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## THE CARDINALITY OF THE SET OF INVARIANT MEANS ON A LOCALLY COMPACT TOPOLOGICAL SEMIGROUP

Heneri A.M. Dzinotyiweyi

### Abstract

For a large class of locally compact topological semigroups, which include all non-compact,  $\sigma$ -compact and locally compact topological groups as a very special case, we show that the cardinality of the set of all invariant means on the space of weakly uniformly continuous functions is either 0 or  $\geq 2^c$ ; where  $c$  denotes the cardinality of continuum.

### 1. Preliminaries

Throughout this paper, let  $S$  denote a (Hausdorff, jointly continuous and) locally compact topological semigroup,  $C(S)$  the space of all bounded complex-valued continuous functions on  $S$  and  $M(S)$  the Banach algebra of all complex-valued bounded Radon measures on  $S$  with convolution multiplication given by

$$\nu * \mu(f) = \iint f(xy) d\nu(x) d\mu(y) = \iint f(xy) d\mu(y) d\nu(x)$$

for all  $\nu, \mu \in M(S)$  and  $f \in C(S)$ . Corresponding to each function  $f$  in  $C(S)$ , measure  $\nu$  in  $M(S)$  and point  $x$  in  $S$  we have the functions  ${}_x f, f_x, \nu \circ f$  and  $f \circ \nu$  in  $C(S)$  given by

$${}_x f(y) = f(xy) \quad \text{and} \quad f_x(y) = f(yx) \quad (y \in S)$$

$$\nu \circ f(y) = \nu(f_y) \quad \text{and} \quad f \circ \nu(y) = \nu({}_y f) \quad (y \in S).$$

A function in  $C(S)$  is said to be *weakly uniformly continuous* if it belongs to the set  $WUC(S) = \{f \in C(S) : \text{the maps } x \rightarrow {}_x f \text{ and } x \rightarrow f_x \text{ of } S \text{ into } C(S) \text{ are weakly continuous}\}$ .

For any subsets  $A$  and  $B$  of  $S$  and point  $x$  in  $S$ , we write  $AB = \{ab : a \in A \text{ and } b \in B\}$ ,  $A^{-1}B = \{y \in S : ay \in B \text{ for some } a \in A\}$ ,  $x^{-1}B = \{x\}^{-1}B$  and  $A^{-1}x = A^{-1}\{x\}$ . By symmetry we similarly define  $BA^{-1}$ ,  $Bx^{-1}$  and  $xA^{-1}$ . An object playing a pivotal role in this paper is the

algebra of all *absolutely continuous measures* on  $S$  - namely  $M_a(S) := \{ \nu \in M(S) : \text{the maps } x \rightarrow |\nu|(x^{-1}K) \text{ and } x \rightarrow |\nu|(Kx^{-1}) \text{ of } S \text{ into } \mathbb{R} \text{ are continuous for all compact } K \subseteq S \}$ . (Here  $|\nu|$  denotes the measure arising from the total variation of  $\nu$ .) For various studies on the algebra  $M_a(S)$ , see e.g. [1], [5] and the references cited there. Taking  $\text{supp}(\nu) := \{ x \in S : |\nu|(X) > 0 \text{ for every open neighbourhood } X \text{ of } x \}$  we define the *foundation* of  $M_a(S)$ , namely  $F_a(S)$ , to be the closure of the set  $\cup \{ \text{supp}(\nu) : \nu \in M_a(S) \}$ . For a function  $f$  in  $C(S)$  we also write  $\text{supp}(f) := \text{closure of } \{ x \in S : f(x) \neq 0 \}$ .

If  $h \in M(S)^*$  and  $\nu \in M(S)$  we define the functional  $\nu \circ h$  on  $M(S)$  and function  $\nu \circ h$  on  $S$  by

$$\nu \circ h(\mu) := h(\nu * \mu) \quad \text{and} \quad \nu \circ h(x) := h(\nu * \bar{x})$$

$$(\mu \in M(S) \quad \text{and} \quad x \in S);$$

where  $\bar{x}$  stands for the point mass at  $x$ . By symmetry one similarly defines  $h \circ \nu$  and  $h \circ \nu$ . By a *mean*  $m$  on  $M_a(S)^*$  (or  $\text{WUC}(S)$ ) we mean a functional such that  $m(1) = 1$  and  $m(f) \geq 0$  if  $f \geq 0$ ; for all  $f$  in  $M_a(S)^*$  (or  $\text{WUC}(S)$ , respectively). A mean  $m$  on  $\text{WUC}(S)$  is said to be *invariant* (or *topologically invariant*) if  $m({}_x f) = m(f_x) = m(f)$  (or  $m(\nu \circ f) = m(f \circ \nu) = m(f)\nu(S)$ , respectively), for all  $f \in \text{WUC}(S)$ ,  $x \in S$  and  $\nu \in M(S)$ . Similarly one defines a (topologically) invariant mean on  $M_a(S)^*$ . We denote the set of all invariant means on  $\text{WUC}(S)$  by  $\text{IM}(\text{WUC}(S))$ .

For ease of reference we mention the following special case of a result proved in [6] and which can also be deduced from the two main theorems of [14].

1.1. PROPOSITION: *If  $S$  has an identity element and coincides with the foundation of  $M_a(S)$ , the following items are equivalent:*

- (a) *There exists a topological invariant mean on  $M_a(S)^*$ .*
- (b) *There exists an invariant mean on  $\text{WUC}(S)$ .*

The following result is also proved in [6].

1.2. PROPOSITION: *Every invariant mean on  $\text{WUC}(S)$  is topologically invariant.*

For any subsets  $A, B, C$  of  $S$  we write  $A \otimes B := \{ AB, A^{-1}B, AB^{-1} \}$  and  $A \otimes B \otimes C := (\cup \{ A \otimes D : D \in B \otimes C \}) \cup (\cup \{ D \otimes C : D \in A \otimes B \})$ . So inductively we can define  $A_1 \otimes A_2 \otimes \dots \otimes A_n$  for any subsets  $A_1, \dots, A_n$  of  $S$ . Following [7], a subset  $B$  of  $S$  is said to be *relatively neo-compact* if  $B$  is contained in a (finite) union of sets in  $A_1 \otimes A_2 \otimes \dots \otimes A_n$  for some compact subsets  $A_1, A_2, \dots, A_n$  of  $S$ . In particular, as noted in [7]:

For a topological semigroup  $S$  such that  $C^{-1}D$  and  $DC^{-1}$  are compact whenever  $C$  and  $D$  are compact subsets of  $S$ , we have that  $B \subseteq S$  is relatively neo-compact if and only if  $B$  is relatively compact. (So relatively neo-compact subsets of a topological group are precisely the relatively compact subsets.)

The *centre* of  $E \subset S$  is the set  $Z(E) := \{x \in S: xy = yx \text{ for all } y \in E\}$ . Let  $P(M_a(S)) := \{\nu \in M_a(S): \nu \geq 0 \text{ and } \|\nu\| = 1\}$ . A net or sequence  $(\mu_\alpha) \subset P(M_a(S))$  is said to be *weakly* (or *strongly*) *convergent to topological invariance* if  $\nu^* \mu_\alpha - \mu_\alpha \rightarrow 0$  and  $\mu_\alpha^* \nu - \mu_\alpha \rightarrow 0$  weakly (or strongly, respectively) in  $M_a(S)$ , for all  $\nu \in P(M_a(S))$ .

Let  $F := \{\phi \in (I^\infty): \phi > 0, \|\phi\| = 1 \text{ and } \phi(g) = 0 \text{ for all } g \in I^\infty \text{ such that } g(k) \rightarrow 0 \text{ as } k \rightarrow \infty\}$  and  $c$  be the cardinality of continuum.

Our aim in this paper is to prove the following result.

**THEOREM:** *Let  $S$  be a  $\sigma$ -compact locally compact topological semigroup such that  $S$  is not relatively neo-compact,  $M_a(S)$  is non-zero and there exists an invariant mean on  $\text{WUC}(S)$ . Suppose either (a)  $S$  has an identity element and coincides with the foundation of  $M_a(S)$ , or (b) the centre of  $F_a(S)$  is not  $M_a(S)$ -negligible. Then there exists a linear isometry  $\tau: (I^\infty)^* \rightarrow \text{WUC}(S)^*$  such that*

$$\tau(F) \subset \text{IM}(\text{WUC}(S)) \quad \text{and so} \quad \text{card}(\text{IM}(\text{WUC}(S))) \geq 2^c.$$

## 2. Proof of the theorem

We partition part of our proof into some lemmas. In the following lemma  $C_0(S)$  denotes the space of functions in  $C(S)$  which are arbitrarily small outside compact sets and  $C_{00}(S) := \{f \in C(S): f \text{ vanishes outside some compact subset of } S\}$ .

2.1. **LEMMA:** *If  $S$  is not relatively neo-compact, then*

$$m(f) = 0 \quad \text{for all } f \in C_0(S) \quad \text{and} \quad m \in \text{IM}(\text{WUC}(S))$$

**PROOF:** Let  $f \in C_{00}(S)$  be positive and take  $K$  to be the support of  $f$  (i.e.  $\text{supp}(f)$ ). Since  $S$  is not relatively neo-compact, we can choose a sequence  $\{x_n\}$  in  $S$  such that

$$x_{n+1} \notin K(x_1^{-1}K \cup \dots \cup x_n^{-1}K)^{-1} \quad \text{for all } n \in \mathbb{N}.$$

So if  $n \neq k$  we have  $x_n^{-1}K \cap x_k^{-1}K = \emptyset$  or, equivalently,

$$\text{supp}(x_n f) \cap \text{supp}(x_k f) = \emptyset.$$

consequently

$$nm(f) = m\left(\sum_{x_1} f + \dots + \sum_{x_n} f\right) \leq \|f\|_S \quad \text{for all } n \in \mathbb{N},$$

and so  $m(f) = 0$ .

Since  $C_{00}(S)$  is dense in  $C_0(S)$ , the remainder of our result follows trivially.

**2.2. LEMMA:** *There is a net in  $P(M_a(S))$  weakly convergent to topological invariance if and only if there is a net in  $P(M_a(S))$  strongly convergent to topological invariance.*

**PROOF:** Suppose  $(\eta_\alpha) \subseteq P(M_a(S))$  converges weakly to topological invariance. Setting  $M_{\nu, \mu} := M_a(S)$  we form the locally convex product space  $M := \Pi \{M_{\nu, \mu} : (\nu, \mu) \in P(M_a(S)) \times P(M_a(S))\}$  with the product of norm topologies and define the linear map  $L: M_a(S) \rightarrow M$  by

$$L(\phi)(\nu, \mu) := \nu^* \phi^* \mu - \phi$$

$$\text{for all } \phi \in M_a(S) \quad \text{and } \nu, \mu \in P(M_a(S)).$$

As noted in [11, page 160], the weak topology on  $M$  coincides with the product of the weak topologies on the  $M_{\nu, \mu}$ 's. Since  $\nu^* \eta_\alpha^* \mu - \eta_\alpha \rightarrow 0$  weakly, for all  $\nu, \mu \in P(M_a(S))$ ,  $0 \in \text{weak-closure } L(P(M_a(S)))$ . Since  $L(P(M_a(S)))$  is a convex subset of the locally convex space  $M$ , we have  $\text{weak-closure } L(P(M_a(S))) = \text{strong-closure } L(P(M_a(S)))$ , by the Hahn-Banach Theorem. So there exists a net  $(\rho_\beta) \subseteq P(M_a(S))$  such that  $L(\rho_\beta) \rightarrow 0$  in  $M$  or, equivalently, such that

$$\|\nu^* \rho_\beta^* \mu - \rho_\beta\| \rightarrow 0 \quad \text{for all } \nu, \mu \in P(M_a(S)).$$

Hence

$$\begin{aligned} \|\nu^* \rho_\beta - \rho_\beta\| &\leq \|\nu^* \rho_\beta - \nu^* \nu^* \rho_\beta^* \nu\| + \|(\nu^* \nu)^* \rho_\beta^* \nu - \rho_\beta\| \\ &\leq \|\rho_\beta - \nu^* \rho_\beta^* \nu\| + \|(\nu^* \nu)^* \rho_\beta^* \nu - \rho_\beta\| \rightarrow 0 \end{aligned}$$

for all  $\nu \in P(M_a(S))$ . Similarly  $\|\rho_\beta^* \nu - \rho_\beta\| \rightarrow 0$ , and the remainder of our lemma follows trivially.

**2.3. LEMMA:** *Let  $S$  be not relatively neo-compact and let  $\{\mu_n\}$  be a sequence in  $P(M_a(S))$  strongly convergent to topological invariance and such that  $T_n := \text{supp}(\mu_n)$  is compact, for all  $n \in \mathbb{N}$ . Then, given  $n_0 \in \mathbb{N}$  and  $\epsilon > 0$ , we can find  $n > n_0$  such that*

$$\mu_n\left((T_1 \cup \dots \cup T_{n_0}) \cap T_n\right) < \epsilon.$$

PROOF: Suppose, on the contrary, there exists an  $n_0$  and  $\epsilon > 0$  such that

$$\mu_n((T_1 \cup \dots \cup T_{n_0}) \cap T_n) > \epsilon \quad \text{for all } n > n_0.$$

Let  $f \in C_0(S)$  be a positive function with  $f = 1$  on  $T_1 \cup \dots \cup T_{n_0}$  and note that

$$\mu_n(f) > \epsilon \quad \text{for all } n > n_0.$$

Let  $m$  be any weak\* - cluster point of  $\{\mu_n\}$  in  $\text{WUC}(S)^*$  and note that  $m$  is such that  $m(f) > \epsilon$  and  $m(\nu \circ g) = m(g)(\nu \in P(M_a(S)))$  and  $g \in \text{WUC}(S)$ . Since  $M_a(S)$  is an ideal of  $M(S)$ , see e.g. [1] or [5], we have  $\bar{x}^*\nu \in P(M_a(S))$  whenever  $x \in S$  and  $\nu \in P(M_a(S))$ . Now  $x^*\nu \circ g = \nu \circ_x g$  and so  $m(g) = m(x^*\nu \circ g) = m(\nu \circ_x g) = m({}_x g)$ ; similarly  $m(g_x) = m(g)(g \in \text{WUC}(S)$  and  $x \in S)$ . By Lemma 2.1, we must have  $m(f) = 0$ . This contradiction implies our result.

2.4. LEMMA: *Let  $S$  be  $\sigma$ -compact with  $M_a(S)$  non-zero and let there be an invariant mean on  $\text{WUC}(S)$ . Then there exists a sequence  $(\rho_n)$  in  $P(M_a(S))$  converging strongly to topological invariance and such that  $K_n := \text{supp}(\rho_n)$  is compact, for all  $n \in \mathbb{N}$ , if any one of the following conditions holds:*

- (a)  *$S$  is the foundation of  $M_a(S)$  and  $S$  has an identity element.*
- (b) *The centre of  $F_a(S)$  is not  $M_a(S)$ -negligible.*

PROOF: First we show that there exists a net in  $P(M_a(S))$  weakly convergent to topological invariance, if (a) or (b) holds. To this end, suppose (a) holds. Then there exists a topological invariant mean  $m$  on  $M_a(S)^*$ , by Proposition 1.1. Now  $m \in \text{weak}^*$ -closure  $(P(M_a(S)))$  in  $M_a(S)^{**}$ . Consequently, there exists a net  $(\mu_\alpha)$  in  $P(M_a(S))$  such that  $\mu_\alpha(h) \rightarrow m(h)$  for all  $h \in M_a(S)^*$ . In particular, for each  $\nu \in P(M_a(S))$  we have

$$|h(\nu^*\mu_\alpha - \mu_\alpha)| = |\mu_\alpha(\nu \circ h) - \mu_\alpha(h)| \rightarrow |m(\nu \circ h) - m(h)| = 0,$$

for all  $h \in M_a(S)^*$ . Similarly  $\mu_\alpha^*\nu - \mu_\alpha \rightarrow 0$  weakly. Thus  $((\mu_\alpha) \subseteq P(M_a(S)))$  is weakly convergent to topological invariance.

Next suppose condition (b) holds. Then we can choose  $\tau \in P(M_a(S))$  such that  $\text{supp}(\tau) \subset Z(F_a(S))$  (, where  $Z(F_a(S))$  denotes the centre of  $F_a(S)$ ). We then have  $\tau^*\nu = \nu^*\tau$  for all  $\nu \in P(M_a(S))$ . Now if  $m_0$  is any invariant mean on  $\text{WUC}(S)$  we have that  $m_0$  is topologically invariant, by proposition 1.2. Let  $(\eta_\alpha)$  be a net in  $P(M_a(S))$  such that  $\eta_\alpha(f) \rightarrow m_0(f)$ , for all  $f \in \text{WUC}(S)$ . Then for any  $\nu \in P(M_a(S))$  and  $h \in M_a(S)^*$ , we have that  $\tau^*\nu \circ h \circ \tau, \tau \circ h \circ \tau \in \text{WUC}(S)$  by [7, Lemma 4.1].

Hence, if  $\mu_\alpha := \tau^* \eta_\alpha^* \tau$ ,

$$\begin{aligned}
 h(\nu^* \mu_\alpha - \mu_\alpha) &= h(\nu^* \tau^* \eta_\alpha^* \tau - \tau^* \eta_\alpha^* \tau) \\
 &= h(\tau^* \nu^* \eta_\alpha^* \tau - \tau^* \eta_\alpha^* \tau) \\
 &= \eta_\alpha(\tau^* \nu \circ h \circ \tau) - \eta_\alpha(\tau \circ h \circ \tau) \\
 &\rightarrow m_0(\tau^* \nu \circ h \circ \tau) - m_0(\tau \circ h \circ \tau) \\
 &= m_0(\nu \circ (\tau \circ h \circ \tau)) - m_0(\tau \circ h \circ \tau) = 0.
 \end{aligned}$$

Similarly  $h(\mu_\alpha^* \nu - \mu_\alpha) \rightarrow 0$ . Thus  $(\mu_\alpha)$  is weakly convergent to topological invariance.

Now suppose either (a) or (b) holds. Then there exists a net  $(\eta_\beta) \subset P(M_a(S))$  strongly convergent to topological invariance, by Lemma 2.2. Fix any  $\lambda \in P(M_a(S))$  and set  $\mu_\beta := \lambda^* \eta_\beta^* \lambda$ . Note that  $(\mu_\beta)$  is also strongly convergent to topological invariance. As  $S$  is  $\sigma$ -compact, we can choose an increasing sequence of compact neighbourhoods,  $D_1 \subset D_2 \subset \dots$ , such that  $S = \cup_{n=1}^\infty D_n$ . Noting that the maps  $(x, y) \rightarrow \bar{x}^* \mu_\beta^* \bar{y}$  of  $S \times S$  into  $M_a(S)$  are norm continuous (see e.g. [5, Corollary 4.10 (ii)]) we can choose a sequence  $\{\mu_{\beta_1}, \mu_{\beta_2}, \dots\}$  from the  $\mu_\beta$ 's such that

$$\|\bar{x}^* \mu_{\beta_n}^* \bar{y} - \mu_{\beta_n}\| < \frac{1}{5n} \quad \text{for all } x, y \in D_n.$$

Choose compact sets  $K_n$  such that

$$\mu_{\beta_n}(S \setminus K_n) < \frac{1}{5n} \quad \text{for all } n \in \mathbb{N}.$$

Setting  $\rho := (\mu_{\beta_n}(K_n))^{-1} \mu_{\beta_n}|_{K_n}$  we have  $(\rho_n) \subset P(M_a(S))$ ,  $\text{supp}(\rho_n) \subset K_n$  and a standard technical argument shows that

$$\|\mu_{\beta_n} - \rho_n\| < 2\mu_{\beta_n}(S \setminus K_n) < \frac{2}{5n}.$$

Consequently

$$\begin{aligned}
 \|\bar{x}^* \rho_n^* \bar{y} - \rho_n\| &< \|\bar{x}^* \rho_n^* \bar{y} - \bar{x}^* \mu_{\beta_n}^* \bar{y}\| + \|\bar{x}^* \mu_{\beta_n}^* \bar{y} - \mu_{\beta_n}\| + \|\mu_{\beta_n} - \rho_n\| \\
 &< 2\|\rho_n - \mu_{\beta_n}\| + \|\bar{x}^* \mu_{\beta_n}^* \bar{y} - \mu_{\beta_n}\| \\
 &< \frac{4}{5n} + \frac{1}{5n} = \frac{1}{n} \quad \text{for all } x, y \in D_n.
 \end{aligned}$$

Now for any  $\nu, \eta \in P(M_a(S))$  with compact supports we have  $\text{supp}(\nu) \cup \text{supp}(\eta) \subset D_n$  for  $n$  larger than some  $n_0$  and hence

$$\begin{aligned} \|\nu^* \rho_n^* \eta - \rho_n\| &= \sup \{ |\nu^* \rho_n^* \eta(f) - \rho_n(f)| : f \in C_0(S), \|f\|_S \leq 1 \} \\ &\leq \sup \left\{ \int |\bar{x}^* \rho_n^* \bar{y}(f) - \rho_n(f)| d\nu(x) d\eta(y) : \right. \\ &\quad \left. f \in C_0(S), \|f\|_S \leq 1 \right\} \\ &< \frac{1}{n}. \end{aligned}$$

It is now trivial to complete the proof of our lemma.

**2.5. PROOF OF OUR THEOREM:** Note that we have the hypothesis of Lemma 2.4 met and so let  $\{\rho_n\}$  and  $\{K_n\}$  be as in Lemma 2.4. Choose  $\nu \in P(M_a(S))$  with  $C = \text{supp}(\nu)$  compact and note that the sequence  $\{\nu^* \rho_n^* \nu\}$  also converges strongly to topological invariance.

Observing that  $T_n = \text{supp}(\nu^* \rho_n^* \nu) = CK_n C$  is compact ( $n \in \mathbb{N}$ ) and recalling Lemma 2.3, there exist subsequences  $\{T_{n_k}\}$  of  $\{T_n\}$  and  $\{\nu^* \rho_{n_k}^* \nu\}$  of  $\{\nu^* \rho_n^* \nu\}$  such that

$$\nu^* \rho_{n_k}^* \nu (T_{n_k} \setminus (T_{n_0} \cup \dots \cup T_{n_{k-1}})) > \frac{1}{2}$$

for  $k \in \mathbb{N}$ , where  $T_{n_0} = \emptyset$ .

Let  $F_k = T_{n_k} \setminus (T_{n_1} \cup \dots \cup T_{n_{k-1}})$  and  $\mu_k = \nu^* \rho_{n_k}^* \nu$ , for all  $k \in \mathbb{N}$ .

Let  $\Pi: \text{WUC}(S) \rightarrow l^\infty$  be the linear mapping defined by

$$\Pi(f)(k) = \rho_{n_k}(f)$$

for all  $f \in \text{WUC}(S)$  and  $k \in \mathbb{N}$ .

To see that  $\Pi$  is onto, let  $g \in l^\infty$  be fixed. Since members of the sequence  $\{F_k\}$  are clearly pairwise disjoint and  $\mu_k(F_k) > \frac{1}{2}$ , the function

$$h = \sum_{k=1}^{\infty} \frac{g(k)}{\mu_k(F_k)} \chi_{F_k}$$

is a linear functional in  $M(S)^*$ . (Here  $\chi_{F_k}$  is the characteristic function of



$F_k$ .) Consequently  $\nu \circ h \circ \nu \in \text{WUC}(S)$ , by [7, Lemma 4.1]. Now, for all  $k \in \mathbb{N}$ , we have

$$\begin{aligned} \Pi(\nu \circ h \circ \nu)(k) &:= \rho_{n_k}(\nu \circ h \circ \nu) \\ &= \nu^* \rho_{n_k}^* \nu(h) \\ &= \mu_k \left( \sum_{i=1}^{\infty} \frac{g(i)}{\mu_i(F_i)} \chi_{F_i} \right) \\ &= g(k). \end{aligned}$$

Thus  $\Pi$  maps the function  $\nu \circ h \circ \nu$  onto  $g$  and  $\Pi$  is onto. Further, we clearly have

$$\|g\|_{\infty} = \|\Pi(\nu \circ h \circ \nu)\|_{\infty} = \|\nu \circ h \circ \nu\|_S.$$

It follows that the dual map  $\Pi^*: (l^{\infty})^* \rightarrow \text{WUC}(S)^*$  is a linear isometry.

To see that  $\Pi^*F \subset \text{IM}(\text{WUC}(S))$ , let  $\phi \in F$  be fixed. Then, clearly

$$\Pi^*\phi > 0 \quad \text{and} \quad \Pi^*\phi(1) = \phi(1) = 1.$$

Now, for any  $\eta \in P(M_a(S))$  and  $f \in \text{WUC}(S)$ , we have

$$\begin{aligned} \Pi(\eta \circ f - f)(k) &:= \rho_{n_k}(\eta \circ f - f) \\ &= (\eta^* \rho_{n_k} - \rho_{n_k})(f) \rightarrow 0 \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Recalling the definition of  $F$  we have

$$\Pi^*\phi(\eta \circ f - f) = 0.$$

Similarly  $\Pi^*\phi(f \circ \eta - f) = 0$  and so  $\Pi^*\phi \in \text{IM}(\text{WUC}(S))$ .

Taking  $\beta\mathbb{N}$  to be the Stone-Cech compactification of  $\mathbb{N}$ , we have  $\beta\mathbb{N} \setminus \mathbb{N} \subset F$ . Since  $\text{card}(\beta\mathbb{N} \setminus \mathbb{N}) = 2^c$  and  $\Pi^*$  is an isometry, we thus get  $\text{card}(\text{IM}(\text{WUC}(S))) \geq \text{card}(F) \geq 2^c$ . So  $\tau = \Pi^*$  is the required map and our Theorem is proved.

### 3. Note on references

Many results on the sizes of sets of invariant means can be found in the literature: for discrete semigroups see e.g. Chou ([2] and [3]), Granirer [8] and Klawe [12]; and for locally compact topological groups see e.g. Chou ([3] and [4]) and Granirer [8]. Related results dealing with the difference between an invariant and a topologically invariant mean can be found in

the papers of Rosenblatt [15] and Liu and Van Rooij [13]. Our main theorem generalizes some of these results and our techniques are inspired by the paper of Chou [4]. The proof of lemma 2.2 is closely related to that given in Greenleaf [9, Theorem 2.42].

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