_{m+1}$ $(m \geq 1)$ given by $X$: $x_1x_2 + x_3^2 + \ldots + x_{n+1}^2 = 0$ and $f = x_1 + x_2^m$ there is a basis of thimbles $\Delta_0, \ldots, \Delta_m$ of $H_n(\tilde{X}, \tilde{X}_0)$ and a basis of vanishing cycles $\delta_1, \ldots, \delta_m$ of $H_{n-1}(\tilde{X}_0)$ such that $\partial\Delta_0 = \delta_1 + \ldots + \delta_m$ and $\partial\Delta_k = \delta_k$ for any $k = 1, \ldots, m$.

(Note that $C_2 \equiv B_2$).

**Proposition 2.9:** For the simple function of type $F_4$ given by $X$: $x_1^3 + x_2^2 + \ldots + x_{n+1}^2 = 0$ and $f = x_2$ there is a basis of thimbles $\Delta_1, \ldots, \Delta_4$ of $H_n(\tilde{X}, \tilde{X}_0)$ and a basis of vanishing cycles $\delta_1, \delta_2$ of $H_{n-1}(\tilde{X}_0)$ such that

$\partial\Delta_1 = \delta_1, \quad \partial\Delta_3 = \delta_2, \quad \partial\Delta_2 = \partial\Delta_4 = \delta_1 + \delta_2$.

**Remark 2.10:** It will follow from the results in the next section, that for $n \equiv 3 \pmod{4}$ the monodromy group $G_0(f)$ (defined in the introduction) is a symmetric group for any $\mathcal{R}$-simple function $f$. More precisely

$G_0(B_m) = S_2, \quad G_0(C_m) = S_m, \quad G_0(F_4) = S_3$.

On the other hand, in these cases the monodromy groups $G(f)$ are all infinite (see 3.7 ii).

Therefore one cannot establish a simple connection between these monodromy groups and the Weyl groups associated to the root systems of type $B_m, C_m$ and $F_4$.

**Remark 2.11:** It is easy to see that the action of the path $w_0$ on $H_{n-1}(\tilde{X}_0)$ is precisely the dual of the monodromy operator in cohomology $h^*$ introduced in [5].
3. The monodromy groups $G_0(f)$ and $G(f)$

Let $(X_0, 0) \subset (Y, 0) \rightarrow (B, 0)$ be a versal deformation of the complete intersection $X_0$, with a smooth base space $B$ and let us denote by $\Delta \subset B$ the discriminant hypersurface of $F$ [3].

For a base point $b \in B \setminus \Delta$, the fundamental group $\pi_1(B \setminus \Delta, b)$ acts on the homology of the smooth fiber $F^{-1}(b) \sim \tilde{X}_0$ and we obtain in this way the monodromy group of $X_0$

$$G(X_0) = \text{Im}\{ \pi_1(B \setminus \Delta, b) \rightarrow \text{Aut} \, H_{n-1}(\tilde{X}_0) \}.$$

This group is independent of the choice of the versal deformation $F$ and of the base point $b$ (provided we take $B$ to be a small enough open ball in some $\mathbb{C}^N$).

Suppose we fix a morsification $f_q: X_\epsilon \rightarrow \mathbb{C}$ of the given function $f$ as in (1.4). Then there is a versal deformation $F$ of $X_0$ as above and a line $l$ in the base space $B$ such that after a natural identification $l = \mathbb{C}$ we have a commutative diagram

$$f_q^{-1}(D_b) \cong F^{-1}(D_b)$$

$$\downarrow f_q \quad \downarrow F$$

$$D_b$$

(3.1)

To obtain such a versal deformation $F$ it is enough to take a system of generators of the $\mathbb{C}$-vector space $\mathcal{O}_{X_0}^{\oplus p+1}/\partial G/\partial x_1 \cdot \mathcal{O}_{X_0} + \ldots + \partial G/\partial x_{n+p} \cdot \mathcal{O}_{X_0}$ (where $\partial G/\partial x_i = (\partial g_1/\partial x_i, \ldots, \partial g_p/\partial x_i, \partial f/\partial x_i)$) including the constant vectors $e_1, \ldots, e_{p+1}$ and the vector $(0, \ldots, 0, q)$.

The set $C$ of critical values of $f_q$ corresponds via (3.1) to the intersection $l \cap \Delta$ and since $f_q$ is a Morse function it follows that all the points $c_k \in l \cap \Delta$ are simple points on $\Delta$ and that the intersection $l \cap \Delta$ is transverse (situation denoted in the sequel by $l \pitchfork \Delta$). ([3], 1.3.i).

The number $s$ of intersection points in $l \cap \Delta$ is equal to the intersection multiplicity $(\Delta, l_0)_0$, where $l_0$ is the line through $0 \in B$ with the same direction as $l$ [10].

**Example 3.2:** For the simple function of type $B_m$ introduced in (2.7) one can take $F: (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}^2, 0)$

$$F(x) = (x_1^m + x_2^2 + \ldots + x_{n+1}^2, x_1)$$
Then the discriminant $\Delta$ is given by the equation $y_1 = y_2^m$ and the morsification $f_0 = x_1: X \to \mathbb{C}$ corresponds to the line $l: y_1 = r$. Hence in this case $s = m$, though $\Delta$ is smooth at 0. It follows that the direction $l_0$: $y_1 = 0$ is not generic with respect to the discriminant, as mentioned in the introduction.

The main result of this section is the following.

**PROPOSITION 3.3:**

$$G_0(f) = G(X_0).$$

**PROOF:** Suppose that $B$ is an open neighbourhood of 0 in $\mathbb{C}^N$ for some $N \geq 2$ and let $h = 0$ be the equation of the discriminant hypersurface $\Delta$ in $B$.

We denote here by $B_\rho$ the closed ball of radius $\rho$ centered at 0 in $\mathbb{C}^N$ and by $d_a$ the line determined by a direction $d \in \mathbb{P}(\mathbb{C}^N)$ and a point $a \in B$.

The results of Hamm-Lê [6] prove the existence of a Zariski open set $U \subset \mathbb{P}(\mathbb{C}^N)$ such that for any $d \in U$ there is a $\rho_0 = \rho(d) > 0$ with the property that for any $\rho$ with $0 < \rho \leq \rho_0$ there is a $\theta_\rho > 0$ such that the homomorphism

$$\pi_1((B_\rho \setminus \Delta) \cap d_a, b) \to \pi_1(B_\rho \setminus \Delta, b) \quad (3.4)$$

induced by the inclusion is an epimorphism for any point $a$ with $0 < |a| \leq \theta_\rho$ and $b \in (B_\rho \setminus \Delta) \cap d_a$.

We cannot apply this result to the line $l$ in our construction above, since $l$ is not in general position with respect to the discriminant $\Delta$ (3.2).

That is why we need the following.

**LEMMA 3.5:** Suppose that the direction $d \in \mathbb{P}(\mathbb{C}^N)$ is chosen such that $d_0 \subset \Delta$. Then there is $\rho$, $\delta > 0$ such that (3.4) is an epimorphism for any point $a$ with $|a| \leq \delta$ and $d_a \cap \Delta$.

**PROOF:** Let $\rho > 0$ be chosen such that

(i) $B_\rho \cap d_0 \cap \Delta = \{0\}$,

(ii) Inside the ball $B_\rho$ we have a conical topological structure for $\Delta$, i.e.

$$(B_\rho, \Delta \cap B_\rho) \simeq C(S_\rho, K)$$

where $S_\rho = \partial B_\rho$, $K = \Delta \cap S_\rho$ as in [11] (2.10).

There is a connected open neighbourhood $V$ of $d$ in $\mathbb{P}(\mathbb{C}^N)$ such that $d' \in \overline{V}$ implies $d'_0 \cap K = \emptyset$.

We choose $\delta > 0$ small enough, such that $d'_a \cap K = \emptyset$ for any $d' \in \overline{V}$ and any point $a$ with $|a| \leq \delta$. 
Take now a point $a$ with $|a| \leq \delta$ and $d_a \cap \Delta$. Using a linear parametrization $\gamma: (\mathbb{C}, 0) \rightarrow (d_a, a)$, we define the function $\varphi = h \gamma$.

Then $\varphi$ is defined on a neighbourhood of $0 \in \mathbb{C}$ which contains the disc $D = d_a \cap B_\rho$ (if $\rho$ and $\delta$ are chosen small enough) and $\varphi^{-1}(0) = \{x_1, \ldots, x_s\}$ where the roots $x_i$ are all in $D$ and have multiplicity one.

We choose now a direction $d' \in V \cap U$ such that

$$(d'_0, \Delta)_0 = m(\Delta)$$

where $m(\Delta)$ is the multiplicity of the discriminant $\Delta$ at the origin. An explicit formula for $m(\Delta)$ can be found in [3], [10] and it follows that $m(\Delta) \geq \mu(x_0)$ with equality iff $x_0$ is a hypersurface singularity.

Note that a path connecting $d$ with $d'$ within $V$ gives rise to a homotopy $q_t: D \rightarrow \mathbb{C}, 0 \leq t \leq 1$ of $q = q_0$ with $q_1$, the function defined as above with respect to $d'_0$.

Since the direction $d'$ is in $U$, there is a $\rho' > 0$ and a $\theta' > 0$ such that, for any $a'$ with $0 < |a'| \leq \theta'$, the corresponding homomorphism (3.4) is an epimorphism.

Choose a path $a(t) 1 \leq t \leq 2$ in $B_\delta$ such that $a(1) = a$, $a(2) = a'$ with $0 < |a'| \leq \theta'$ and $d_a(t) \cap \Delta$ for any $t$. This gives rise as above to a homotopy $q_t: D \rightarrow \mathbb{C}, 1 \leq t \leq 2$. Since all the functions $q_t$ have only simple roots $x_k(t)$ in $\text{Int} D$, we obtain in this way $s$ paths $x_1(t), \ldots, x_s(t)$ for $0 \leq t \leq 2$.

We choose the order on the paths such that $x_1(2), \ldots, x_m(2)$ are precisely the end points within the disc $B_\rho \cap d'_a \subset D$ where $m = m(\Delta)$ (Note the identification $D = d_a \cap B_\rho$ for any $t$).

Consider the following commutative diagram.

The isomorphism $c_*$ is induced by a path in $B_\rho \setminus \Delta$ from $b$ to $b'$ and $\tilde{\varphi}$ is obtained via the homotopy $q_\gamma$.

If we denote by $w_k$ (resp. $w'_k$) the elementary path in $D \setminus \{x_1(t), \ldots, x_s(t)\}$ encircling the point $x_k(t)$ for $t = 0$ (resp. $t = 2$), then the left hand side of the diagram corresponds to

$$F(w'_1, \ldots, w'_m) \sim F(w'_1, \ldots, w'_s) \sim F(w_1, \ldots, w_s)$$
where $F(a_1, \ldots, a_p)$ denotes the free group generated by $a_1, \ldots, a_p$.

This ends the proof of (3.5) and hence of (3.3).

**Corollary 3.6:** Suppose $X_0$ is a hypersurface singularity and let $m = m(\Delta) = \mu(X_0)$. Then in the monodromy exact sequence (2.3) of the function $f$ (up to a change of indexes) the vanishing cycles $\delta_k = \partial \Delta_k$ ($k = 1, \ldots, m$) form a basis of $H_{n-1}(\tilde{X}_0)$ and the Picard-Lefschetz transformations associated to the elementary paths $w_k$ ($k = 1, \ldots, m$) generate the group $G_0(f)$.

**Proof:** The proof of (3.5) implies that (up to a change of indexes) the images of $w_1, \ldots, w_m$ generate the group $G_0(f) = G(X_0)$.

The monodromy group $G(X_0)$ acts transitively on the set of vanishing cycles in $H_{n-1}(\tilde{X}_0)$ [4], (2.58).

Hence for any such cycle $\delta$ there is an element $g \in G_0(f)$ such that $\delta = \pm g \cdot \delta_1$.

Since $g$ is a product of Picard-Lefschetz transformations associated to $w_1, \ldots, w_m$, it follows that

$$\delta \in \mathbb{Z} \langle \delta_1, \ldots, \delta_m \rangle$$

i.e. $\delta_1, \ldots, \delta_m$ form a basis of $H_{n-1}(\tilde{X}_0)$.

Finally we give some information about the other monodromy group of $f$, namely $G(f)$.

**Proposition 3.7:**

(i) There is an exact sequence of groups

$$0 \rightarrow \mathbb{Z}^\alpha \rightarrow G(f) \rightarrow G_0(f) \rightarrow 1$$

for some $\alpha \in \mathbb{N}$ with $0 \leq \alpha \leq \mu(X) \cdot \mu(X_0)$.

(ii) Suppose that $X_0$ is a hypersurface singularity and the intersection form on $H_{n-1}(\tilde{X}_0)$ is nondegenerate.

Then $\alpha \geq \mu(X)$.

If moreover the action of $G_0(f)$ on $H_{n-1}(\tilde{X}_0) \otimes \mathbb{C}$ is irreducible, then $\alpha = \mu(X) \cdot \mu(X_0)$.

**Proof:** Put $m = \mu(X_0)$, $m' = \mu(X)$ and $s = m + m'$. Assume that $\{ \Delta_i \}$ is a basis of $H_n(\tilde{X}_0)$ (made of thimbles only in the proof of (ii)!\) such that $\delta_k = \partial \Delta_k$ for $k = 1, \ldots, m$ form a basis for $H_{n-1}(\tilde{X}_0)$.

Then for any $k > m$ there is a combination

$$v_k = \Delta_k + \sum_{i=1}^{m} a_k, \Delta_i$$

such that $\partial v_k = 0$. 


In the basis $v_{m+1}, \ldots, v_s, \Delta_1, \ldots, \Delta_m$ the action of $w_k$ on $H_n(\bar{X}, \bar{X}_0)$ is given by a matrix

$$T_k = \begin{pmatrix} 1 & A_k \\ 0 & B_k \end{pmatrix}$$

We define an epimorphism $\rho: G(f) \to G_0(f)$ by associating to an $s \times s$ matrix as above the $m \times m$ matrix in the lower right corner. We get thus an exact sequence

$$1 \to \ker \rho \to G(f) \overset{\rho}{\to} G_0(f) \to 1$$

where $\ker \rho$ is a subgroup in the (abelian!) multiplicative group of all the matrices

$$M = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}$$

It follows that $\ker \rho \subset \mathbb{Z}^{m \cdot m'}$ and this gives us (i). To prove (ii) we assume the basis $\delta_k$ chosen as in (3.6). Note that the matrix $A_k$ defined above is zero for $k \leq m$ and has a single nonzero row (that corresponding to the vector $v_k$) for $m < k \leq s$ if the intersection form is nondegenerate. This proves the first part of (ii).

Moreover, note that if

$$\begin{pmatrix} 1 \\ 0 \\ \vdots \\ u \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \ker \rho$$

for some row vector $u \neq 0$, then the same is true for the vector $u \cdot B$ for any $B \in G_0(f)$.

If the action of $G_0(f)$ on the homology group $H_{n-1}(\bar{X}_0; \mathbb{C})$ is irreducible, then it follows that

$$\dim \mathbb{C}<u \cdot B; B \in G_0(f)> = m$$

Hence $\ker \rho$ contains in this case $m \cdot m'$ $\mathbb{C}$-linearly independent vectors and this implies the result in the second part of (ii).

**Remarks 3.8:**

a. The condition about the intersection form in (3.7.ii) is necessary. For instance, if $f$ is a simple function of type $B_k$ and $n$ is even, it follows from (2.7) that $G_0(f) = G(f) = 0$. 

\[ \square \]
On the other hand, note that both assumptions in (3.7.ii) hold when $X_0$ is one of Arnold simple hypersurface singularities $A_n$, $D_n$, $E_6$, $E_7$ or $E_8$ and $n \equiv 3 \pmod{4}$ ([12], §8).

b. In general the subgroup $\ker \rho \subset \mathbb{Z}^{mm'}$ is not the whole group, even when they have the same rank.

For instance, for a function of type $B_k$ and $n$ odd, $\ker \rho = 2 \cdot \mathbb{Z}^{k-1} \subset \mathbb{Z}^{k-1}$.

References


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