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# MONODROMY OF FUNCTIONS DEFINED ON ISOLATED SINGULARITIES OF COMPLETE INTERSECTIONS

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A basic tool in the study of an analytic function germ $f$: $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at the origin (or of the corresponding hypersurface germ $Y = f^{-1}(0)$) is the wellknown local monodromy group ([4], [8], [12]).

This widely studied monodromy group can be defined in two equivalent ways:

(i) Using a morsification of the function $f$.

(ii) Using a line in the base space $B$ of a versal deformation for $Y$, in general position with respect to the discriminant hypersurfaces $\Delta \subset B$.

In this paper we extend the construction (i) above to function germs $f$: $(X, 0) \to (\mathbb{C}, 0)$ defined on a complete intersection $(X, 0) \subset (\mathbb{C}^{n+p}, 0)$ with an isolated singular point at the origin and such that $X_0 = f^{-1}(0)$ is also a complete intersection with an isolated singularity at 0 (here $n = \dim X > 0$).

In this way we obtain an action of a fundamental group $\pi = \pi_1(\text{disc} \setminus \{\text{s points}\})$ on the exact sequence of the pair $(\tilde{X}, \tilde{X}_0)$ in homology (with $Z$-coefficients):

$$0 \to H_n(\tilde{X}) \to H_n(\tilde{X}, \tilde{X}_0) \overset{\partial}{\to} H_{n-1}(\tilde{X}_0) \to 0 \qquad (*)$$

where $\tilde{X}$, $\tilde{X}_0$ are the Milnor fibers of $X$ and $X_0$ ([5]) chosen such that $\tilde{X}_0 \subset \tilde{X}$ and $s = \mu(X) + \mu(X_0)$ is the sum of their Milnor numbers.

More precisely, the action of $\pi$ on $H_n(\tilde{X})$ is trivial, while the actions on the other two homology groups can be described in terms of Picard-Lefschetz formulas with respect to thimbles $\Delta_k \in H_n(\tilde{X}, \tilde{X}_0)$ and corresponding vanishing cycles $\delta_k = \partial\Delta_k \in H_{n-1}(\tilde{X}_0)$.

The $\pi$-exact sequence $(*)$ is proved to be a contact invariant of the function $f$ i.e. it depends only on the isomorphism class (in a natural sense) of the pair of complete intersections $(X, X_0)$. This fact, as well as the independence of the sequence $(*)$ on the choice of the morsification for $f$ is obtained by a simple application of the Thom-Mather Second Isotopy Lemma.

To give some explicit examples, we compute next the $\pi$-sequence $(*)$ for all the $\mathscr{R}$-simple functions $f$ defined on an isolated hypersurface singularity $X$ of dimension $n > 1$, as listed in [1].

Note that the $\pi$-sequence $(*)$ gives us in particular two monodromy groups

$$G_0(f) = \operatorname{Im}\{\pi \to \operatorname{Aut} H_{n-1}(\tilde{X}_0)\}$$

$$G(f) = \operatorname{Im}\{\pi \to \operatorname{Aut} H_n(\tilde{X}, \tilde{X}_0)\}.$$

We prove that $G_0(f)$ is precisely the monodromy group of the complete intersection $X_0$ defined as in (ii). In fact the morsification process used above gives rise to a line in the base space $B$ of a (suitable chosen) versal deformation of $X_0$, whose direction depends on the function $f$ and is *not* generic with respect to the discriminant $\Delta \subset B$.

That is why we need a slightly modified version of a result of Hamm-Lê on the fundamental group $\pi_1(B \setminus \Delta)$ (see Lemma 3.5).

Then we show that the other monodromy group $G(f)$ is a semidirect product of $G_0(f)$ with a free abelian group $\mathbb{Z}^\alpha$ and we also give some estimates for the rank $\alpha$.

Finally we remark that constructions similar to some of ours (i.e. morsifications and connections with versal deformations) have been used many a time before (e.g. by Iomdin [7] and Lê [10]) but always with different aims in view, as far as we known.

We would like to express our deep gratitude to Professor V.I. Arnold for a very stimulating discussion.

## §1. Morsifications and monodromy map of pairs

Let $X$: $g_1 = \ldots = g_p = 0$ be an analytic complete intersection in a neighbourhood of the origin of $\mathbb{C}^{n+p}$, with an isolated singular point at 0. ($n \geqslant 1, p \geqslant 0$). Consider also an analytic function germ

$$f\colon (\mathbb{C}^{n+p}, 0) \to (\mathbb{C}, 0)$$

such that $X_0 = f^{-1}(0) \cap X$ is again a complete intersection with an isolated singularity at 0.

For $\epsilon \gg \delta > 0$ chosen sufficiently small, it is known that the Milnor fiber of $X$

$$X_r = \{x \in B_\epsilon;\ g(x) = r\}$$

is a compact $C^\infty$-manifold with boundary for any $r \in \mathbb{C}^p$ sufficiently general with $0 < |r| \leqslant \delta$, where

$$B_\epsilon = \{x \in \mathbb{C}^{n+p};\ |x| \leqslant \epsilon\}. \qquad [5]$$

The space $X_r$ (denoted in the introduction by $\tilde{X}$) has the homotopy type of a bouquet of $n$-spheres, the number of which is by definition the Milnor number $\mu(X)$ of the complete intersection $X$.

For $r$ small enough, it is easy to see that $f' = f|\text{Int } X_r$ has only a finite number of critical points $a_1, \dots, a_k$ and moreover $a_i \to 0$ when $r \to 0$ for any $i = 1, \dots, k$.

Let us denote by $\mu(f', a_i)$ the Milnor number of the function $f'$ at the critical point $a_i$.

One has the following property, in analogy with a result of Lê ([10], (3.6.4)).

PROPOSITION 1.1:

$$\sum_{i=1,k} \mu(f', a_i) = \mu(X) + \mu(X_0).$$

PROOF: Let $D_\delta$ denote the open disc $\{z \in \mathbb{C};\ |z| < \delta\}$. For $\epsilon$, $\delta$ and $r$ suitable chosen, the inclusion

$$E = X_r \cap f^{-1}(D_\delta) \hookrightarrow X_r \tag{1.2}$$

is a homotopy equivalence (see for instance [10] (3.5)) and moreover the restriction

$$f|_{\partial E}\colon \partial E \to D_\delta \qquad \text{where} \quad \partial E = \partial X_r \cap f^{-1}(D_\delta) \tag{1.3}$$

is a submersion.

Let $b \in D_\delta$ be a regular value of $\tilde{f} = f|E$ and let $c_i = f(a_i) \in D_\delta$ be the (not necessarily distinct) critical values of $\tilde{f}$.

Then $F = \tilde{f}^{-1}(b)$ is the Milnor fiber of the complete intersection $X_0$ and the exact sequence of the pair $(E, F)$ shows that $H_n(E, F)$ is a free abelian group of rank $s = \mu(X) + \mu(X_0)$. ($\mathbb{Z}$-coefficients for homology are used throughout in this paper).

We compute now this group in a different way, following ([9], §5). Choose small disjoint closed discs $D_i$ centered at the critical values $c_i$ and fix some points $b_i \in \partial D_i$.

For each $i$, take a $C^\infty$-embedded interval $l_i$ from $b$ to $b_i$ such that $l = \cup l_i$ can be contracted within itself to $b$ and $D_\delta$ can be contracted to $C = \cup D_i \cup l$.

Since $\tilde{f}$ induces a (proper) locally trivial fibration

$$E \backslash \tilde{f}^{-1}\{c_i\}_i \to D_\delta \backslash \{c_i\}_i$$

these retractions can be lifted to the corresponding subsets of $E$ and we get the following isomorphisms

$$H_n(E,F) \xleftarrow{\sim} H_n(\tilde{f}^{-1}(C),F) \xrightarrow{\sim} H_n(\tilde{f}^{-1}(C),\tilde{f}^{-1}(l)).$$

By excision, the last group is equal to

$$\bigoplus_i H_n(\tilde{f}^{-1}(D_i),\tilde{f}^{-1}(b_i))$$

Assume that $a_{i1},\ldots,a_{im}$ are the critical points of $\tilde{f}$ in the fiber over $c_i$. Let $B_j$ be the intersection of a small closed ball centered at $a_{ij}$ with $\tilde{f}^{-1}(D_i)$ and denote with $F_i$ the fiber $\tilde{f}^{-1}(b_i)$.

It follows that

$$H_n(\tilde{f}^{-1}(D_i), F_i) \simeq H_n\left(\bigcup_{j=1}^{m} B_j \cup F_i, F_i\right) \simeq \bigoplus_{j=1}^{m} H_n(B_j, B_j \cap F_i).$$

Moreover

$$H_n(B_j, B_j \cap F_i) \underset{\sim}{\overset{\partial}{\rightarrow}} H_{n-1}(B_j \cap F_i)$$

is a free abelian group of rank $\mu(f', a_{ij})$ by the definition of the Milnor numbers of $f'$, if the discs $D_i$ and the balls $B_j$ are chosen small enough. □

We consider now the problem of the existence of morsifications of the function $f'\colon X_r \to \mathbf{C}$, i.e. small deformations of $f'$ having only nondegenerate critical points with distinct critical values.

If $P$ denotes the vector space of polynomials in $x_1,\ldots,x_{n+p}$ of degree $\leqslant 3$, it is easy to show by standard transversality arguments that there is a Zariski open subset $U \subset P$ such that the function

$$f_q = (f+q)|X_r$$

is a Morse function for any $q \in U$.

Moreover, if we have chosen already $\epsilon \gg \delta > 0$ such that (1.2) and (1.3) hold true for any generic $r \in \mathbf{C}^p$ with $|r| \leqslant \delta$, then there is an $\eta > 0$ such that $|q| < \eta$ implies similar properties for $f_q$.

Suppose now we have two polynomials $q_0$, $q_1 \in U$ such that $|q_i| < \eta$. We can find a $C^\infty$-path $q_t$ in $U$ such that $q_t = q_0$ for $0 \leqslant t \leqslant a$, $q_t = q_1$ for $1-a \leqslant t \leqslant 1$ and $|q_t| < \eta$ for any $t \in [0,1]$, where $a \in (0, 1/3)$.

Consider the spaces

$$\tilde{D} = D_\delta \times (0,1) \quad \text{and} \quad \tilde{E} = \{(x,t) \in X_r \times (0,1); f_{q_t}(x) \in D_\delta\}$$

and the proper map

$$\varphi\colon \tilde{E} \to \tilde{D}, \quad \varphi(x, t) = \left(f_{q_t}(x), t\right).$$

If $a_i(t)$ (resp. $c_i(t)$) denote the critical points (resp. critical values) of $f_{q_t}$ for $i = 1,\dots,s = \mu(X) + \mu(X_0)$, then we can stratify the map $\varphi$ as follows ([2], Chap. I). The strata in $\tilde{D}$ are given by

$$\tilde{D}_1 = \left\{(c_i(t), t);\ t \in (0, 1), i = 1,\dots,s\right\} \quad \text{and} \quad \tilde{D}_3 = \tilde{D} \setminus \tilde{D}_1.$$

The strata in $\tilde{E}$ are given by

$$\tilde{E}_1 = \left\{(a_i(t), t);\ t \in (0, 1), i = 1,\dots,s\right\}$$

$$\tilde{E}_{2n-2} = \left\{(x, t);\ t \in (0, 1), x \in \left(f_{q_t}\right)^{-1}(c_i(t)) \cap \partial X_r, i = 1,\dots,s\right\}$$

$$\tilde{E}_{2n-1} = \left\{(x, t);\ t \in (0, 1), x \in \left(f_{q_t}\right)^{-1}(c_i(t)) \cap \operatorname{Int} X_r, i = 1,\dots,s\right\}$$

$$\tilde{E}_{2n} = (\partial X_r \times (0, 1)) \cap (\tilde{E} \setminus \tilde{E}_{2n-2})$$

$\tilde{E}_{2n+1} = \tilde{E} \setminus$ the union of the other strata $\tilde{E}_k$ defined above.

The lower index gives the real dimension of the stratum. (These definitions work for $n \geqslant 2$. The simpler case $n = 1$ is left to the reader.)

The Whitney-Thom regularity conditions are obviously satisfied for any pair of strata.

By Thom-Mather Second Isotopy Lemma ([2], II, (5.8)) we obtain a commutative diagram

$$\begin{array}{ccc}
\varphi^{-1}(D_\delta \times \alpha) & \xrightarrow{H} & \varphi^{-1}(D_\delta \times (1 - \alpha)) \\
\downarrow f_{q_0} & & \downarrow f_{q_1} \\
D_\delta \times \alpha & \xrightarrow{h} & D_\delta \times (1 - \alpha)
\end{array}$$

where $\alpha \in (0, a)$ and $H$, $h$ are homomorphisms compatible with the induced stratifications.

In particular we get the following result.

LEMMA 1.4: *The topological type of the map of pairs*

$$f_q\colon \left(f_q^{-1}(D_\delta), f_q^{-1}(C)\right) \to (D_\delta, C)$$

*where $C$ is the set of critical values of the function $f_q$ is independent of the polynomial $q \in U$, $|q| < \eta$.*

It is also clear the independence of the topological type of the map above of the choice of (suitable) $\epsilon$, $\delta$ and $r$. Moreover, if we change the function $f$ to a function $f_1 = f + k$, where $k$ is a function in the ideal $(g_1, \dots, g_p)$ of the complete intersection $X$, note that the distance $\|f_1 - f\|_{X_r}$ can be made as small as we want by taking $r$ small enough.

Using a stratification argument as above it follows that the topological type of the map of pairs in (1.4) depends only on the restriction $f|X$ i.e. on a function in $m_X = m/(g_1, \dots, g_p)$, where $m \subset \mathcal{O}_{n+p}$ is the maximal ideal.

(We shall consider throughout in this paper only functions $f \in m_X$ such that $X_0 = f^{-1}(0)$ is a complete intersection with an isolated singularity at 0).

The discussion below will also imply independence from the defining equations $g_i = 0$ of $X$, and hence we can give the following.

DEFINITION 1.5: The topological type of the map of pairs in (1.4) will be called the *monodromy map of pairs* of the function $f \in m_X$ and will be denoted simply by

$$f^*: (E^*, E_c^*) \to (D, C).$$

This topological object is constant in $\mu$-constant families in the following precise sense (compare to [12], §9).

Let $(X_t, 0) \subset (\mathbb{C}^{n+p}, 0)$ be a smooth family of complete intersections with isolated singular points at the origin such that $\dim X_t = n$ and $\mu(X_t) = $ const. for $t \in [0, 1]$. Assume that $f_t \in m_{X_t}$ is a smooth family of function germs such that $\mu(f_t^{-1}(0)) = $ const.

Using the construction of morsifications and stratification arguments as above, one can then show that the monodromy map of pairs of the function $f_t$ is independent of $t$.

A special case of this situation is the following.

DEFINITION 1.6 [1]: We say that two function germs $f_1, f_2 \in m_X$ defined on the complete intersection $(X, 0)$ are *$\mathcal{K}$-(contact)-equivalent* if there is an automorphism $u$ of the local $\mathbb{C}$-algebra $\mathcal{O}_X$ such that $(u(f_1)) = (f_2)$, where $(a)$ means the ideal generated by $a$ in $\mathcal{O}_X$.

Since the complete intersections $X$ and $X_{0i} = f_i^{-1}(0)$ $i = 1, 2$ have isolated singularities at the origin, the question of $\mathcal{K}$-equivalence of $f_1$ and $f_2$ can be settled in a jet space $J^k(n+p, p+1)$, via the action of a *connected* algebraic group $G_{\mathcal{K}}^k$ (the particular case when $X$ is a hypersurface is treated in detail in [1]).

It follows that $(X, f_1)$ and $(X, f_2)$ can be connected by a $\mu$-constant family $(X_t, f_t)$ as above and we get thus the following.

COROLLARY 1.7: *If two function germs* $f_1, f_2 \in m_X$ *are* $\mathcal{K}$*-equivalent then their associated monodromy maps* $f_1^*$ *and* $f_2^*$ *are the same.*

## §2. Monodromy exact sequence. Examples

Let $f^*: (E^*, E_c^*) \to (D, C)$ be the monodromy map of pairs of a function $f \in m_X$ as in §1.

If $b \in D \setminus C$ and $F = (f^*)^{-1}(b)$, then the locally trivial fibration $E^* \setminus E_c^* \to D \setminus C$ defines in the usual way an action of the fundamental group $\pi = \pi_1(D \setminus C)$ on the middle homology group $H_{n-1}(F)$ of the fiber.

Moreover, for any homotopy class $w \in \pi$ there is a well defined homomorphism

$$\tau_w: H_{n-1}(F) \to H_n(E^*, F)$$

called *the extension along the path w*. For a detailed construction and the main properties of $\tau_w$ we send to ([9], (6.4)).

We can define an action of the fundamental group $\pi$ on the homology group $H_n(E^*, F)$ by the formula

$$w \cdot x = x + (-1)^{n-1} \tau_w(\partial x) \tag{2.1}$$

where $\partial$ is the connecting homomorphism in the exact sequence of the pair $(E^*, F)$

$$0 \to H_n(E^*) \xrightarrow{i} H_n(E^*, F) \xrightarrow{\partial} H_{n-1}(F) \to 0. \tag{2.2}$$

If we consider the trivial action of $\pi$ on $H_n(E^*)$, then this exact sequence is a *$\pi$-exact sequence*, i.e. the homomorphisms $i$ and $\partial$ are $\pi$-equivariant.

Let $\tilde{X}$ (say equal to $X_r$ in §1) and $\tilde{X}_0$ (say equal to $X_r \cap f^{-1}(b)$) denote the associated Milnor fibers of the complete intersections $X$ and $X_0$.

The corresponding exact sequence

$$0 \to H_n(\tilde{X}) \to H_n(\tilde{X}, \tilde{X}_0) \xrightarrow{\partial} H_{n-1}(\tilde{X}_0) \to 0 \tag{2.3}$$

is isomorphic to the exact sequence (2.2) and via this isomorphism we can transfer the $\pi$-actions on the homology groups in (2.3).

DEFINITION 2.4: The $\pi$-exact sequence (2.3) constructed as above is called *the monodromy exact sequence* of the function $f$.

EXAMPLE 2.5: If the complete intersection $X$ is smooth, then the sequence (2.3) becomes

$$0 \to 0 \to H_n(\tilde{X}, \tilde{X}_0) \underset{\sim}{\overset{\partial}{\to}} H_{n-1}(\tilde{X}_0) \to 0$$

and hence it contains the same information as the action of $\pi$ on $H_{n-1}(\tilde{X}_0)$ i.e. the classical monodromy action for the hypersurface $X_0$. □

Put again $s = \mu(X) + \mu(X_0) = rkH_n(\tilde{X}, \tilde{X}_0)$ and let $C = \{c_1, \dots, c_s\}$. We denote by $w_k \in \pi$ the elementary path encircling $c_k$ ([9] (6.1)) and chose the order of these paths such that

$$w_s \cdot \ldots \cdot w_1 = w_0$$

where $w_0$ is the class of the path $w_0(t) = b \cdot e^{2\pi i t}$, $0 \leqslant t \leqslant 1$ (we assume here $|b| > |c_k|$ for any $k = 1, \dots, s$).

We recall from the proof of (1.1) the isomorphisms

$$H_n(\tilde{X}, \tilde{X}_0) \simeq H_n(E^*, F) \simeq \bigoplus_{k=1}^{s} H_n\big((f^*)^{-1}(D_k), (f^*)^{-1}(b_k)\big)$$

Since $f^*$ is a morsification, each of the last homology groups is free abelian of rank one.

We shall denote by $\Delta_1, \dots, \Delta_s$ the corresponding generators of the group $H_n(\tilde{X}, \tilde{X}_0)$, which are precisely the *thimbles* of Lefschetz ([9] (6.2)).

With these notations, the $\pi$-actions in the exact sequence (2.3) can be described in terms of Picard-Lefschetz formulas.

LEMMA 2.6:

$$\textit{For} \quad x \in H_n(\tilde{X}, \tilde{X}_0) \colon w_k \cdot x = x + (-1)^{n(n+1)/2} (\partial x, \partial \Delta_k) \Delta_k$$

$$\textit{For} \quad x \in H_{n-1}(\tilde{X}_0) \colon w_k \cdot x = x + (-1)^{n(n+1)/2} (x, \partial \Delta_k) \partial \Delta_k$$

*where* ( , ) *denotes the intersection form on* $H_{n-1}(\tilde{X}_0)$ *and* $k = 1, \dots, s$.

PROOF. The second formula is the usual Picard-Lefschetz formula (see for instance ([8], §5)). The first one follows from (2.1) and the formula for $\tau_w$ given in ([9], (6.7.1)). □

It follows that in order to determine the monodromy exact sequence it is enough to fix a basis $\{\delta_k\}$ of the group $H_{n-1}(\tilde{X}_0)$ and to compute with respect to it the *vanishing cycles* $\partial \Delta_t$ and the intersection form.

As examples of this method, we give the description of the monodromy exact sequences of the $\mathscr{R}$-simple functions defined on an isolated hypersurface singularity $X$ with $\dim X > 1$ which were classified in ([1], §3).

In all these cases $X_0$ is an isolated hypersurface singularity of type $A_k$ for some $k$ and we can chose a distinguished basis of vanishing cycles $\{\delta_i\}$ for $H_{n-1}(\tilde{X}_0)$ corresponding to a Dynkin diagram of type $A_k$ ([4], (2.4)).

Moreover, using the stabilization of singularities (i.e. addition of a sum of squares to the given equation of $X_0$ as described in [4] (2.3)), we can assume $n = 1$ when we compute $\partial\Delta_i$.

The results are given below, without these tedious computations.

PROPOSITION 2.7: *For the simple function of type $B_m$ ($m \geqslant 2$) given by $X$: $x_1^m + x_2^2 + \ldots + x_{n+1}^2 = 0$ and $f = x_1$ there is a basis of thimbles $\Delta_1, \ldots, \Delta_m$ of $H_n(\tilde{X}, \tilde{X}_0)$ and a vanishing cycle $\delta$ which generates $H_{n-1}(\tilde{X}_0)$ such that $\partial\Delta_k = \delta$ for any $k = 1, \ldots, m$.*

PROPOSITION 2.8: *For the simple function of type $C_{m+1}$ ($m \geqslant 1$) given by $X$: $x_1x_2 + x_3^2 + \ldots + x_{n+1}^2 = 0$ and $f = x_1 + x_2^m$ there is a basis of thimbles $\Delta_0, \ldots, \Delta_m$ of $H_n(\tilde{X}, \tilde{X}_0)$ and a basis of vanishing cycles $\delta_1, \ldots, \delta_m$ of $H_{n-1}(\tilde{X}_0)$ such that $\partial\Delta_0 = \delta_1 + \ldots + \delta_m$ and $\partial\Delta_k = \delta_k$ for any $k = 1, \ldots, m$. (Note that $C_2 \equiv B_2$).*

PROPOSITION 2.9: *For the simple function of type $F_4$ given by $X$: $x_1^3 + x_2^2 + \ldots + x_{n+1}^2 = 0$ and $f = x_2$ there is a basis of thimbles $\Delta_1, \ldots, \Delta_4$ of $H_n(\tilde{X}, \tilde{X}_0)$ and a basis of vanishing cycles $\delta_1$, $\delta_2$ of $H_{n-1}(\tilde{X}_0)$ such that*

$$\partial\Delta_1 = \delta_1, \quad \partial\Delta_3 = \delta_2, \quad \partial\Delta_2 = \partial\Delta_4 = \delta_1 + \delta_2.$$

REMARK 2.10: It will follow from the results in the next section, that for $n \equiv 3 \pmod 4$ the monodromy group $G_0(f)$ (defined in the introduction) is a symmetric group for any $\mathscr{R}$-simple function $f$. More precisely

$$G_0(B_m) = S_2, \quad G_0(C_m) = S_m, \quad G_0(F_4) = S_3.$$

On the other hand, in these cases the monodromy groups $G(f)$ are all infinite (see 3.7 ii).

Therefore one cannot establish a simple connection between these monodromy groups and the Weyl groups associated to the root systems of type $B_m$, $C_m$ and $F_4$.

REMARK 2.11: It is easy to see that the action of the path $w_0$ on $H_{n-1}(\tilde{X}_0)$ is precisely the dual of the monodromy operator in cohomology $h^*$ introduced in [5].

## 3. The monodromy groups $G_0(f)$ and $G(f)$

Let $(X_0, 0) \subset (Y, 0) \xrightarrow{F} (B, 0)$ be a versal deformation of the complete intersection $X_0$, with a smooth base space $B$ and let us denote by $\Delta \subset B$ the discriminant hypersurface of $F$ [3].

For a base point $b \in B \backslash \Delta$, the fundamental group $\pi_1(B \backslash \Delta, b)$ acts on the homology of the smooth fiber $F^{-1}(b) \sim \tilde{X}_0$ and we obtain in this way the *monodromy group of* $X_0$

$$G(X_0) = \mathrm{Im}\{\pi_1(B \backslash \Delta, b) \to \mathrm{Aut}\, H_{n-1}(\tilde{X}_0)\}.$$

This group is independent of the choice of the versal deformation $F$ and of the base point $b$ (provided we take $B$ to be a small enough open ball in some $\mathbb{C}^N$).

Suppose we fix a morsification $f_q$: $X_r \to \mathbb{C}$ of the given function $f$ as in (1.4). Then there is a versal deformation $F$ of $X_0$ as above and a line $l$ in the base space $B$ such that after a natural identification $l \simeq \mathbb{C}$ we have a commutative diagram

$$\begin{array}{ccc} & f_q^{-1}(D_\delta) \simeq F^{-1}(D_\delta) & \\ {}_{f_q}\searrow & & \swarrow_{F} \\ & D_\delta & \end{array} \tag{3.1}$$

To obtain such a versal deformation $F$ it is enough to take a system of generators of the $\mathbb{C}$-vector space $\mathcal{O}_{X_0}^{p+1}/\partial G/\partial x_1 \cdot \mathcal{O}_{X0} + \dots + \partial G/\partial x_{n+p} \cdot \mathcal{O}_{X_0}$ (where $\partial G/\partial x_i = (\partial g_1/\partial x_i, \dots, \partial g_p/\partial x_i, \partial f/\partial x_i)$) including the constant vectors $e_1, \dots, e_{p+1}$ and the vector $(0, \dots, 0, q)$.

The set $C$ of critical values of $f_q$ corresponds via (3.1) to the intersection $l \cap \Delta$ and since $f_q$ is a Morse function it follows that all the points $c_k \in l \cap \Delta$ are simple points on $\Delta$ and that the intersection $l \cap \Delta$ is transverse (situation denoted in the sequel by $l \pitchfork \Delta$). ([3], 1.3.i).

The number $s$ of intersection points in $l \cap \Delta$ is equal to the intersection multiplicity $(\Delta, l_0)_0$, where $l_0$ is the line through $0 \in B$ with the same direction as $l$ [10].

Example 3.2: For the simple function of type $B_m$ introduced in (2.7) one can take $F$: $(\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^2, 0)$

$$F(x) = \left(x_1^m + x_2^2 + \dots + x_{n+1}^2, x_1\right)$$

Then the discriminant $\Delta$ is given by the equation $y_1 = y_2^m$ and the morsification $f_0 = x_1$: $X_r \to \mathbb{C}$ corresponds to the line $l$: $y_1 = r$. Hence in this case $s = m$, though $\Delta$ is smooth at 0. It follows that the direction $l_0$: $y_1 = 0$ is not generic with respect to the discriminant, as mentioned in the introduction. □

The main result of this section is the following.

PROPOSITION 3.3:

$$G_0(f) = G(X_0).$$

PROOF: Suppose that $B$ is an open neighbourhood of 0 in $\mathbb{C}^N$ for some $N \geqslant 2$ and let $h = 0$ be the equation of the discriminant hypersurface $\Delta$ in $B$.

We denote here by $B_\rho$ the closed ball of radius $\rho$ centered at 0 in $\mathbb{C}^N$ and by $d_a$ the line determined by a direction $d \in P(\mathbb{C}^N)$ and a point $a \in B$.

The results of Hamm-Lê [6] prove the existence of a Zariski open set $U \subset P(\mathbb{C}^N)$ such that for any $d \in U$ there is a $\rho_0 = \rho(d) > 0$ with the property that for any $\rho$ with $0 < \rho \leqslant \rho_0$ there is a $\theta_\rho > 0$ such that the homomorphism

$$\pi_1((B_\rho \setminus \Delta) \cap d_a, b) \to \pi_1(B_\rho \setminus \Delta, b) \tag{3.4}$$

induced by the inclusion is an epimorphism for any point $a$ with $0 < |a| \leqslant \theta_\rho$ and $b \in (B_\rho \setminus \Delta) \cap d_a$.

We cannot apply this result to the line $l$ in our construction above, since $l$ is not in general position with respect to the discriminant $\Delta$ (3.2).

That is why we need the following.

LEMMA 3.5: *Suppose that the direction $d \in P(\mathbb{C}^N)$ is chosen such that $d_0 \not\subset \Delta$. Then there is $\rho$, $\delta > 0$ such that* (3.4) *is an epimorphism for any point $a$ with $|a| \leqslant \delta$ and $d_a \pitchfork \Delta$.*

PROOF: Let $\rho > 0$ be chosen such that

(i) $B_\rho \cap d_0 \cap \Delta = \{0\}$.

(ii) Inside the ball $B_\rho$ we have a conical topological structure for $\Delta$, i.e.

$$(B_\rho, \Delta \cap B_\rho) \simeq C(S_\rho, K)$$

where $S_\rho = \partial B_\rho$, $K = \Delta \cap S_\rho$ as in [11] (2.10).

There is a connected open neighbourhood $V$ of $d$ in $P(\mathbb{C}^N)$ such that $d' \in \overline{V}$ implies $d'_0 \cap K = \emptyset$.

We choose $\delta > 0$ small enough, such that $d'_a \cap K = \emptyset$ for any $d' \in \overline{V}$ and any point $a$ with $|a| \leqslant \delta$.

Take now a point $a$ with $|a| \leqslant \delta$ and $d_a \pitchfork \Delta$. Using a linear parametrization $\gamma$: $(\mathbb{C}, 0) \to (d_a, a)$, we define the function $\varphi = h\gamma$.

Then $\varphi$ is defined on a neighbourhood of $0 \in \mathbb{C}$ which contains the disc $D = d_a \cap B_\rho$ (if $\rho$ and $\delta$ are chosen small enough) and $\varphi^{-1}(0) = \{x_1, \ldots, x_s\}$ where the roots $x_i$ are all in $D$ and have multiplicity one.

We choose now a direction $d' \in V \cap U$ such that

$$(d_0', \Delta)_0 = m(\Delta)$$

where $m(\Delta)$ is the multiplicity of the discriminant $\Delta$ at the origin. An explicit formula for $m(\Delta)$ can be found in [3], [10] and it follows that $m(\Delta) \geqslant \mu(X_0)$ with equality iff $X_0$ is a hypersurface singularity.

Note that a path connecting $d$ with $d'$ within $V$ gives rise to a homotopy $\varphi_t$: $D \to \mathbb{C}$, $0 \leqslant t \leqslant 1$ of $\varphi = \varphi_0$ with $\varphi_1$, the function defined as above with respect to $d_a'$.

Since the direction $d'$ is in $U$, there is a $\rho' > 0$ and a $\theta' > 0$ such that, for any $a'$ with $0 < |a'| \leqslant \theta'$, the corresponding homomorphism (3.4) is an epimorphism.

Choose a path $a(t)$ $1 \leqslant t \leqslant 2$ in $B_\delta$ such that $a(1) = a$, $a(2) = a'$ with $0 < |a'| \leqslant \theta'$ and $d'_{a(t)} \pitchfork \Delta$ for any $t$. This gives rise as above to a homotopy $\varphi_t$: $D \to \mathbb{C}$ $1 \leqslant t \leqslant 2$. Since all the functions $\varphi_t$ have only simple roots $x_k(t)$ in Int $D$, we obtain in this way $s$ paths $x_1(t), \ldots, x_s(t)$ for $0 \leqslant t \leqslant 2$.

We choose the order on the paths such that $x_1(2), \ldots, x_m(2)$ are precisely the end points within the disc $B_{\rho'} \cap d_{a'}' \subset D$, where $m = m(\Delta)$ (Note the identification $D \simeq d'_{a(t)} \cap B_\rho$ for any $t$).

Consider the following commutative diagram.

$$\begin{array}{ccc}
\pi_1((B_\rho \setminus \Delta) \cap d_a, b) & \xrightarrow{\quad i_\# \quad} & \pi_1(B_\rho \setminus \Delta, b) \\
\tilde{\varphi} \downarrow & & \downarrow c_* \\
\pi_1((B_\rho \setminus \Delta) \cap d_{a'}', b') & \longrightarrow & \pi_1(B_\rho \setminus \Delta, b') \\
i_\# \uparrow & & \uparrow i_\# \\
\pi_1((B_{\rho'} \setminus \Delta) \cap d_{a'}', b') & \xrightarrow{i_\#} \pi_1(B_{\rho'} \setminus \Delta, b') &
\end{array}$$

The isomorphism $c_*$ is induced by a path in $B_\rho \setminus \Delta$ from $b$ to $b'$ and $\tilde{\varphi}$ is obtain via the homotopy $\varphi_t$.

If we denote by $w_k$ (resp. $w_k'$) the elementary path in $D \setminus \{x_1(t), \ldots, x_s(t)\}$ encircling the point $x_k(t)$ for $t = 0$ (resp. $t = 2$), then the left hand side of the diagram corresponds to

$$F(w_1', \ldots, w_m') \overset{i_\#}{\hookrightarrow} F(w_1', \ldots, w_s') \underset{\sim}{\overset{\tilde{\varphi}}{\to}} F(w_1, \ldots, w_s)$$

where $F(a_1,\dots,a_p)$ denotes the free group generated by $a_1,\dots,a_p$.

This ends the proof of (3.5) and hence of (3.3). □

COROLLARY 3.6: *Suppose $X_0$ is a hypersurface singularity and let $m = m(\Delta) = \mu(X_0)$. Then in the monodromy exact sequence* (2.3) *of the function $f$ (up to a change of indexes) the vanishing cycles $\delta_k = \partial\Delta_k$ $(k = 1,\dots,m)$ form a basis of $H_{n-1}(\tilde{X}_0)$ and the Picard-Lefschetz transformations associated to the elementary paths $w_k$ $(k = 1,\dots,m)$ generate the group $G_0(f)$.*

PROOF: The proof of (3.5) implies that (up to a change of indexes) the images of $w_1,\dots,w_m$ generate the group $G_0(f) = G(X_0)$.

The monodromy group $G(X_0)$ acts transitively on the set of vanishing cycles in $H_{n-1}(\tilde{X}_0)$ [4], (2.58).

Hence for any such cycle $\delta$ there is an element $g \in G_0(f)$ such that $\delta = \pm g \cdot \delta_1$.

Since $g$ is a product of Picard-Lefschetz transformations associated to $w_1,\dots,w_m$, it follows that

$$\delta \in \mathbb{Z}\langle \delta_1,\dots,\delta_m \rangle$$

i.e. $\delta_1,\dots,\delta_m$ form a basis of $H_{n-1}(\tilde{X}_0)$. □

Finally we give some information about the other monodromy group of $f$, namely $G(f)$.

PROPOSITION 3.7:

(*i*) *There is an exact sequence of groups*

$$0 \to \mathbb{Z}^\alpha \to G(f) \to G_0(f) \to 1$$

*for some $\alpha \in \mathbb{N}$ with $0 \leqslant \alpha \leqslant \mu(X) \cdot \mu(X_0)$.*

(*ii*) *Suppose that $X_0$ is a hypersurface singularity and the intersection form on $H_{n-1}(\tilde{X}_0)$ is nondegenerate.*

*Then $\alpha \geqslant \mu(X)$.*

*If moreover the action of $G_0(f)$ on $H_{n-1}(\tilde{X}_0) \otimes \mathbb{C}$ is irreducible, then $\alpha = \mu(X) \cdot \mu(X_0)$.*

PROOF: Put $m = \mu(X_0)$, $m' = \mu(X)$ and $s = m + m'$. Assume that $\{\Delta_i\}$ is a basis of $H_n(\tilde{X}, \tilde{X}_0)$ (made of thimbles only in the proof of (ii)!) such that $\delta_k = \partial\Delta_k$ for $k = 1,\dots,m$ form a basis for $H_{n-1}(\tilde{X}_0)$.

Then for any $k > m$ there is a combination

$$v_k = \Delta_k + \sum_{i=1}^{m} a_{ki}\Delta_i \quad \text{such that} \quad \partial v_k = 0.$$

In the basis $v_{m+1}, \dots, v_s, \Delta_1, \dots, \Delta_m$ the action of $w_k$ on $H_n(\tilde{X}, \tilde{X}_0)$ is given by a matrix

$$T_k = \begin{pmatrix} 1 & A_k \\ 0 & B_k \end{pmatrix}$$

We define an epimorphism $\rho$: $G(f) \to G_0(f)$ by associating to an $s \times s$ matrix as above the $m \times m$ matrix in the lower right corner. We get thus an exact sequence

$$1 \to \ker \rho \to G(f) \overset{\rho}{\to} G_0(f) \to 1$$

where $\ker \rho$ is a subgroup in the (abelian!) multiplicative group of all the matrices

$$M = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}$$

It follows that $\ker \rho \subset \mathbb{Z}^{m \cdot m'}$ and this gives us (i). To prove (ii) we assume the basis $\delta_k$ chosen as in (3.6). Note that the matrix $A_k$ defined above is zero for $k \leqslant m$ and has a single nonzero row (that corresponding to the vector $v_k$) for $m < k \leqslant s$ if the intersection form is nondegenerate. This proves the first part of (ii).

Moreover, note that if

$$\left(\begin{array}{c|c} & 0 \\ 1 & \dots u \dots \\ & 0 \\ \hline 0 & 1 \end{array}\right) \in \ker \rho$$

for some row vector $u \neq 0$, then the same is true for the vector $u \cdot B$ for any $B \in G_0(f)$.

If the action of $G_0(f)$ on the homology group $H_{n-1}(\tilde{X}_0; \mathbb{C})$ is irreducible, then it follows that

$$\dim \mathbb{C}\langle u \cdot B;\ B \in G_0(f) \rangle = m$$

Hence $\ker \rho$ contains in this case $m \cdot m'$ $\mathbb{C}$-linearly independent vectors and this implies the result in the second part of (ii). □

REMARKS 3.8:

a. The condition about the intersection form in (3.7.ii) is necessary. For instance, if $f$ is a simple function of type $B_k$ and $n$ is even, it follows from (2.7) that $G_0(f) = G(f) = 0$.