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THE PERIOD MAP FOR HYPERSURFACE SECTIONS OF HIGH DEGREE OF AN ARBITRARY VARIETY

Mark L. Green *

Introduction

In this paper, for a fixed projective variety Y , we will say that a property holds for *sufficiently ample* analytic line bundles $L \rightarrow Y$ if there exists an ample bundle $L_0 \rightarrow Y$ so that the property holds for all L on Y with $L > L_0$, i.e. $L \otimes L_0^{-1}$ ample. We will denote this by saying the property holds for $L \gg 0$.

We will prove two theorems:

THEOREM 0.1: *Let Y be a smooth complete algebraic variety of dimension ≥ 2 . Then for $L \rightarrow Y$ a sufficient ample line bundle, the Local Torelli Theorem is true for any smooth Z in the linear system $|L|$.*

REMARK: What we will actually show is that the map

$$H^1(Z, \Theta_Z) \xrightarrow{P^*} \text{Hom}(H^0(Z, K_Z), H^1(Z, \Omega_Z^{n-1})) \quad (0.2)$$

is injective, where $n = \dim Z$; this is one piece of the derivative of the period map. Note that we are considering all first order deformations of Z and not just those arising by varying Z to first order in the linear system $|L|$.

THEOREM 0.3: *Let Y be a smooth complete algebraic variety of dimension ≥ 2 , $L \rightarrow Y$ a sufficiently ample line bundle. Assume K_Y is very ample. Let*

$$G = \{f \in \text{Aut}_{\text{hol}}(Y) \mid f^*(L) \simeq L\} \quad (0.4)$$

Then the period map has degree 1 on its domain in

$$\mathbb{P}(H^0(Y, L))/G.$$

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What we will actually show is stronger than (0.3). To state exactly what is proved, we make the following definition:

DEFINITION 0.5: Let V_1, \dots, V_k be vector spaces. We say that two elements

$$\alpha, \beta \in V_1 \otimes V_2 \otimes \dots \otimes V_k$$

are GL -equivalent if they lie in the same orbit of $GL(V_1) \times \dots \times GL(V_k)$.

Let

$$T \subseteq H^1(Z, \Theta_Z)$$

be the image of $H^0(Y, L)$. The highest piece of the derivative of the period map $P_{*,Z}$ may be regarded as an element

$$P_{*,Z} \in T^* \otimes H^0(Z, K_Z)^* \otimes H^1(Z, \Omega_Z^{n-1})$$

Let

$$\mathbb{P}(H^0(Y, L))_{ns} \leftrightarrow \{\text{smooth, reduced } Z \in |L|\}.$$

Then what will actually be shown is that

$$\text{If } |K_Y| \text{ is very ample, then for } L \text{ a sufficiently ample line bundle, the map} \tag{0.6}$$

$$\begin{aligned} &\mathbb{P}(H^0(Y, L))_{ns}/G \\ &\rightarrow \frac{T^* \otimes H^0(Z, K_Z)^* \otimes H^1(Z, \Omega_Z^{n-1})}{GL(T^*) \times GL(H^0(Z, K_Z)^*) \times GL(H^1(Z, \Omega_Z^{n-1}))} \end{aligned} \tag{0.7}$$

is injective.

These theorems settle a conjecture in [C-G-Gr-H]; the method of proof of (0.1) is essentially a resurgence of an idea that occurs there. The proof of (0.3) is inspired by Donagi's proof of a similar result for hypersurfaces in projective space [D].

The author is grateful to Ron Donagi and Phillip Griffiths for their help and encouragement.

§1. Hodge theory on a hypersurface of high degree

Given a short exact sequence of vector spaces

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

there is a long exact sequence

$$0 \rightarrow S^k A \rightarrow B \otimes S^{k-1} A \rightarrow \dots \rightarrow \Lambda^{k-1} B \otimes A \rightarrow \Lambda^k B \rightarrow \Lambda^k C \rightarrow 0 \quad (1.1)$$

for any $k \geq 1$. If Y is a compact complex manifold of dimension m and $Z \subset Y$ is a smooth submanifold of dimension n with normal bundle N_Z in Y , we have a short exact sequence

$$0 \rightarrow N_Z^* \rightarrow \Omega_Y^1 \otimes \mathcal{O}_Z \rightarrow \Omega_Z^1 \rightarrow 0. \quad (1.2)$$

Thus we have, for any $p \geq 1$, long exact sequences

$$0 \rightarrow S^p N_Z^* \rightarrow \dots \rightarrow \Omega_Y^{p-1} \otimes N_Z^* \rightarrow \Omega_Y^p \otimes \mathcal{O}_Z \rightarrow \Omega_Z^p \rightarrow 0. \quad (1.3)$$

The exact sequences (1.3) turn out to be quite useful in computations whenever we have an explicit form for N_Z , most notably in the case of complete intersections. To use these sequences, we need:

LEMMA 1.4: *Let*

$$0 \rightarrow F_1 \xrightarrow{f_1} F_2 \xrightarrow{f_2} \dots \rightarrow F_{k-1} \xrightarrow{f_{k-1}} F_k \rightarrow 0$$

be an exact sequence of vector bundles on a compact complex manifold Z . Then there is a spectral sequence abutting to zero with

$$E_1^{p,q} = H^q(Z, F_p)$$

$$E_2^{p,q} = \frac{\ker \left(H^q(Z, F_p) \xrightarrow{f_p^*} H^q(Z, F_{p+1}) \right)}{\operatorname{im} \left(H^q(Z, F_{p-1}) \xrightarrow{f_{p-1}^*} H^q(Z, F_p) \right)}.$$

PROOF: Consider the bigraded complex

$$B^{p,q} = \mathcal{A}^{0,q}(Z, F_p)$$

where $\mathcal{A}^{0,q}$ denotes \mathcal{C}^∞ $(0, q)$ -forms, with maps

$$B^{p,q} \xrightarrow{d} B^{p+1,q} \quad d = f_p^*$$

$$B^{p,q} \xrightarrow{\delta} B^{p,q+1} \quad \delta = \bar{\delta}.$$

There is then (see [G-H]) a pair of spectral sequences $'E_r^{p,q}, ''E_r^{p,q}$ having the same abutment, with

$$\begin{aligned} 'E_1^{p,q} &= H_d^p(B^{\cdot,q}), & 'E_2^{p,q} &= H_8^p H_d^p(B^{\cdot\cdot}) \\ ''E_1^{p,q} &= H_8^q(B^{p,\cdot}), & ''E_2^{p,q} &= H_d^p H_8^q(B^{\cdot\cdot}). \end{aligned}$$

The rows of $B^{\cdot\cdot}$ are exact, so

$$'E_1^{p,q} = 0 \quad \text{for all } p, q.$$

Thus the spectral sequence $'E_r^{p,q}$ abuts to zero, and hence so does $''E_r^{p,q}$. Furthermore,

$$\begin{aligned} ''E_1^{p,q} &= H^q(Z, F_p) \\ ''E_2^{p,q} &= \frac{\ker\left(H^q(Z, F_p) \xrightarrow{f_p^*} H^q(Z, F_{p+1})\right)}{\text{im}\left(H^q(Z, F_{p-1}) \xrightarrow{f_{p-1}^*} H^q(Z, F_p)\right)} \end{aligned}$$

This completes the proof of the lemma. \square

LEMMA 1.5: *Let Y be a compact Kähler manifold and $Z \subset Y$ a complex submanifold of dimension n . If*

$$H^i(Z, \Omega_Y^j \otimes S^m N_Z^*) = 0 \quad \text{for all } i < n, 0 \leq j \leq n, 1 \leq m \leq n \quad (1.6)$$

then

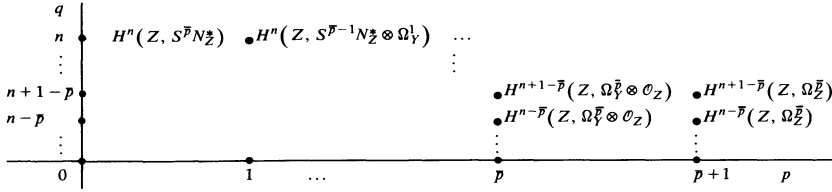
$$H^{p+q}(Z) \simeq H^q(Z, \Omega_Y^p \otimes \mathcal{O}_Z) \quad \text{if } p+q < n \quad (1.7)$$

and there is a short exact sequence

$$\begin{aligned} 0 &\rightarrow \left(\frac{H^{n-p}(Z, \Omega_Z^p)}{\text{im } H^{n-p}(Z, \Omega_Y^p \otimes \mathcal{O}_Z)} \right) \\ &\rightarrow \left(\frac{H^0(Z, S^p N_Z \otimes K_Z)}{\text{im } H^0(Z, S^{p-1} N_Z \otimes \Theta_Y \otimes K_Z)} \right)^* \\ &\rightarrow (\ker H^{n+1-p}(Z, \Omega_Y^p \otimes \mathcal{O}_Z)) \rightarrow H^{n+1-p}(Z, \Omega_Z^p) \rightarrow 0. \end{aligned} \quad (1.8)$$

PROOF: We apply Lemma 1.4 to the exact sequence (1.3), taking $p = \bar{p}$.

We obtain a spectral sequence which abuts to zero whose E_1 term looks as follows:



The only non-zero differentials emerging from the position $(0, n)$ are d_1 , d_{p-1} , and d_p . The only non-zero differentials whose target is the position $(\bar{p} + 1, n - \bar{p})$ are d_1 and d_p . We thus obtain maps

$$\begin{cases} \ker(H^n(Z, S^p N_Z^*) \rightarrow H^n(Z, S^{p-1} N_Z^* \otimes \Omega_Y^1)) \xrightarrow{d_{p-1}} \\ \ker(H^{n+1-p}(Z, \Omega_Y^p \otimes \mathcal{O}_Z) \rightarrow H^{n+1-p}(Z, \Omega_Z^{\bar{p}})) \\ \ker d_{p-1} \xrightarrow{d_p} \left(\frac{H^{n-p}(Z, \Omega_Z^{\bar{p}})}{\text{im } H^{n-p}(Z, \Omega_Y^p \otimes \mathcal{O}_Z)} \right) \end{cases} \quad (1.9)$$

where the second map is an isomorphism because the spectral sequence abuts to zero. Using Serre Duality, (1.9) gives (1.8).

There are no non-zero differentials other than d_1 coming into the positions (\bar{p}, \bar{q}) and $(\bar{p} + 1, \bar{q})$ if $\bar{q} < n - \bar{p}$. This shows (1.7) and completes the proof of Lemma 1.5. \square

A result similar to this is:

LEMMA 1.10: *Let Y be a compact Kähler manifold and $Z \subset Y$ a complex submanifold of dimension n . If*

$$H^i(Z, \Omega_Y^j \otimes S^m N_Z^* \otimes K_Z^{-1}) = 0$$

for $0 < i < n, 1 \leq j \leq n, 1 \leq m \leq n - 2$

then

$$H^1(Z, \Theta_Z) \simeq \left(\frac{H^0(Z, S^{n-1} N_Z \otimes K_Z^2)}{\text{im } H^0(Z, S^{n-2} N_Z \otimes \Theta_Y \otimes K_Z^2)} \right)^* \quad (1.12)$$

PROOF: We take the exact sequence (1.3) for $p = n - 1$ and tensor with

K_Z^{-1} . This yields the exact sequence

$$\begin{aligned} 0 \rightarrow S^{n-1}N_Z^* \otimes K_Z^{-1} \rightarrow S^{n-2}N_Z^* \otimes \Omega_Y^1 \otimes K_Z^{-1} \rightarrow \dots \\ \rightarrow \Omega_Y^{n-1} \otimes K_Z^{-1} \rightarrow \Theta_Z \rightarrow 0. \end{aligned} \quad (1.13)$$

Now apply Lemma 1.4 and observe that

$$\begin{aligned} \ker(H^n(Z, S^{n-1}N_Z^* \otimes K_Z^{-1})) \\ \rightarrow H^n(Z, S^{n-2}N_Z^* \otimes \Omega_Y^1 \otimes K_Z^{-1}) \xrightarrow[\cong]{d_{n-1}} H^1(Z, \Theta_Z) \end{aligned}$$

Now (1.12) follows by Serre Duality, completing the proof of Lemma 1.10.

LEMMA 1.14: *Let Y be a smooth $(n+1)$ -fold, and $L \rightarrow Y$ a sufficiently ample analytic line bundle. If Z is a smooth reduced element of the linear system $|L|$, then for $n \geq 1$,*

$$H^1(Z, \Theta_Z) \simeq \left(\frac{H^0(Z, L^{(n-1)} \otimes K_Z^2)}{\text{im } H^0(Z, L^{(n-2)} \otimes \Theta_Y \otimes K_Z^2)} \right)^* \quad (1.15)$$

and there is a short exact sequence

$$\begin{aligned} 0 \rightarrow \left(\frac{H^1(Z, \Omega_Z^{n-1})}{\text{im } H^1(Z, \Omega_Y^{n-1} \otimes \mathcal{O}_Z)} \right) \rightarrow \left(\frac{H^0(Z, L^{(n-1)} \otimes K_Z)}{\text{im } H^0(Z, L^{(n-2)} \otimes \Theta_Y \otimes K_Z)} \right)^* \\ \rightarrow (\ker H^2(Z, \Omega_Y^{n-1} \otimes \mathcal{O}_Z) \rightarrow H^2(Z, \Omega_Z^{n-1})) \rightarrow 0 \end{aligned} \quad (1.16)$$

Furthermore, there is a commutative diagram

$$\begin{array}{ccc} H^1(Z, \Omega_Z^{n-1})^* \otimes H^0(Z, K_Z) & \leftarrow & H^0(Z, L^{(n-1)} \otimes K_Z) \otimes H^0(Z, K_Z) \\ \downarrow & & \downarrow \\ H^1(Z, \Theta_Z)^* & \leftarrow & H^0(Z, L^{(n-1)} \otimes K_Z^2) \end{array}$$

where the horizontal maps are induced by (1.15) and (1.16), the vertical map on the left is the dual of the highest piece

$$H^1(Z, \Theta_Z) \xrightarrow{P^*} H^0(Z, K_Z)^* \otimes H^1(Z, \Omega_Z^{n-1}) \quad (1.18)$$

of the derivative of the period map, and the vertical map on the right is multiplication.

PROOF: From the restriction sequence

$$0 \rightarrow \mathcal{O}_Y(L^{-1}) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_Z \rightarrow 0 \quad (1.19)$$

and the isomorphism of bundles

$$N_Z \simeq L|_Z \quad (1.20)$$

we have a long exact sequence

$$\begin{aligned} \dots &\rightarrow H^i(Y, \Omega_Y^i \otimes L^m) \rightarrow H^i(Z, \Omega_Z^i \otimes L^m \otimes \mathcal{O}_Z) \\ &\rightarrow H^{i+1}(Y, \Omega_Y^i \otimes L^{(m-1)}) \rightarrow \dots \end{aligned} \quad (1.21)$$

By the Kodaira Vanishing Theorem, we conclude that (1.6) holds when L is sufficiently ample.

By the adjunction formula

$$K_Z \simeq K_Y \otimes L|_Z \quad (1.22)$$

and (1.19) we have a long exact sequence

$$\begin{aligned} \dots &\rightarrow H^i(Y, \Omega_Y^i \otimes K_Y^{-1} \otimes L^{-m-1}) \rightarrow H^i(Z, \Omega_Z^i \otimes S^m N_Z^* \otimes K_Z^{-1}) \\ &\rightarrow H^{i+1}(Y, \Omega_Y^i \otimes K_Y^{-1} \otimes L^{-m-2}) \rightarrow \dots \end{aligned} \quad (1.23)$$

so (1.11) holds when L is sufficiently ample. Thus (1.15) and (1.16) follow from Lemmas (1.5) and (1.10). Since multiplication by $H^0(Z, K_Z)$ commutes with all the differentials of the spectral sequence, we conclude that (1.17) commutes, proving the lemma.

LEMMA 1.24: *Let Y be a smooth $(n+1)$ -fold with $n \geq 1$ and $L \rightarrow Y$ a sufficiently ample analytic line bundle. Then the Local Torelli Theorem holds for any smooth, reduced $Z \in |L|$ if the multiplication map*

$$H^0(Y, K_Y \otimes L^n) \otimes H^0(Y, K_Y \otimes L) \rightarrow H^0(Y, K_Y^2 \otimes L^{(n+1)}) \quad (1.25)$$

is surjective.

PROOF: To show that the map

$$H^1(Z, \Theta_Z) \xrightarrow{P^*} H^0(Z, K_Z)^* \otimes H^1(Z, \Omega_Z^{n-1})$$

is injective, it is equivalent to show that the dual map

$$H^1(Z, \Omega_Z^{n-1})^* \otimes H^0(Z, K_Z) \rightarrow H^1(Z, \Theta_Z)^*$$

is surjective. By Lemma 1.14, it is enough to show that

$$H^0(Z, K_Z \otimes L^{(n-1)}) \otimes H^0(Z, K_Z) \rightarrow H^0(Z, K_Z^2 \otimes L^{(n-1)}) \tag{1.26}$$

is surjective. From the restriction sequence (1.19), we have the long exact sequence

$$\begin{aligned} \dots \rightarrow H^0(Y, K_Y^2 \otimes L^{(n+1)}) &\rightarrow H^0(Z, K_Z^2 \otimes L^{(n-1)}) \\ &\rightarrow H^1(Y, K_Y^2 \otimes L^n) \rightarrow \dots \end{aligned} \tag{1.27}$$

By the Kodaira Vanishing Theorem,

$$H^1(Y, K_Y^2 \otimes L^n) = 0$$

for L sufficiently ample, and thus the map

$$H^0(Y, K_Y^2 \otimes L^{(n+1)}) \rightarrow H^0(Z, K_Z^2 \otimes L^{(n-1)})$$

is surjective. The lemma follows.

LEMMA 1.28: *Let Y be a smooth $(n + 1)$ -fold, E_1, E_2 analytic vector bundles over Y . For L a sufficiently ample line bundle on Y , the multiplication map*

$$H^0(Y, E_1 \otimes L^a) \otimes H^0(Y, E_2 \otimes L^b) \rightarrow H^0(Y, E_1 \otimes E_2 \otimes L^{a+b}) \tag{1.29}$$

is surjective when $a \geq 1$ and $b \geq 1$.

PROOF *: Let Δ be the diagonal on $Y \times Y$, and π_1, π_2 the canonical projections. We then have a commutative diagram

$$\begin{array}{ccc} H^0(Y \times Y, \pi_1^*(E_1 \otimes L^a) \otimes \pi_2^*(E_2 \otimes L^b)) & \rightarrow & H^0(Y \times Y, \mathcal{O}_\Delta(\pi_1^*(E_1 \otimes L^a) \otimes \pi_2^*(E_2 \otimes L^b))) \\ \cong & & \cong \\ H^0(Y, E_1 \otimes L^a) \otimes H^0(Y, E_2 \otimes L^b) & \rightarrow & H^0(Y, E_1 \otimes E_2 \otimes L^{a+b}) \end{array} \tag{1.30}$$

* This argument was suggested by Ron Donagi and replaces an earlier, more complicated proof.

From the restriction sequence

$$0 \rightarrow \mathcal{I}_\Delta \rightarrow \mathcal{O}_{Y \times Y} \rightarrow \mathcal{O}_\Delta \rightarrow 0$$

we see that to prove the surjectivity of (1.29), it suffices to prove

$$H^1(Y \times Y, \mathcal{I}_\Delta \otimes \pi_1^*(E_1 \otimes L^a) \otimes \pi_2^*(E_2 \otimes L^b)) = 0 \tag{1.31}$$

Since $\pi_1^*L \otimes \pi_2^*L$ is sufficiently ample on $Y \times Y$ if L is sufficiently ample on Y , (1.31) holds when $a \geq 1$ and $b \geq 1$. \square

We conclude this section by noting that Theorem 0.1 is a direct consequence of Lemmas (1.24) and (1.28). The need to take $L \gg 0$ arose in satisfying (1.6), (1.11), and (1.25). In explicit situations, e.g. when Y is a complete intersection in \mathbb{P}_N , these may be verified directly to yield many known results.

§2. A global Torelli theorem

Let Y be a smooth algebraic variety of dimension $n + 1$ and $L \rightarrow Y$ a sufficiently ample line bundle. Let

$$\begin{cases} s \in H^0(Y, L) \\ Z = \text{div } s, \quad Z \text{ smooth and reduced.} \end{cases} \tag{2.1}$$

We have

$$T_s(\mathbb{P}(H^0(Y, L))) \simeq H^0(Y, L)/(s) \tag{2.2}$$

and let T be the image of the Kodaira-Spencer map

$$T_s(\mathbb{P}(H^0(Y, L))) \rightarrow H^1(Z, \Theta_Z). \tag{2.3}$$

The first derivative of the period map at Z has as its leading piece

$$T \xrightarrow{P_{*,Z}} \text{Hom}(H^0(Z, K_Z), H^1(Z, \Omega_Z^{n-1})) \tag{2.4}$$

which we may alternatively regard as an element

$$\hat{P}_{*,Z} \in T^* \otimes H^0(Z, K_Z)^* \otimes H^1(Z, \Omega_Z^{n-1}). \tag{2.5}$$

Using the notations introduced in the introduction, in order to show that P has degree one on

$$\mathcal{M} = \mathbb{P}(H^0(Y, L))_{ns}/G \tag{2.6}$$

it is sufficient to show that

$$\text{For } Z \in |L| \text{ generic, the GL-equivalence class of } \hat{P}_{*,Z} \text{ determines } Z. \tag{2.7}$$

Let Σ_Y denote the *first prolongation bundle* (see [A-C-G-H]) of L ; it sits in the exact sequence

$$0 \rightarrow \mathcal{O}_Y \rightarrow \Sigma_Y \rightarrow \Theta_Y \rightarrow 0 \tag{2.8}$$

with extension class

$$-c_1(L) \in H^1(Y, \Omega_Y^1).$$

We can differentiate s to obtain

$$\widetilde{ds} \in H^0(Y, \Sigma_Y^* \otimes L). \tag{2.9}$$

For any coherent analytic sheaf $\mathcal{F} \rightarrow Y$, we obtain a map

$$H^0(Y, \mathcal{F} \otimes \Sigma_Y \otimes L^{-1}) \xrightarrow{\widetilde{ds}} H^0(Y, \mathcal{F}) \tag{2.10}$$

and we define the *pseudo-Jacobian system*

$$J_{\mathcal{F}} \subseteq H^0(Y, \mathcal{F}) \tag{2.11}$$

to be the image of the map (2.10). If $\dim Y \geq 2$, then for L sufficiently ample, from the exact sequence

$$H^0(Y, \Theta_Y) \rightarrow H^0(Z, \Theta_Y) \rightarrow H^1(Y, \Theta_Y \otimes L^{-1})$$

and the Kodaira Vanishing Theorem we have that

$$H^0(Y, \Theta_Y) \rightarrow H^0(Z, \Theta_Y). \tag{2.12}$$

From the exact sequence

$$H^0(Z, \Theta_Y) \xrightarrow{\widetilde{ds}|_Z} H^0(Z, L) \rightarrow H^1(Z, \Theta_Z) \tag{2.13}$$

we conclude from (2.3) that

$$T \simeq \frac{H^0(Y, L)}{J_L} \tag{2.14}$$

for L sufficiently ample.

It is useful to have the following *Duality Theorem* or *Generalized Macaulay's Theorem*:

THEOREM 2.15: *For Y a smooth $(n+1)$ -fold, $E \rightarrow Y$ a fixed analytic vector bundle, and $L \rightarrow Y$ a sufficiently ample line bundle,*

$$\frac{H^0(Y, K_Y^2 \otimes L^{(n+2)})}{J_{K_Y^2 \otimes L^{(n+2)}}} \simeq \mathbf{C} \quad (2.16)$$

and the map

$$\begin{aligned} & \frac{H^0(Y, E \otimes L^a)}{J_E} \otimes \frac{H^0(Y, E^* \otimes K_Y^2 \otimes L^{(n+2-a)})}{J_{E^* \otimes K_Y^2 \otimes L^{(n+2-a)}}} \\ & \rightarrow \frac{H^0(Y, K_Y^2 \otimes L^{(n+2)})}{J_{K_Y^2 \otimes L^{(n+2)}}} \simeq \mathbf{C} \end{aligned} \quad (2.17)$$

is a perfect pairing provided

$$H^{a-1}(Y, E \otimes \Lambda^a \Sigma_Y) = H^a(Y, E \otimes \Lambda^a \Sigma_Y) = 0 \quad \text{or} \quad a = 0. \quad (2.18)$$

If only

$$H^{a-1}(Y, E \otimes \Lambda^a \Sigma_Y) = 0 \quad (2.19)$$

then the pairing (2.17) has no left kernel.

PROOF OF 2.15: Using

$$\widetilde{ds} \in H^0(Y, \Sigma_Y^* \otimes L)$$

we may construct the Koszul complex

$$\begin{aligned} 0 \rightarrow \Lambda^{n+2} \Sigma_Y \otimes L^{-(n+2)} & \xrightarrow{\widetilde{ds}} \Lambda^{n+1} \Sigma_Y \otimes L^{-(n+1)} \xrightarrow{\widetilde{ds}} \dots \\ & \xrightarrow{\widetilde{ds}} \Sigma_Y \otimes L^{-1} \xrightarrow{\widetilde{ds}} \mathcal{O}_Y \rightarrow 0. \end{aligned} \quad (2.20)$$

Tensoring (2.20) with $E \otimes L^a$ and applying Lemma 1.4, we obtain a spectral sequence abutting to zero. For L sufficiently ample, using the hypothesis (2.18), we get that

$$\begin{aligned} & \ker(H^{n+1}(Y, \Lambda^{n+2} \Sigma_Y \otimes E \otimes L^{a-(n+2)}) \\ & \rightarrow H^{n+1}(Y, \Lambda^{n+1} \Sigma_Y \otimes E \otimes L^{a-(n+1)}) \\ & \xrightarrow{d_{n+2}} \frac{H^0(Y, E \otimes L^a)}{J_{E \otimes L^a}} \\ & = \end{aligned}$$

and thus

$$\left(\frac{H^0(Y, E^* \otimes K_Y^2 \otimes L^{(n+2)-a})}{J_{E^* \otimes K_Y^2 \otimes L^{(n+2)-a}}} \right)^* \simeq \frac{H^0(Y, E \otimes L^a)}{J_{E \otimes L^a}}.$$

When $E = 1$, $a = 0$, this gives (2.16). Moreover, because multiplication with $H^0(Y, E^* \otimes K_Y^2 \otimes L^{(n+2)-a})$ gives a map of the entire spectral sequence, we conclude that (2.17) gives the duality. The case where we have only the hypothesis (2.19) is similar. This proves (2.15).

We next generalize *Donagi's Symmetrizer Lemma* with the following two results:

THEOREM 2.21 (Generalized Symmetrizer Lemma): *Let Y be a smooth $n + 1$ fold, $L \rightarrow Y$ an analytic line bundle, $M \rightarrow Y$ an analytic line bundle with $|M|$ base-point free, and $E \rightarrow Y$ an analytic vector bundle. Then for L sufficiently ample, the Koszul complex*

$$\begin{aligned} 0 \rightarrow \frac{H^0(Y, E \otimes L)}{J_{E \otimes L}} &\rightarrow H^0(Y, M)^* \otimes \frac{H^0(Y, E \otimes M \otimes L)}{J_{E \otimes M \otimes L}} \\ &\rightarrow \Lambda^2 H^0(Y, M)^* \otimes \frac{H^0(Y, E \otimes M^2 \otimes L)}{J_{E \otimes M^2 \otimes L}} \end{aligned} \tag{2.22}$$

is exact as far as written above, provided that

$$H^1(Y, E \otimes \Sigma) \rightarrow H^0(Y, M)^* \otimes H^1(Y, E \otimes M \otimes \Sigma) \tag{2.23}$$

is injective.

THEOREM 2.24: *For L sufficiently ample, the Koszul complex*

$$\begin{aligned} 0 \rightarrow H^0(Y, K_Y) &\rightarrow \left(\frac{H^0(Y, L)}{J_L} \right)^* \otimes \frac{H^0(Z, K_Z)}{H^0(Y, \Omega_Y^n)} \\ &\rightarrow \Lambda^2 \left(\frac{H^0(Y, L)}{J_L} \right)^* \otimes H^1(Z, \Omega_Z^{n-1}) \end{aligned} \tag{2.25}$$

is exact as far as written, and

$$\begin{aligned} 0 \rightarrow \frac{H^0(Y, L \otimes K_Y^{-1})}{J_{L \otimes K_Y^{-1}}} &\rightarrow H^0(Y, K_Y)^* \otimes \frac{H^0(Y, L)}{J_L} \\ &\rightarrow \Lambda^2 H^0(Y, K_Y)^* \otimes \frac{H^0(Z, K_Z)}{H^0(Y, \Omega_Y^n)} \end{aligned} \tag{2.26}$$

is exact as far as written provided that $|K_Y|$ is base-point free and $\dim \varphi_K(Y) \geq 2$.

Proof of Theorems (2.21) and (2.24): Using the Generalized Macaulay's Theorem (2.15) and its proof, we have for L sufficiently ample that the sequence (2.22) is dual to

$$\begin{aligned}
 & \Lambda^2 H^0(Y, M) \otimes \frac{H^0(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-2})}{J_{K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-2}} + H^1(Y, E \otimes M^2 \otimes \Sigma)^*} \\
 & \rightarrow H^0(Y, M) \otimes \frac{H^0(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-1})}{J_{K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-1}} + H^1(Y, E \otimes M \otimes \Sigma)^*} \\
 & \rightarrow \frac{H^0(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1})}{J_{K_Y^2 \otimes L^{(n+1)} \otimes E^{-1}} + H^1(Y, E \otimes \Sigma)^*} \rightarrow 0
 \end{aligned} \tag{2.27}$$

The sequence

$$\begin{aligned}
 & \Lambda^2 H^0(Y, M) \otimes H^0(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-2}) \\
 & \rightarrow H^0(Y, M) \otimes H^0(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-1}) \\
 & \rightarrow H^0(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1}) \rightarrow 0
 \end{aligned} \tag{2.28}$$

is exact by considering the Koszul complex

$$\begin{aligned}
 & \dots \rightarrow \Lambda^2 H^0(Y, M) \otimes K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-2} \\
 & \rightarrow H^0(Y, M) \otimes K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-1} \\
 & \rightarrow K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \rightarrow 0
 \end{aligned}$$

and applying Lemma 1.4 and the Kodaira Vanishing Theorem. Likewise,

$$H^0(Y, M) \otimes J_{K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-1}} \rightarrow J_{K_Y^2 \otimes L^{(n+1)} \otimes E^{-1}}$$

by a similar argument, while

$$H^0(Y, M) \otimes H^1(Y, E \otimes M \otimes \Sigma)^* \rightarrow H^1(Y, E \otimes \Sigma)$$

by the dual of (2.23).

We are now done by

LEMMA 2.29: Let V be a vector space, $S(V)$ the symmetric algebra on V , and $A = \bigoplus_{q \in \mathbb{Z}} A_q \subseteq B = \bigoplus_{q \in \mathbb{Z}} B_q$ graded $S(V)$ -modules. The Koszul complex

$$\Lambda^2 V \otimes (B_q/A_q) \xrightarrow{F} V \otimes (B_{q+1}/A_{q+1}) \xrightarrow{G} B_{q+2}/A_{q+2} \rightarrow 0 \quad (2.30)$$

is exact as far as written provided that

$$\Lambda^2 V \otimes B_q \xrightarrow{\tilde{F}} V \otimes B_{q+1} \xrightarrow{\tilde{G}} B_{q+2} \rightarrow 0 \quad (2.31)$$

is exact as far as written and

$$V \otimes A_{q+1} \rightarrow A_{q+2}. \quad (2.32)$$

PROOF OF (2.29): As \tilde{G} is surjective, so is G . If $\alpha \in \ker G$, then choosing $\tilde{\alpha} \in V \otimes B_{q+1}$ representing α ,

$$\tilde{G}(\tilde{\alpha}) \in A_{q+2}$$

By (2.32), we may modify $\tilde{\alpha}$ to $\tilde{\tilde{\alpha}}$ representing α so that

$$\tilde{G}(\tilde{\tilde{\alpha}}) = 0$$

So

$$\tilde{\tilde{\alpha}} = \tilde{F}(\tilde{\beta})$$

for some $\tilde{\beta} \in \Lambda^2 V \otimes B_q$, and then

$$\alpha = F(\beta)$$

where β is the projection of $\tilde{\beta}$ to $\Lambda^2 V \otimes (B_q/A_q)$. \square

We have now proved (2.21). To prove (2.24), we will prove first the exactness of the sequence

$$\begin{aligned} 0 \rightarrow H^0(Y, K_Y) &\rightarrow \left(\frac{H^0(Y, L)}{J_L} \right)^* \otimes \frac{H^0(Z, K_Z)}{H^0(Y, \Omega_Y^n)} \\ &\rightarrow \Lambda^2 \left(\frac{H^0(Y, L)}{J_L} \right)^* \otimes \frac{H^1(Z, \Omega_Z^{n-1})}{H^1(Z, \Omega_Y^{n-1} \otimes \mathcal{O}_Z)} \end{aligned}$$

which is stronger than showing exactness for (2.25). Using the dualities of §1, and the fact $H^0(Y, \Omega_Y^n) \rightarrow H^0(Z, \Omega_Y^n \otimes \mathcal{O}_Z)$ if $\dim Y \geq 2$ if $L \gg 0$

and the generalized Macauley's Theorem, we may dualize the above sequence to

$$\begin{aligned} & \Lambda^2 \left(\frac{H^0(Y, L)}{J} \right) \otimes \frac{H^0(Y, K_Y \otimes L^n)}{J_{K_Y \otimes L^n} + \text{more}} \\ & \rightarrow \frac{H^0(Y, L)}{J_L} \otimes \frac{H^0(Y, K_Y \otimes L^{(n+1)})}{J_{K_Y \otimes L^{(n+1)}} + \text{more}} \\ & \rightarrow \frac{H^0(Y, K_Y \otimes L^{(n+2)})}{J_{K_Y \otimes L^{(n+2)}}} \rightarrow 0 \end{aligned}$$

and we now proceed analogously to the preceding case, using Lemma 1.8, the Kodaira Vanishing Theorem, and Lemma 2.29. The one additional fact we require is that the sequence

$$\begin{aligned} & \Lambda^2 H^0(Y, L) \otimes H^0(Y, K_Y \otimes L^n) \\ & \rightarrow H^0(Y, L) \otimes H^0(Y, K_Y \otimes L^{(n+1)}) \\ & \rightarrow H^0(Y, K_Y \otimes L^{(n+2)}) \rightarrow 0 \end{aligned}$$

is exact as far as written for L sufficiently ample. This follows from Lemma 2.47, which we have put at the end of this section.

The dual of (2.26) is

$$\begin{aligned} & \Lambda^2 H^0(Y, K_Y) \otimes \frac{H^0(Y, K_Y \otimes L^{(n+1)})}{J_{K_Y \otimes L^{(n+1)}} + \text{more}} \\ & \rightarrow H^0(Y, K_Y) \otimes \frac{H^0(Y, K_Y^2 \otimes L^{(n+1)})}{J_{K_Y^2 \otimes L^{(n+1)}} + H^1(Y, \Sigma)^*} \\ & \rightarrow \frac{H^0(Y, K_Y^3 \otimes L^{(n+1)})}{J_{K_Y^3 \otimes L^{(n+1)}} + H^1(Y, \Sigma \otimes K_Y^{-1})^*} \rightarrow 0 \end{aligned}$$

and again we are done provided that

$$H^1(Y, \Sigma \otimes K_Y^{-1}) \rightarrow H^0(Y, K_Y)^* \otimes H^1(Y, \Sigma) \quad (2.33)$$

is injective. Applying Lemma 1.8 to the Koszul complex

$$0 \rightarrow \Sigma \otimes K_Y^{-1} \rightarrow H^0(Y, K_Y)^* \otimes \Sigma \rightarrow \Lambda^2 H^0(Y, K_Y)^* \otimes \Sigma \otimes K_Y \rightarrow \dots$$

we conclude that the injectivity of (2.33) is equivalent to proving

$$\begin{aligned} H^0(Y, K_Y)^* \otimes H^0(Y, \Sigma) &\rightarrow \Lambda^2 H^0(Y, K_Y)^* \otimes H^0(Y, \Sigma \otimes K_Y) \\ &\rightarrow \Lambda^3 H^0(Y, K_Y)^* \otimes H^0(Y, \Sigma \otimes K_Y^2) \end{aligned}$$

is exact at the middle term. From the exact sequence

$$0 \rightarrow \mathcal{O}_Y \rightarrow \Sigma_Y \rightarrow \Theta_Y \rightarrow 0$$

we have an exact sequence

$$0 \rightarrow H^0(Y, K_Y) \rightarrow H^0(Y, \Sigma \otimes K_Y) \rightarrow H^0(Y, \Omega_Y^n) \xrightarrow{\cup c_1(L)} H^1(Y, K_Y)$$

and the last map is an isomorphism by the Strong Lefschetz Theorem, so

$$H^0(Y, K_Y) \xrightarrow{\cong} H^0(Y, \Sigma \otimes K_Y)$$

and we are now reduced to showing

$$\begin{aligned} H^0(Y, K_Y)^* \otimes H^0(Y, \mathcal{O}_Y) &\rightarrow \Lambda^2 H^0(Y, K_Y)^* \otimes H^0(Y, K_Y) \\ &\rightarrow \Lambda^3 H^0(Y, K_Y)^* \otimes H^0(Y, K_Y^2) \end{aligned}$$

is exact at the middle term. However, for $\dim \varphi_{K_Y} \geq 2$ this is true by the $\mathcal{X}_{p,1}$ Theorem of [G]. \square

We have the following corollary of Theorem 2.21:

COROLLARY 2.34: *If $|K_Y|$ is base-point free and Y is of general type, then for any $m \geq 1$ the Koszul complex*

$$\begin{aligned} 0 \rightarrow \frac{H^0(Y, L \otimes K_Y^{-m})}{J_{L \otimes K_Y^{-m}}} &\rightarrow H^0(Y, K_Y)^* \otimes \frac{H^0(Y, L \otimes K_Y^{1-m})}{J_{L \otimes K_Y^{1-m}}} \\ &\rightarrow \Lambda^2 H^0(Y, K_Y)^* \otimes \frac{H^0(Y, L \otimes K_Y^{2-m})}{J_{L \otimes K_Y^{2-m}}} \end{aligned}$$

is exact as far as written for k sufficiently large if $\dim \varphi_{K_Y}(Y) \geq 2$.

PROOF: Using (2.21), we need only show that hypothesis (2.23) holds in this case, i.e. that

$$H^1(Y, K_Y^{-m} \otimes \Sigma) \rightarrow H^0(Y, K_Y)^* \otimes H^1(Y, K_Y^{1-m} \otimes \Sigma)$$

is injective. As seen above, this is equivalent to the sequence

$$\begin{aligned}
 & H^0(Y, K_Y)^* \otimes H^0(Y, K_Y^{1-m} \otimes \Sigma) \\
 & \rightarrow \Lambda^2 H^0(Y, K_Y)^* \otimes H^0(Y, K_Y^{2-m} \otimes \Sigma) \\
 & \rightarrow \Lambda^3 H^0(Y, K_Y)^* \otimes H^0(Y, K_Y^{3-m} \otimes \Sigma)
 \end{aligned} \tag{2.36}$$

being exact at the middle term. For $m = 1$, we have already proved this.

As is well known,

$$H^0(Y, \Theta_Y) = 0 \quad \text{if } Y \text{ is of general type}$$

and thus

$$H^0(Y, K_Y^{2-m} \otimes \Sigma) = 0 \quad \text{for } m \geq 3$$

and

$$H^0(Y, K_Y^{2-m} \otimes \Sigma) \simeq H^0(Y, \mathcal{O}_Y) \quad \text{for } m = 2.$$

We are done in case $m \geq 3$, while for $m = 2$ we are reduced to the exactness of

$$0 \rightarrow \Lambda^2 H^0(Y, K_Y)^* \otimes H^0(Y, \mathcal{O}_Y) \rightarrow \Lambda^3 H^0(Y, K_Y)^* \otimes H^0(Y, K_Y)$$

at the middle term, and this follows from the fact that Koszul map $\Lambda^2 V \rightarrow \Lambda^3 V \otimes V^*$ is injective for any vector space. \square

We are now ready to begin proving Theorem (0.3). The image of the derivative of the period map gives us the GL -equivalence class of the map

$$\frac{H^0(Y, L)}{J_L} \otimes H^0(Z, K_Z) \rightarrow H^1(Z, \Omega_Z^{n-1}). \tag{2.37}$$

We require

LEMMA 2.38: *The right kernel of (2.37) is the image of $H^0(Y, \Omega_Y^n)$ for L sufficiently ample.*

PROOF: We know by Hodge theory that the image of $H^0(Y, \Omega_Y^n)$ in $H^0(Z, K_Z)$ is invariant as we deform Z on Y . It remains to show that the map

$$\frac{H^0(Z, K_Z)}{H^0(Y, \Omega_Y^n)} \rightarrow \left(\frac{H^0(Y, L)}{J_L} \right)^* \otimes H^1(Z, \Omega_Z^{n-1})$$

is injective. A fortiori, it would be enough to show that

$$\frac{H^0(Z, K_Z)}{H^0(Y, \Omega_Y^n)} \rightarrow \left(\frac{H^0(Y, L)}{J_L} \right)^* \otimes \frac{H^1(Z, \Omega_Z^{n-1})}{H^1(Z, \Omega_Y^{n-1} \otimes \mathcal{O}_Z)}$$

is injective. By the results of §1, for L sufficiently ample this dualizes to the (quotiented) multiplication map

$$\frac{H^0(Y, L)}{J_L} \otimes \frac{H^0(Y, K_Y \otimes L^n)}{J_{K_Y \otimes L^n} + \text{more}} \rightarrow \frac{H^0(Y, K_Y \otimes L^{n+1})}{J_{K_Y \otimes L^{n+1}} + \text{more}}$$

which we must show is surjective. It suffices to show that

$$H^0(Y, L) \otimes H^0(Y, K_Y \otimes L^n) \rightarrow H^0(Y, K_Y \otimes L^{n+1})$$

This follows from Lemma 1.28, for L sufficiently ample. \square

Thus, from (2.37), we can construct the map

$$\frac{H^0(Y, L)}{J_L} \otimes \frac{H^0(Z, K_Z)}{H^0(Y, \Omega_Y^n)} \rightarrow H^1(Z, \Omega_Z^{n-1}).$$

From this, we can construct the second map in the sequence (2.25), and thus can reconstruct the GL -equivalence class of the map

$$H^0(Y, K_Y) \otimes \frac{H^0(Y, L)}{J_L} \rightarrow \frac{H^0(Z, K_Z)}{H^0(Y, \Omega_Y^n)}. \quad (2.39)$$

From (2.39), we can construct the second map in the sequence (2.26), and thus can reconstruct the vector space

$$\frac{H^0(Y, L \otimes K_Y^{-1})}{J_{L \otimes K_Y^{-1}}}$$

and the GL -equivalence class of the map

$$\frac{H^0(Y, L \otimes K_Y^{-1})}{J_{L \otimes K_Y^{-1}}} \otimes H^0(Y, K_Y) \rightarrow \frac{H^0(Y, L)}{J_L}.$$

For any m_0 chosen in advance, we can choose L sufficiently ample so that we can recover inductively using Corollary (2.24) the GL -equivalence classes of the maps

$$\frac{H^0(Y, L \otimes K_Y^{-m})}{J_{L \otimes K_Y^{-m}}} \otimes H^0(Y, K_Y) \rightarrow \frac{H^0(Y, L \otimes K_Y^{1-m})}{J_{L \otimes K_Y^{1-m}}} \quad (2.40)$$

for all $m \leq m_0$.

However,

$$J_{L \otimes K_Y^{-m}} \simeq H^0(Y, \Sigma \otimes K_Y^{-m}).$$

If K_Y is ample, we conclude that for some m_1 ,

$$J_{L \otimes K_Y^{-m}} = 0 \text{ for } m \geq m_1 \tag{2.41}$$

and the m_1 may be chosen independent of L . We now have the maps

$$H^0(Y, L \otimes K_Y^{-m}) \otimes H^0(Y, K_Y) \rightarrow H^0(Y, L \otimes K_Y^{1-m}) \tag{2.42}$$

for $m_0 \geq m \geq m_1 + 1$, where we can make m_0 as large as we like at the expense of our choice of how ample L must be. For

$$W \subseteq H^0(Y, K_Y)$$

a linear subspace with base locus B , we have by making L sufficiently ample that

$$H^0(Y, L \otimes K_Y^{-m}) \otimes W \twoheadrightarrow H^0(Y, L \otimes K_Y^{1-m} \otimes \mathcal{I}_B). \tag{2.43}$$

Thus we can detect from the map (2.42) which W 's have a base locus, and thus can determine the Chow form of $\varphi_{K_Y}(Y)$. Since $\varphi_{K_Y}(Y) \simeq Y$ by hypothesis, for each $p \in Y$ we can determine by (2.42) the subspaces

$$H^0(Y, L \otimes K_Y^{1-m} \otimes \mathcal{I}_p) \subset H^0(Y, L \otimes K_Y^{1-m})$$

for $m_0 \geq m \geq m_1 + 1$. From this, we can determine the Chow form of $\varphi_{L \otimes K_Y^{1-m}}(Y)$. We thus can reconstruct the map (2.42) not merely up to GL -equivalence, but up to the action of G . Now, operating our induction in reverse, we can construct the projections

$$H^0(Y, L \otimes K_Y^{-m}) \rightarrow \frac{H^0(Y, L \otimes K_Y^{-m})}{J_{L \otimes K_Y^{-m}}} \tag{2.44}$$

for all $m \leq m_0$, modulo the action of G . Thus in particular, we can construct

$$J_L \subset H^0(Y, L)$$

modulo the action of G .

We now wish to recover Z from J_L . In the case at hand, this is easy as $H^0(Y, \Theta_Y) = 0$; however, it is interesting to give the general argument.

To do this, we consider the group \tilde{G} consisting of pairs of analytic isomorphisms

$$\begin{array}{ccc} L & \xrightarrow{\tilde{u}} & L \\ \downarrow & & \downarrow \\ Y & \xrightarrow{u} & Y \end{array}$$

so that the diagram commutes and \tilde{u} is linear on each fiber. As

$$H^0(Y, \text{Hom}(L, L)) \simeq \mathbb{C}$$

there is a natural exact sequence of Lie groups

$$0 \rightarrow \mathbb{C}^* \rightarrow \tilde{G} \rightarrow G \rightarrow 0. \tag{2.45}$$

Furthermore, we may make the identifications of the tangent spaces at the identity

$$\begin{array}{ccc} T_e(\mathbb{C}^*) & \rightarrow & T_e(\tilde{G}) & \rightarrow & T_e(G) \\ & & \parallel & & \parallel \\ H^0(Y, \mathcal{O}_Y) & \rightarrow & H^0(Y, \Sigma_Y) & \rightarrow & \ker \left(\overbrace{H^0(Y, \Theta_Y) \rightarrow H^1(Y, \mathcal{O}_Y)}^{\cup_{c_1(L)}} \right) \end{array}$$

so that the above diagram commutes. Further, \tilde{G} acts on $H^0(Y, L)$ by pullback, so that the tangent space to the orbit of s is

$$T_s(\tilde{G}s) \simeq J_L. \tag{2.46}$$

Gives this, the argument given by Donagi [D] adapting techniques of Mather and Yau [M-Y] goes through verbatim. This completes the proof of Theorem (0.3), once we have shown the lemma needed to prove the exactness of (2.25).

LEMMA 2.47: *Let Y be a smooth $(n + 1)$ -fold, $E \rightarrow Y$ an analytic vector bundle. If $L \rightarrow Y$ is a sufficiently ample vector bundle, then for any $a \geq 1$, the Koszul sequence*

$$\begin{array}{ccc} \Lambda^2 H^0(Y, L) \otimes H^0(Y, E \otimes L^a) & \xrightarrow{\alpha_a} & H^0(Y, L) \otimes H^0(Y, E \otimes L^{a+1}) \\ & & \downarrow \beta_a \\ & & H^0(Y, E \otimes L^{a+2}) \rightarrow 0 \end{array} \tag{2.48}$$

is exact as far as written.

PROOF: Surjectivity of β_a follows from Lemma 1.28. Exactness at the middle term would follow from the surjectivity of the map

$$H^0(Y, L) \otimes \ker \beta_{a-1} \xrightarrow{\gamma_a} \ker \beta_a \tag{2.49}$$

defined by

$$\gamma_a \left(l \otimes \left(\sum_i s_i \otimes r_i \right) \right) = \sum_i s_i \otimes (lr_i)$$

where $l \in H^0(Y, L)$, s_0, \dots, s_r a basis for $H^0(Y, L)$ and $r_i \in H^0(Y, E \otimes L^a)$. To see this implication, we note that there is a commutative diagram

$$\begin{array}{ccc} & & H^0(Y, L) \otimes \ker \beta_{a-1} \\ & \delta_a \swarrow & \downarrow \gamma_a \\ \Lambda^2 H^0(Y, L) \otimes H^0(Y, E \otimes L^a) & \xrightarrow{\alpha_a} & \ker \beta_a \end{array}$$

where

$$\delta_a \left(l \otimes \left(\sum_i s_i \otimes r_i \right) \right) = \sum_i (l \wedge s_i) \otimes r_i.$$

Thus

$$\text{im } \gamma_a \subseteq \text{im } \alpha_a$$

so it will suffice for our purposes to show that γ_a is surjective.

On $Y \times Y$, let Δ be the diagonal and π_1, π_2 the canonical projections. From the exact sequence

$$0 \rightarrow \mathcal{O}_{Y \times Y}(-\Delta) \rightarrow \mathcal{O}_{Y \times Y} \rightarrow \mathcal{O}_\Delta \rightarrow 0 \tag{2.51}$$

tensored with $\pi_1^*(L) \otimes \pi_2^*(E \otimes L^{a+1})$, we conclude that

$$\ker \beta_a \simeq H^0(Y \times Y, \pi_1^*(L) \otimes \pi_2^*(E \otimes L^{a+1}) \otimes \mathcal{O}_{Y \times Y}(-\Delta)). \tag{2.52}$$

Further, γ_a in these terms is the map

$$\begin{aligned} & H^0(Y \times Y, \pi_2^*(L)) \otimes H^0(Y \times Y, \pi_1^*(L) \\ & \otimes \pi_2^*(E \otimes L^a) \otimes \mathcal{O}_{Y \times Y}(-\Delta)) \\ & \xrightarrow{\tilde{\gamma}_a} H^0(Y \times Y, \pi_1^*(L) \otimes \pi_2^*(E \otimes L^{a+1}) \otimes \mathcal{O}_{Y \times Y}(-\Delta)). \end{aligned} \tag{2.53}$$

On $Y \times Y \times Y$, let

$$\Delta_{i,j} = \{(y_1, y_2, y_3) \in Y \times Y \times Y \mid y_i = y_j\}$$

and let π_1, π_2, π_3 be the projections. We may rewrite $\tilde{\gamma}_a$ equivalently as the map

$$\begin{aligned} & H^0(Y \times Y \times Y, \pi_3^*(L)) \otimes H^0(Y \times Y \times Y, \pi_1^*(L)) \\ & \quad \otimes \pi_2^*(E \otimes L^a) \otimes \mathcal{O}_{Y \times Y \times Y}(-\Delta_{12}) \\ & \xrightarrow{\tilde{\gamma}_a} H^0(Y \times Y \times Y, \pi_1^*(L) \otimes \pi_2^*(E \otimes L^a) \otimes \pi_3^*(L) \\ & \quad \otimes \mathcal{O}_{Y \times Y \times Y}(-\Delta_{12}) \otimes \mathcal{O}_{\Delta_{23}}). \end{aligned} \tag{2.54}$$

From tensoring the restriction sequence for Δ_{23} on $Y \times Y \times Y$ appropriately, we see that $\tilde{\gamma}_a$ is surjective if

$$\begin{aligned} & H^1(Y \times Y \times Y, \pi_1(L) \otimes \pi_2(E \otimes L^a) \otimes \pi_3(L) \\ & \quad \otimes \mathcal{O}_{Y \times Y \times Y}(-\Delta_{12} - \Delta_{23})) = 0. \end{aligned} \tag{2.55}$$

For L sufficiently ample, (2.55) holds for all $a \geq 1$. This proves Lemma 2.47. \square

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