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THE PERIOD MAP FOR HYPERSURFACE SECTIONS OF HIGH DEGREE OF AN ARBITRARY VARIETY

Mark L. Green *

Introduction

In this paper, for a fixed projective variety $Y$, we will say that a property holds for sufficiently ample analytic line bundles $L \to Y$ if there exists an ample bundle $L_0 \to Y$ so that the property holds for all $L$ on $Y$ with $L > L_0$, i.e. $L \otimes L_0^{-1}$ ample. We will denote this by saying the property holds for $L \gg 0$.

We will prove two theorems:

**THEOREM 0.1:** Let $Y$ be a smooth complete algebraic variety of dimension $\geq 2$. Then for $L \to Y$ a sufficient ample line bundle, the Local Torelli Theorem is true for any smooth $Z$ in the linear system $|L|$.

**REMARK:** What we will actually show is that the map

$$H^1(Z, \Theta_Z) \overset{p^*}{\to} \text{Hom}(H^0(Z, K_Z), H^1(Z, \Omega_Z^{n-1}))$$

(0.2)

is injective, where $n = \dim Z$; this is one piece of the derivative of the period map. Note that we are considering all first order deformations of $Z$ and not just those arising by varying $Z$ to first order in the linear system $|L|$.

**THEOREM 0.3:** Let $Y$ be a smooth complete algebraic variety of dimension $\geq 2$, $L \to Y$ a sufficiently ample line bundle. Assume $K_Y$ is very ample. Let

$$G = \{ f \in \text{Aut}_{\text{hol}}(Y) | f^*(L) = L \}$$

(0.4)

Then the period map has degree 1 on its domain in

$$\mathbb{P}(H^0(Y, L))/G.$$
What we will actually show is stronger than (0.3). To state exactly what is proved, we make the following definition:

**Definition 0.5:** Let $V_1, \ldots, V_k$ be vector spaces. We say that two elements

$$\alpha, \beta \in V_1 \otimes V_2 \otimes \ldots \otimes V_k$$

are $GL$-equivalent if they lie in the same orbit of $GL(V_1) \times \ldots \times GL(V_k)$.

Let

$$T \subseteq H^1(Z, \Theta_Z)$$

be the image of $H^0(Y, L)$. The highest piece of the derivative of the period map $P_{*, Z}$ may be regarded as an element

$$P_{*, Z} \in T^* \otimes H^0(Z, K_Z)^* \otimes H^1(Z, \Omega_Z^{n-1})$$

Let

$$\mathcal{P}(H^0(Y, L))_{ns}/G$$

be the image of $H^0(Y, L)$. The highest piece of the derivative of the period map $P_{*, Z}$ may be regarded as an element

$$P_{*, Z} \in T^* \otimes H^0(Z, K_Z)^* \otimes H^1(Z, \Omega_Z^{n-1})$$

be the image of $H^0(Y, L)$. The highest piece of the derivative of the period map $P_{*, Z}$ may be regarded as an element

Then what will actually be shown is that

$$\text{If } |K_Y| \text{ is very ample, then for } L \text{ a sufficiently ample line bundle, the map}$$

$$\mathcal{P}(H^0(Y, L))_{ns}/G$$

is injective.

These theorems settle a conjecture in [C-G-Gr-H]; the method of proof of (0.1) is essentially a resurgence of an idea that occurs there. The proof of (0.3) is inspired by Donagi's proof of a similar result for hypersurfaces in projective space [D].

The author is grateful to Ron Donagi and Phillip Griffiths for their help and encouragement.

§1. Hodge theory on a hypersurface of high degree

Given a short exact sequence of vector spaces

$$0 \to A \to B \to C \to 0$$
there is a long exact sequence

\[ 0 \to S^k A \to B \otimes S^{k-1} A \to \ldots \to \Lambda^{k-1} B \otimes A \to \Lambda^k B \to \Lambda^k C \to 0 \quad (1.1) \]

for any \( k \geq 1 \). If \( Y \) is a compact complex manifold of dimension \( m \) and \( Z \subset Y \) is a smooth submanifold of dimension \( n \) with normal bundle \( N_Z \) in \( Y \), we have a short exact sequence

\[ 0 \to N^*_Z \to \Omega^1_Y \otimes \mathcal{O}_Z \to \Omega^1_Z \to 0. \quad (1.2) \]

Thus we have, for any \( p \geq 1 \), long exact sequences

\[ 0 \to S^p N^*_Z \to \ldots \to \Omega^{p-1}_Y \otimes N^*_Z \to \Omega^p_Y \otimes \mathcal{O}_Z \to \Omega^p_Z \to 0. \quad (1.3) \]

The exact sequences (1.3) turn out to be quite useful in computations whenever we have an explicit form for \( N_Z \), most notably in the case of complete intersections. To use these sequences, we need:

**Lemma 1.4:** Let

\[ 0 \to F_1 \xrightarrow{f_1} F_2 \xrightarrow{f_2} \ldots \xrightarrow{f_{k-1}} F_k \to 0 \]

be an exact sequence of vector bundles on a compact complex manifold \( Z \). Then there is a spectral sequence abutting to zero with

\[ E_1^{p, q} = H^q(Z, F_p) \]

\[ E_2^{p, q} = \frac{\ker \left( H^q(Z, F_p) \xrightarrow{f_p} H^q(Z, F_{p+1}) \right)}{\text{im} \left( H^q(Z, F_{p-1}) \xrightarrow{f_{p-1}} H^q(Z, F_p) \right)} \]

**Proof:** Consider the bigraded complex

\[ B^{p, q} = \mathcal{A}^{0, q}(Z, F_p) \]

where \( \mathcal{A}^{0, q} \) denotes \( \mathcal{C}^{\infty} (0, q) \)-forms, with maps

\[ B^{p, q} \xrightarrow{d} B^{p+1, q} \quad d = f_p^* \]

\[ B^{p, q} \xrightarrow{\delta} B^{p, q+1} \quad \delta = \delta. \]
There is then (see [G-H]) a pair of spectral sequences \( E^{p,q}_\ast \) and \( E^{\prime p,q}_\ast \) having the same abutment, with

\[
E^{p,q}_1 = H^p_d(B^{\cdot\cdot\cdot}) \quad \text{and} \quad E^{\prime p,q}_1 = H^p_d(B^{\cdot\cdot\cdot})
\]

The rows of \( B^{\cdot\cdot\cdot} \) are exact, so

\[
E^{p,q}_1 = 0 \quad \text{for all} \quad p, q.
\]

Thus the spectral sequence \( E^{p,q}_\ast \) abuts to zero, and hence so does \( E^{\prime p,q}_\ast \). Furthermore,

\[
E^{\prime p,q}_2 = H^q(Z, F_p)
\]

This completes the proof of the lemma. \( \square \)

**Lemma 1.5:** Let \( Y \) be a compact Kähler manifold and \( Z \subset Y \) a complex submanifold of dimension \( n \). If

\[
H^i(Z, \Omega^j_Y \otimes S^m\mathcal{N}_Z^\ast) = 0 \quad \text{for all} \quad i < n, 0 \leq j \leq n, 1 \leq m \leq n
\]

then

\[
H^{p,q}(Z) = H^q(Z, \Omega^p_Y \otimes \mathcal{O}_Z) \quad \text{if} \quad p + q < n
\]

and there is a short exact sequence

\[
0 \to \left( \frac{H^{n-p}(Z, \Omega^n_Y)}{\text{im} H^{n-p}(Z, \Omega^n_Y)} \right) \to \left( \frac{H^0(Z, S^p\mathcal{N}_Z \otimes K_Z)}{\text{im} H^0(Z, S^{p-1}\mathcal{N}_Z \otimes \Theta_Y \otimes K_Z)} \right) \ast \to \left( \ker H^{n+1-p}(Z, \Omega^n_Y \otimes \mathcal{O}_Z) \right) \to H^{n+1-p}(Z, \Omega^n_Y) \to 0.
\]

**Proof:** We apply Lemma 1.4 to the exact sequence (1.3), taking \( p = \bar{p} \).
We obtain a spectral sequence which abuts to zero whose $E_1$ term looks as follows:

\[
\begin{array}{cccccccc}
\text{n} & n+1-p & \cdots & \cdots & \cdots & \cdots & \cdots & \text{p+1} & \text{p} \\
\text{q} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{array}
\]

The only non-zero differentials emerging from the position $(0, n)$ are $d_1$, $d_{p-1}$, and $d_p$. The only non-zero differentials whose target is the position $(p+1, n-p)$ are $d_1$ and $d_p$. We thus obtain maps

\[
\begin{align*}
\text{ker}(H^n(Z, S^p N_Z^2)) & \to H^n(Z, S^{p-1} N_Z^2 \otimes \Omega_Y^1) \\
\text{ker}(H^{n+1-p}(Z, \Omega_Y^p \otimes \mathcal{O}_Z)) & \to H^{n+1-p}(Z, \Omega_X^p) \\
\text{ker } d_{p-1} & \to \frac{H^{n-p}(Z, \Omega_Y^p)}{\text{im } H^{n-p}(Z, \Omega_Y^p \otimes \mathcal{O}_Z)}
\end{align*}
\]

where the second map is an isomorphism because the spectral sequence abuts to zero. Using Serre Duality, (1.9) gives (1.8).

There are no non-zero differentials other than $d_1$ coming into the positions $(\bar{p}, \bar{q})$ and $(\bar{p}+1, \bar{q})$ if $\bar{q} < n - \bar{p}$. This shows (1.7) and completes the proof of Lemma 1.5. □

A result similar to this is:

**Lemma 1.10:** Let $Y$ be a compact Kähler manifold and $Z \subset Y$ a complex submanifold of dimension $n$. If

\[
H^i(Z, \Omega_Y^j \otimes S^m N_Z^2 \otimes K_Z^{-1}) = 0
\]

for $0 < i < n$, $1 \leq j \leq n$, $1 \leq m \leq n - 2$

then

\[
H^1(Z, \Theta_Z) = \left( \frac{H^0(Z, S^{n-1} N_Z \otimes K_Z^2)}{\text{im } H^0(Z, S^{n-2} N_Z \otimes \Theta_Y \otimes K_Z^2)} \right)^* \quad (1.12)
\]

**Proof:** We take the exact sequence (1.3) for $p = n - 1$ and tensor with
This yields the exact sequence
\[ 0 \to S^{n-1}_Z \otimes K_z^{-1} \to S^{n-2}_Z \otimes \Omega^1_Y \otimes K_z^{-1} \to \ldots \]
\[ \to \Omega^{-1}_Y \otimes K_z^{-1} \to \Theta_z \to 0. \tag{1.13} \]

No apply Lemma 1.4 and observe that
\[
\ker \left( H^n(Z, S^{n-1}_Z \otimes K_z^{-1}) \right)
\to H^n(Z, S^{n-2}_Z \otimes \Omega^1_Y \otimes K_z^{-1}) \xrightarrow{d_{n-1}} H^1(Z, \Theta_z)
\]

Now (1.12) follows by Serre Duality, completing the proof of Lemma 1.10.

**Lemma 1.14:** Let \( Y \) be a smooth \((n + 1)\)-fold, and \( L \to Y \) a sufficiently ample analytic line bundle. If \( Z \) is a smooth reduced element of the linear system \(|L|\), then for \( n \geq 1 \),
\[
H^1(Z, \Theta_z) \simeq \left( \frac{H^0(Z, L^{(n-1)} \otimes K_z^2)}{\text{im} \ H^0(Z, L^{(n-2)} \otimes \Theta_Y \otimes K_z^2)} \right)^* \tag{1.15}
\]
and there is a short exact sequence
\[
0 \to \left( \frac{H^1(Z, \Omega^{-1}_Z)}{\text{im} \ H^1(Z, \Omega^{-1}_Y \otimes \Theta_z)} \right) \to \left( \frac{H^0(Z, L^{(n-1)} \otimes K_z)}{\text{im} \ H^0(Z, L^{(n-2)} \otimes \Theta_Y \otimes K_z)} \right)^* \to \ker H^2(Z, \Omega^{-1}_Z \otimes \Theta_z) \to H^2(Z, \Omega^{-1}_Z) \to 0 \tag{1.16}
\]

Furthermore, there is a commutative diagram
\[
\begin{array}{ccc}
H^1(Z, \Omega^{-1}_Z)^* \otimes H^0(Z, K_z) & \leftarrow & H^0(Z, L^{(n-1)} \otimes K_z) \otimes H^0(Z, K_z) \\
H^1(Z, \Theta_z)^* & \downarrow & H^0(Z, L^{(n-1)} \otimes K_z^2) \\
& & \\
\end{array}
\]

where the horizontal maps are induced by (1.15) and (1.16), the vertical map on the left is the dual of the highest piece
\[
H^1(Z, \Theta_z)^* \to H^0(Z, K_z)^* \otimes H^1(Z, \Omega^{-1}_Z) \tag{1.18}
\]
of the derivative of the period map, and the vertical map on the right is multiplication.
PROOF: From the restriction sequence
\[ 0 \to \mathcal{O}_Y(L^{-1}) \to \mathcal{O}_Y \to \mathcal{O}_Z \to 0 \] (1.19)
and the isomorphism of bundles
\[ N_Z \cong L|_Z \] (1.20)
we have a long exact sequence
\[ \ldots \to H^i(Y, \Omega_Y^i \otimes L^m) \to H^i(Z, \Omega_Y^i \otimes L^m \otimes \mathcal{O}_Z) \to \]
\[ \to H^{i+1}(Y, \Omega_Y^i \otimes L^{(m-1)}) \to \ldots \] (1.21)
By the Kodaira Vanishing Theorem, we conclude that (1.6) holds when \( L \) is sufficiently ample.

By the adjunction formula
\[ K_Z \cong K_Y \otimes L|_Z \] (1.22)
and (1.19) we have a long exact sequence
\[ \ldots \to H^i(Y, \Omega_Y^i \otimes K_Y^{-1} \otimes L^{-m}) \to H^i(Z, \Omega_Y^i \otimes S^mN_Z^* \otimes K_Z^{-1}) \to \]
\[ \to H^{i+1}(Y, \Omega_Y^i \otimes K_Y^{-1} \otimes L^{-m-1}) \to \ldots \] (1.23)
so (1.11) holds when \( L \) is sufficiently ample. Thus (1.15) and (1.16) follow from Lemmas (1.5) and (1.10). Since multiplication by \( H^0(Z, K_Z) \) commutes with all the differentials of the spectral sequence, we conclude that (1.17) commutes, proving the lemma.

**Lemma 1.24:** Let \( Y \) be a smooth \((n + 1)\)-fold with \( n \geq 1 \) and \( L \to Y \) a sufficiently ample analytic line bundle. Then the Local Torelli Theorem holds for any smooth, reduced \( Z \subseteq |L| \) if the multiplication map
\[ H^0(Y, K_Y \otimes L^n) \otimes H^0(Y, K_Y \otimes L) \to H^0(Y, K_Y^2 \otimes L^{(n+1)}) \] (1.25)
is surjective.

**Proof:** To show that the map
\[ H^1(Z, \Theta_Z) \overset{P^*}{\to} H^0(Z, K_Z)^* \otimes H^1(Z, \Omega_Z^{n-1}) \]
is injective, it is equivalent to show that the dual map
\[ H^1(Z, \Omega_Z^{n-1})^* \otimes H^0(Z, K_Z) \to H^1(Z, \Theta_Z)^* \]
is surjective. By Lemma 1.14, it is enough to show that
\[ H^0(Z, K_Z \otimes L^{(n-1)}) \otimes H^0(Z, K_Z) \to H^0(Z, K_Z^2 \otimes L^{(n-1)}) \] (1.26)
is surjective. From the restriction sequence (1.19), we have the long exact sequence
\[ \cdots \to H^0(Y, K_Y^2 \otimes L^{(n+1)}) \to H^0(Z, K_Z^2 \otimes L^{(n-1)}) \]
\[ \to H^1(Y, K_Y^2 \otimes L^n) \to \cdots \] (1.27)
By the Kodaira Vanishing Theorem,
\[ H^1(Y, K_Y^2 \otimes L^n) = 0 \]
for \( L \) sufficiently ample, and thus the map
\[ H^0(Y, K_Y^2 \otimes L^{(n+1)}) \to H^0(Z, K_Z^2 \otimes L^{(n-1)}) \]
is surjective. The lemma follows.

**Lemma 1.28:** Let \( Y \) be a smooth \( (n + l) \)-fold, \( E_1, E_2 \) analytic vector bundles over \( Y \). For \( L \) a sufficiently ample line bundle on \( Y \), the multiplication map
\[ H^0(Y, E_1 \otimes L^a) \otimes H^0(Y, E_2 \otimes L^b) \to H^0(Y, E_1 \otimes E_2 \otimes L^{a+b}) \]
(1.29)
is surjective when \( a \geq 1 \) and \( b \geq 1 \).

**Proof:** Let \( \Delta \) be the diagonal on \( Y \times Y \), and \( \pi_1, \pi_2 \) the canonical projections. We then have a commutative diagram
\[
\begin{align*}
H^0(Y \times Y, \pi_1^*(E_1 \otimes L^a) \otimes \pi_2^*(E_2 \otimes L^b)) &\cong H^0(Y \times Y, \sigma_\Delta(\pi_1^*(E_1 \otimes L^a) \otimes \pi_2^*(E_2 \otimes L^b))) \\
H^0(Y, E_1 \otimes L^a) \otimes H^0(Y, E_2 \otimes L^b) &\to H^0(Y, E_1 \otimes E_2 \otimes L^{a+b})
\end{align*}
\] (1.30)
* This argument was suggested by Ron Donagi and replaces an earlier, more complicated proof.
From the restriction sequence
$$0 \to \mathcal{I}_\Delta \to \mathcal{O}_{Y \times Y} \to \mathcal{O}_\Delta \to 0$$
we see that to prove the surjectivity of (1.29), it suffices to prove
$$H^1(Y \times Y, \mathcal{I}_\Delta \otimes \pi_1^*((E_1 \otimes L^a) \otimes \pi_2^*((E_2 \otimes L^b))) = 0 \quad (1.31)$$

Since $\pi_1^*L \otimes \pi_2^*L$ is sufficiently ample on $Y \times Y$ if $L$ is sufficiently ample on $Y$, (1.31) holds when $a \geq 1$ and $b \geq 1$. 

We conclude this section by noting that Theorem 0.1 is a direct consequence of Lemmas (1.24) and (1.28). The need to take $L \gg 0$ arose in satisfying (1.6), (1.11), and (1.25). In explicit situations, e.g. when $Y$ is a complete intersection in $\mathbb{P}_N$, these may be verified directly to yield many known results.

§2. A global Torelli theorem

Let $Y$ be a smooth algebraic variety of dimension $n + 1$ and $L \to Y$ a sufficiently ample line bundle. Let

$$\begin{cases} s \in H^0(Y, L) \\ Z = \text{div } s, \text{ } Z \text{ smooth and reduced.} \end{cases} \quad (2.1)$$

We have

$$T_s(\mathbb{P}(H^0(Y, L))) = H^0(Y, L)/(s) \quad (2.2)$$

and let $T$ be the image of the Kodaira-Spencer map

$$T_s(\mathbb{P}(H^0(Y, L))) \to H^1(Z, \Theta_Z). \quad (2.3)$$

The first derivative of the period map at $Z$ has as its leading piece

$$T \to \Phi(\text{Hom}(H^0(Z, K_Z), H^1(Z, \Omega_Z^{n-1}))) \quad (2.4)$$

which we may alternatively regard as an element

$$\tilde{\Phi}_{*,Z} \in T^* \otimes H^0(Z, K_Z)^* \otimes H^1(Z, \Omega_Z^{n-1}). \quad (2.5)$$

Using the notations introduced in the introduction, in order to show that $P$ has degree one on

$$\mathcal{M} = \mathbb{P}(H^0(Y, L))_{ns}/G \quad (2.6)$$
it is sufficient to show that

\begin{equation}
\text{For } Z \in |L| \text{ generic, the } GL\text{-equivalence class of } \hat{P}_{\ast . Z} \text{ determines } Z. \tag{2.7}
\end{equation}

Let \( \Sigma_Y \) denote the first prolongation bundle (see [A-C-G-H]) of \( L \); it sits in the exact sequence

\begin{equation}
0 \to \mathcal{O}_Y \to \Sigma_Y \to \Theta_Y \to 0 \tag{2.8}
\end{equation}

with extension class

\[-c_1(L) \in H^1(Y, \Omega^1_Y).\]

We can differentiate \( s \) to obtain

\begin{equation}
\tilde{ds} \in H^0(Y, \Sigma_Y \otimes L). \tag{2.9}
\end{equation}

For any coherent analytic sheaf \( \mathcal{F} \to Y \), we obtain a map

\begin{equation}
H^0(Y, \mathcal{F} \otimes \Sigma_Y \otimes L^{-1}) \xrightarrow{\tilde{ds}} H^0(Y, \mathcal{F}) \tag{2.10}
\end{equation}

and we define the pseudo-Jacobian system

\begin{equation}
J_\mathcal{F} \subseteq H^0(Y, \mathcal{F}) \tag{2.11}
\end{equation}

to be the image of the map (2.10). If \( \dim Y \geq 2 \), then for \( L \) sufficiently ample, from the exact sequence

\begin{equation}
H^0(Y, \Theta_Y) \to H^0(Z, \Theta_Y) \to H^1(Y, \Theta_Y \otimes L^{-1}) \tag{2.12}
\end{equation}

and the Kodaira Vanishing Theorem we have that

\begin{equation}
H^0(Y, \Theta_Y) \to H^0(Z, \Theta_Y). \tag{2.13}
\end{equation}

From the exact sequence

\begin{equation}
H^0(Z, \Theta_Y) \xrightarrow{\tilde{ds}|_Z} H^0(Z, L) \to H^1(Z, \Theta_Z) \tag{2.14}
\end{equation}

we conclude from (2.3) that

\begin{equation}
T = \frac{H^0(Y, L)}{J_L} \tag{2.15}
\end{equation}

for \( L \) sufficiently ample.
It is useful to have the following Duality Theorem or Generalized Macaulay’s Theorem:

**Theorem 2.15:** For $Y$ a smooth $(n+1)$-fold, $E \to Y$ a fixed analytic vector bundle, and $L \to Y$ a sufficiently ample line bundle,

$$H^0\left(Y, K_Y^2 \otimes L^{(n+2)}\right) / J_{K_Y^2 \otimes L^{(n+2)}} = \mathbb{C}$$  \hfill (2.16)

and the map

$$H^0\left(Y, E \otimes L^a\right) / J_E \otimes H^0\left(Y, E^* \otimes K_Y^2 \otimes L^{(n+2-a)}\right) / J_{E^* \otimes K_Y^2 \otimes L^{(n+2-a)}}$$

$$\to H^0\left(Y, K_Y^2 \otimes L^{(n+2)}\right) / J_{K_Y^2 \otimes L^{(n+2)}} = \mathbb{C}$$  \hfill (2.17)

is a perfect pairing provided

$$H^{a-1}\left(Y, E \otimes \Lambda^a \Sigma_Y\right) = H^a\left(Y, E \otimes \Lambda^a \Sigma_Y\right) = 0 \quad \text{or} \quad a = 0. \quad (2.18)$$

If only

$$H^{a-1}\left(Y, E \otimes \Lambda^a \Sigma_Y\right) = 0 \quad (2.19)$$

then the pairing (2.17) has no left kernel.

**Proof of 2.15:** Using

$$\tilde{d}s \in H^0\left(Y, \Sigma^*_Y \otimes L\right)$$

we may construct the Koszul complex

$$0 \to \Lambda^{n+2} \Sigma_Y \otimes L^{-(n+2)} \xrightarrow{\tilde{d}s} \Lambda^{n+1} \Sigma_Y \otimes L^{-(n+1)} \xrightarrow{\tilde{d}s} \cdots \quad (2.20)$$

Tensoring (2.20) with $E \otimes L^a$ and applying Lemma 1.4, we obtain a spectral sequence abutting to zero. For $L$ sufficiently ample, using the hypothesis (2.18), we get that

$$\ker\left(H^{n+1}\left(Y, \Lambda^{n+2} \Sigma_Y \otimes E \otimes L^{a-(n+2)}\right)\right)$$

$$\to H^{n+1}\left(Y, \Lambda^{n+1} \Sigma_Y \otimes E \otimes L^{a-(n+1)}\right)$$

$$\xrightarrow{d_{n+2}} H^0\left(Y, E \otimes L^a\right) / J_{E \otimes L^a}$$
and thus
\[
\left( \frac{H^0(Y, E^* \otimes K_Y^{2} \otimes L^{(n+2-a)})}{J_{E^* \otimes K_Y^{2} \otimes L^{(n+2-a)}}} \right)^* \cong \frac{H^0(Y, E \otimes L^n)}{J_{E \otimes L^n}}.
\]

When \( E = 1 \), \( a = 0 \), this gives (2.16). Moreover, because multiplication with \( H^0(Y, E^* \otimes K_Y^{2} \otimes L^{(n+2-a)}) \) gives a map of the entire spectral sequence, we conclude that (2.17) gives the duality. The case where we have only the hypothesis (2.19) is similar. This proves (2.15).

We next generalize Donagi's Symmetrizer Lemma with the following two results:

**Theorem 2.21** (Generalized Symmetrizer Lemma): Let \( Y \) be a smooth \( n + 1 \) fold, \( L \to Y \) an analytic line bundle, \( M \to Y \) an analytic line bundle with \( |M| \) base-point free, and \( E \to Y \) an analytic vector bundle. Then for \( L \) sufficiently ample, the Koszul complex

\[
0 \to \frac{H^0(Y, E \otimes L)}{J_{E \otimes L}} \to \frac{H^0(Y, M)^* \otimes H^0(Y, E \otimes M \otimes L)}{J_{E \otimes M, M \otimes L}} \to \Lambda^2 \frac{H^0(Y, M)^* \otimes H^0(Y, E \otimes M^2 \otimes L)}{J_{E \otimes M^2 \otimes L}}
\]

(2.22)

is exact as far as written above, provided that

\[
H^1(Y, E \otimes \Sigma) \to H^0(Y, M)^* \otimes H^1(Y, E \otimes M \otimes \Sigma)
\]

(2.23)

is injective.

**Theorem 2.24**: For \( L \) sufficiently ample, the Koszul complex

\[
0 \to \frac{H^0(Y, K_Y)}{J_L} \to \frac{H^0(Y, L)}{J_L} \otimes \frac{H^0(Z, K_Z)}{H^0(Y, \Omega^n_Y)}
\]

\[
\to \Lambda^2 \left( \frac{H^0(Y, L)}{J_L} \right)^* \otimes H^1(Z, \Omega^{n-1}_Z)
\]

(2.25)

is exact as for as written, and

\[
0 \to \frac{H^0(Y, L \otimes K_Y^{-1})}{J_L \otimes K_Y^{-1}} \to \frac{H^0(Y, K_Y)^* \otimes H^0(Y, L)}{J_L}
\]

\[
\to \Lambda^2 H^0(Y, K_Y)^* \otimes \frac{H^0(Z, K_Z)}{H^0(Y, \Omega^n_Y)}
\]

(2.26)
is exact as far as written provided that $|K_Y|$ is base-point free and $\dim \varphi_{K_Y}(Y) \geq 2$.

Proof of Theorems (2.21) and (2.24): Using the Generalized Macaulay's Theorem (2.15) and its proof, we have for $L$ sufficiently ample that the sequence (2.22) is dual to

$$
\Lambda^2 H^0(Y, M) \otimes \frac{H^0(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-2})}{J_{K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-2}} + H^1(Y, E \otimes M^2 \otimes \Sigma)^*} \rightarrow H^0(Y, M) \otimes \frac{H^0(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-1})}{J_{K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-1}} + H^1(Y, E \otimes M \otimes \Sigma)^*} \rightarrow \frac{H^0(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1})}{J_{K_Y^2 \otimes L^{(n+1)} \otimes E^{-1}} + H^1(Y, E \otimes \Sigma)^*} \rightarrow 0
$$

(2.27)

The sequence

$$
\Lambda^2 H^0(Y, M) \otimes H^0(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-2})
$$

$$
\rightarrow H^0(Y, M) \otimes H^0(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-1})
$$

$$
\rightarrow H^0(Y, K_Y^2 \otimes L^{(n+1)} \otimes E^{-1}) \rightarrow 0
$$

(2.28)

is exact by considering the Koszul complex

$$
\dotsc \rightarrow \Lambda^2 H^0(Y, M) \otimes K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-2}
$$

$$
\rightarrow H^0(Y, M) \otimes K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-1}
$$

$$
\rightarrow K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \rightarrow 0
$$

and applying Lemma 1.4 and the Kodaira Vanishing Theorem. Likewise,

$$
H^0(Y, M) \otimes J_{K_Y^2 \otimes L^{(n+1)} \otimes E^{-1} \otimes M^{-1}} \rightarrow J_{K_Y^2 \otimes L^{(n+1)} \otimes E^{-1}}
$$

by a similar argument, while

$$
H^0(Y, M) \otimes H^1(Y, E \otimes M \otimes \Sigma)^* \rightarrow H^1(Y, E \otimes \Sigma)
$$

by the dual of (2.23).

We are now done by
LEMMA 2.29: Let $V$ be a vector space, $S(V)$ the symmetric algebra on $V$, and $A = \oplus_{q \in \mathbb{Z}} A_q \subseteq B = \oplus_{q \in \mathbb{Z}} B_q$ graded $S(V)$-modules. The Koszul complex

$$\Lambda^2 V \otimes (B_q/A_q) \xrightarrow{F} V \otimes (B_{q+1}/A_{q+1}) \xrightarrow{G} B_{q+2}/A_{q+2} \to 0$$  \hspace{1cm} (2.30)$$

is exact as far as written provided that

$$\Lambda^2 V \otimes B_q \xrightarrow{\tilde{F}} V \otimes B_{q+1} \xrightarrow{\tilde{G}} B_{q+2} \to 0$$  \hspace{1cm} (2.31)$$

is exact as far as written and

$$V \otimes A_{q+1} \to A_{q+2}. \hspace{1cm} (2.32)$$

PROOF OF (2.29): As $\tilde{G}$ is surjective, so is $G$. If $\alpha \in \ker G$, then choosing $\mathbf{a} \in V \otimes B_{q+1}$ representing $\alpha$,

$$\tilde{G}(\mathbf{a}) \in A_{q+2}$$

By (2.32), we may modify $\mathbf{a}$ to $\mathbf{\tilde{a}}$ representing $\alpha$ so that

$$\tilde{G}(\mathbf{\tilde{a}}) = 0$$

So

$$\mathbf{\tilde{a}} = \tilde{F}(\mathbf{\tilde{\beta}})$$

for some $\mathbf{\tilde{\beta}} \in \Lambda^2 V \otimes B_q$, and then

$$\alpha = F(\mathbf{\beta})$$

where $\mathbf{\beta}$ is the projection of $\mathbf{\tilde{\beta}}$ to $\Lambda^2 V \otimes (B_q/A_q)$. \hfill \Box

We have now proved (2.21). To prove (2.24), we will prove first the exactness of the sequence

$$0 \to H^0(Y, K_Y) \to \left( \frac{H^0(Y, L)}{J_L} \right)^* \otimes \frac{H^0(Z, K_Z)}{H^0(Y, \Omega^n_Y)}$$

$$\to \Lambda^2 \left( \frac{H^0(Y, L)}{J_L} \right)^* \otimes \frac{H^1(Z, \Omega^n_{Z-1})}{H^1(Z, \Omega^n_Y \otimes \mathcal{O}_Z)}$$

which is stronger than showing exactness for (2.25). Using the dualities of §1, and the fact $H^0(Y, \Omega^n_Y) \to H^0(Z, \Omega^n_Y \otimes \mathcal{O}_Z)$ if $\dim Y \geq 2$ if $L \gg 0$
and the generalized Macauley's Theorem, we may dualize the above sequence to

$$\Lambda^2 \left( \frac{H^0(Y, L)}{J} \right) \otimes \frac{H^0(Y, K_Y \otimes L^n)}{J_L \otimes L^n + \text{more}}$$

$$\longrightarrow \frac{H^0(Y, L)}{J_L} \otimes \frac{H^0(Y, K_Y \otimes L^{(n+1)})}{J_{K_Y \otimes L^{(n+1)}} + \text{more}}$$

$$\longrightarrow \frac{H^0(Y, K_Y \otimes L^{(n+2)})}{J_{K_Y \otimes L^{(n+2)}}} \rightarrow 0$$

and we now proceed analogously to the preceding case, using Lemma 1.8, the Kodaira Vanishing Theorem, and Lemma 2.29. The one additional fact we require is that the sequence

$$\Lambda^2 H^0(Y, L) \otimes H^0(Y, K_Y \otimes L^n)$$

$$\longrightarrow H^0(Y, L) \otimes H^0(Y, K_Y \otimes L^{(n+1)})$$

$$\longrightarrow H^0(Y, K_Y \otimes L^{(n+2)}) \rightarrow 0$$

is exact as far as written for $L$ sufficiently ample. This follows from Lemma 2.47, which we have put at the end of this section.

The dual of (2.26) is

$$\Lambda^2 H^0(Y, K_Y) \otimes \frac{H^0(Y, K_Y \otimes L^{(n+1)})}{J_{K_Y \otimes L^{(n+1)}} + \text{more}}$$

$$\longrightarrow H^0(Y, K_Y) \otimes \frac{H^0(Y, K_Y^2 \otimes L^{(n+1)})}{J_{K_Y^2 \otimes L^{(n+1)}} + H^1(Y, \Sigma)^*}$$

$$\longrightarrow \frac{H^0(Y, K_Y^3 \otimes L^{(n+1)})}{J_{K_Y^3 \otimes L^{(n+1)}} + H^1(Y, \Sigma \otimes K_Y^{-1})^*} \rightarrow 0$$

and again we are done provided that

$$H^1(Y, \Sigma \otimes K_Y^{-1}) \rightarrow H^0(Y, K_Y)^* \otimes H^1(Y, \Sigma)$$

(2.33)

is injective. Applying Lemma 1.8 to the Koszul complex

$$0 \rightarrow \Sigma \otimes K_Y^{-1} \rightarrow H^0(Y, K_Y)^* \otimes \Sigma \rightarrow \Lambda^2 H^0(Y, K_Y)^* \otimes \Sigma \otimes K_Y \rightarrow \ldots$$
we conclude that the injectivity of (2.33) is equivalent to proving
\[ H^0(Y, K_Y)^* \otimes H^0(Y, \Sigma) \to \Lambda^2 H^0(Y, K_Y)^* \otimes H^0(Y, \Sigma \otimes K_Y) \]
\[ \to \Lambda^3 H^0(Y, K_Y)^* \otimes H^0(Y, \Sigma \otimes K_Y^2) \]
is exact at the middle term. From the exact sequence
\[ 0 \to \mathcal{O}_Y \to \Sigma_Y \to \Theta_Y \to 0 \]
we have an exact sequence
\[ 0 \to H^0(Y, K_Y) \to H^0(Y, \Sigma \otimes K_Y) \to H^0(Y, \Omega^n_Y) \cup c_1(L) \to H^1(Y, K_Y) \]
and the last map is an isomorphism by the Strong Lefschetz Theorem, so
\[ H^0(Y, K_Y) \xrightarrow{=} H^0(Y, \Sigma \otimes K_Y) \]
and we are now reduced to showing
\[ H^0(Y, K_Y)^* \otimes H^0(Y, \mathcal{O}_Y) \to \Lambda^2 H^0(Y, K_Y)^* \otimes H^0(Y, K_Y) \]
\[ \to \Lambda^3 H^0(Y, K_Y)^* \otimes H^0(Y, K_Y^2) \]
is exact at the middle term. However, for \( \dim \varphi_{K_Y} \geq 2 \) this is true by the \( \mathcal{X}_{p,1} \) Theorem of [G]. \( \square \)

We have the following corollary of Theorem 2.21:

**Corollary 2.34:** If \( |K_y| \) is base-point free and \( Y \) is of general type, then for any \( m \geq 1 \) the Koszul complex
\[ 0 \to \frac{H^0(Y, L \otimes K_Y^{-m})}{J_{L \otimes K_Y^{-m}}} \to H^0(Y, K_Y)^* \otimes \frac{H^0(Y, L \otimes K_Y^{-m})}{J_{L \otimes K_Y^{-m}}} \]
\[ \to \Lambda^2 H^0(Y, K_Y)^* \otimes \frac{H^0(Y, L \otimes K_Y^{-m})}{J_{L \otimes K_Y^{-m}}} \]
is exact as far as written for \( k \) sufficiently large if \( \dim \varphi_{K_Y}(Y) \geq 2 \).

**Proof:** Using (2.21), we need only show that hypothesis (2.23) holds in this case, i.e. that
\[ H^1(Y, K_Y^{-m} \otimes \Sigma) \to H^0(Y, K_Y)^* \otimes H^1(Y, K_Y^{-m} \otimes \Sigma) \]
is injective. As seen above, this is equivalent to the sequence

$$H^0(Y, K_Y)^* \otimes H^0(Y, K_Y^{1-m} \otimes \Sigma)$$

$$\rightarrow \Lambda^2 H^0(Y, K_Y)^* \otimes H^0(Y, K_Y^{2-m} \otimes \Sigma)$$

$$\rightarrow \Lambda^3 H^0(Y, K_Y)^* \otimes H^0(Y, K_Y^{3-m} \otimes \Sigma)$$

(2.36)

being exact at the middle term. For $m = 1$, we have already proved this.

As is well known,

$$H^0(Y, \Theta_Y) = 0 \quad \text{if} \quad Y \text{ is of general type}$$

and thus

$$H^0(Y, K_Y^{2-m} \otimes \Sigma) = 0 \quad \text{for} \quad m \geq 3$$

and

$$H^0(Y, K_Y^{2-m} \otimes \Sigma) = H^0(Y, \Theta_Y) \quad \text{for} \quad m = 2.$$

We are done in case $m \geq 3$, while for $m = 2$ we are reduced to the exactness of

$$0 \rightarrow \Lambda^2 H^0(Y, K_Y)^* \otimes H^0(Y, \Theta_Y) \rightarrow \Lambda^3 H^0(Y, K_Y)^* \otimes H^0(Y, K_Y)$$

at the middle term, and this follows from the fact that Koszul map $\Lambda^2 V \rightarrow \Lambda^3 V \otimes V^*$ is injective for any vector space.

We are now ready to begin proving Theorem (0.3). The image of the derivative of the period map gives us the $GL$-equivalence class of the map

$$\frac{H^0(Y, L)}{J_L} \otimes H^0(Z, K_Z) \rightarrow H^1(Z, \Omega_Z^{n-1}).$$

(2.37)

We require

**Lemma 2.38:** The right kernel of (2.37) is the image of $H^0(Y, \Omega_Y^n)$ for $L$ sufficiently ample.

**Proof:** We know by Hodge theory that the image of $H^0(Y, \Omega_Y^n)$ in $H^0(Z, K_Z)$ is invariant as we deform $Z$ on $Y$. It remains to show that the map

$$\frac{H^0(Z, K_Z)}{H^0(Y, \Omega_Y^n)} \rightarrow \left( \frac{H^0(Y, L)}{J_L} \right)^* \otimes H^1(Z, \Omega_Z^{n-1})$$
is injective. A fortiori, it would be enough to show that

\[
\frac{H^0(Z, K_Z)}{H^0(Y, \Omega^n_Y)} \to \left( \frac{H^0(Y, L)}{J_L} \right)^* \otimes \frac{H^1(Z, \Omega^{n-1}_Z)}{H^1(Z, \Omega^{n-1}_Y \otimes \mathcal{O}_Z)}
\]

is injective. By the results of §1, for \( L \) sufficiently ample this dualizes to the (quotiented) multiplication map

\[
\frac{H^0(Y, L)}{J_L} \otimes \frac{H^0(Y, K_Y \otimes L^n)}{J_{K_Y \otimes L^n} + \text{more}} \to \frac{H^0(Y, K_Y \otimes L^{n+1})}{J_{K_Y \otimes L^{n+1}} + \text{more}}
\]

which we must show is surjective. It suffices to show that

\[
H^0(Y, L) \otimes H^0(Y, K_Y \otimes L^n) \to H^0(Y, K_Y \otimes L^{n+1})
\]

This follows from Lemma 1.28, for \( L \) sufficiently ample. \( \square \)

Thus, from (2.37), we can construct the map

\[
\frac{H^0(Y, L)}{J_L} \otimes \frac{H^0(Z, K_Z)}{H^0(Y, \Omega^n_Y)} \to H^1(Z, \Omega^{n-1}_Z).
\]

From this, we can construct the second map in the sequence (2.25), and thus can reconstruct the \( GL \)-equivalence class of the map

\[
H^0(Y, K_Y) \otimes \frac{H^0(Y, L)}{J_L} \to \frac{H^0(Z, K_Z)}{H^0(Y, \Omega^n_Y)}.
\] (2.39)

From (2.39), we can construct the second map in the sequence (2.26), and thus can reconstruct the vector space

\[
\frac{H^0(Y, L \otimes K^{-1}_Y)}{J_{L \otimes K^{-1}_Y}}
\]

and the \( GL \)-equivalence class of the map

\[
\frac{H^0(Y, L \otimes K^{-1}_Y)}{J_{L \otimes K^{-1}_Y}} \otimes H^0(Y, K_Y) \to \frac{H^0(Y, L)}{J_L}.
\]

For any \( m_0 \) chosen in advance, we can choose \( L \) sufficiently ample so that we can recover inductively using Corollary (2.24) the \( GL \)-equivalence classes of the maps

\[
\frac{H^0(Y, L \otimes K^{1-m}_Y)}{J_{L \otimes K^{1-m}_Y}} \otimes H^0(Y, K_Y) \to \frac{H^0(Y, L \otimes K^{1-m}_Y)}{J_{L \otimes K^{1-m}_Y}}
\] (2.40)
for all $m \leq m_0$.

However,

$$J_{L \otimes K_Y^{-m}} \simeq H^0(Y, \Sigma \otimes K_Y^{-m}).$$

If $K_Y$ is ample, we conclude that for some $m_1$,

$$J_{L \otimes K_Y^{-m}} = 0 \text{ for } m \geq m_1 \quad (2.41)$$

and the $m_1$ may be chosen independent of $L$. We now have the maps

$$H^0(Y, L \otimes K_Y^{-m}) \otimes H^0(Y, K_Y) \rightarrow H^0(Y, L \otimes K_Y^{1-m}) \quad (2.42)$$

for $m_0 \geq m \geq m_1 + 1$, where we can make $m_0$ as large as we like at the expense of our choice of how ample $L$ must be. For

$$W \subseteq H^0(Y, K_Y)$$

a linear subspace with base locus $B$, we have by making $L$ sufficiently ample that

$$H^0(Y, L \otimes K_Y^{-m}) \otimes W \rightarrow H^0(Y, L \otimes K_Y^{1-m} \otimes \mathcal{I}_B). \quad (2.43)$$

Thus we can detect from the map (2.42) which $W$'s have a base locus, and thus can determine the Chow form of $\varphi_{K_Y}(Y)$. Since $\varphi_{K_Y}(Y) = Y$ by hypothesis, for each $p \in Y$ we can determine by (2.42) the subspaces

$$H^0(Y, L \otimes K_Y^{1-m} \otimes \mathcal{I}_p) \subseteq H^0(Y, L \otimes K_Y^{1-m})$$

for $m_0 \geq m \geq m_1 + 1$. From this, we can determine the Chow form of $\varphi_{L \otimes K_Y^{1-m}}(Y)$. We thus can reconstruct the map (2.42) not merely up to $GL$-equivalence, but up to the action of $G$. Now, operating our induction in reverse, we can construct the projections

$$H^0(Y, L \otimes K_Y^{-m}) \rightarrow \frac{H^0(Y, L \otimes K_Y^{-m})}{J_{L \otimes K_Y^{-m}}} \quad (2.44)$$

for all $m \leq m_0$, modulo the action of $G$. Thus in particular, we can construct

$$J_L \subseteq H^0(Y, L)$$

modulo the action of $G$.

We now wish to recover $Z$ from $J_L$. In the case at hand, this is easy as $H^0(Y, \Theta_Y) = 0$; however, it is interesting to give the general argument.
To do this, we consider the group $\tilde{G}$ consisting of pairs of analytic isomorphisms $L \xrightarrow{\tilde{u}} L$\
$\downarrow \quad \downarrow$\
$Y \xrightarrow{u} Y$

so that the diagram commutes and $\tilde{u}$ is linear on each fiber. As

$$H^0(Y, \text{Hom}(L, L)) = \mathbb{C}$$

there is a natural exact sequence of Lie groups

$$0 \to \mathbb{C}^* \to \tilde{G} \to G \to 0.$$  \hspace{1cm} (2.45)

Furthermore, we may make the identifications of the tangent spaces at the identity

$$T_e(\mathbb{C}^*) \to T_e(\tilde{G}) \to T_e(G)$$

$$H^0(Y, \mathcal{O}_Y) \to H^0(Y, \Sigma_Y) \to \ker\left(H^0(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y) \bigcup_{\mathcal{O}_L} \right)$$

so that the above diagram commutes. Further, $\tilde{G}$ acts on $H^0(Y, L)$ by pullback, so that the tangent space to the orbit of $s$ is

$$T_s(\tilde{G}s) = J_L.$$  \hspace{1cm} (2.46)

Gives this, the argument given by Donagi [D] adapting techniques of Mather and Yau [M-Y] goes through verbatim. This completes the proof of Theorem (0.3), once we have shown the lemma needed to prove the exactness of (2.25).

**Lemma 2.47:** Let $Y$ be a smooth $(n + 1)$-fold, $E \to Y$ an analytic vector bundle. If $L \to Y$ is a sufficiently ample vector bundle, then for any $a \geq 1$, the Koszul sequence

$$\wedge^2 H^0(Y, L) \otimes H^0(Y, E \otimes L^a) \xrightarrow{\alpha_a} H^0(Y, L) \otimes H^0(Y, E \otimes L^{a+1}) \xrightarrow{\beta_a} H^0(Y, E \otimes L^{a+2}) \to 0$$  \hspace{1cm} (2.48)
is exact as far as written.

PROOF: Surjectivity of $\beta_a$ follows from Lemma 1.28. Exactness at the middle term would follow from the surjectivity of the map

$$H^0(Y, L) \otimes \ker \beta_{a-1} \rightarrow \ker \beta_a$$

(2.49)

defined by

$$\gamma_a \left(l \otimes \left(\sum s_i \otimes r_i \right) \right) = \sum s_i \otimes (lr_i)$$

where $l \in H^0(Y, L)$, $s_0, \ldots, s_r$ a basis for $H^0(Y, L)$ and $r_i \in H^0(Y, E \otimes L^a)$. To see this implication, we note that there is a commutative diagram

$$\begin{array}{c}
\Lambda^2 H^0(Y, L) \otimes H^0(Y, E \otimes L^a) \rightarrow \ker \beta_a \\
\downarrow \gamma_a \\
H^0(Y, L) \otimes \ker \beta_{a-1}
\end{array}$$

where

$$\delta_a \left(l \otimes \left(\sum s_i \otimes r_i \right) \right) = \sum (l \wedge s_i) \otimes r_i.$$

Thus

$$\text{im} \gamma_a \subseteq \text{im} \alpha_a$$

so it will suffice for our purposes to show that $\gamma_a$ is surjective.

On $Y \times Y$, let $\Delta$ be the diagonal and $\pi_1, \pi_2$ the canonical projections. From the exact sequence

$$0 \rightarrow \mathcal{O}_{Y \times Y}(-\Delta) \rightarrow \mathcal{O}_{Y \times Y} \rightarrow \mathcal{O}_{\Delta} \rightarrow 0$$

(2.51)
tensored with $\pi_1^*(L) \otimes \pi_2^*(E \otimes L^{a+1})$, we conclude that

$$\ker \beta_a \approx H^0(Y \times Y, \pi_1^*(L) \otimes \pi_2^*(E \otimes L^{a+1}) \otimes \mathcal{O}_{Y \times Y}(-\Delta)).$$

(2.52)

Further, $\gamma_a$ in these terms is the map

$$H^0(Y \times Y, \pi_1^*(L)) \otimes H^0(Y \times Y, \pi_1^*(L))$$

$$\otimes \pi_2^*(E \otimes L^a) \otimes \mathcal{O}_{Y \times Y}(-\Delta))$$

$$\gamma_a \rightarrow H^0(Y \times Y, \pi_1^*(L) \otimes \pi_2^*(E \otimes L^{a+1}) \otimes \mathcal{O}_{Y \times Y}(-\Delta)).$$

(2.53)
On $Y \times Y \times Y$, let
\[ \Delta_{ij} = \{(y_1, y_2, y_3) \in Y \times Y \times Y \mid y_i = y_j\} \]

and let $\pi_1, \pi_2, \pi_3$ be the projections. We may rewrite $\tilde{\gamma}_a$ equivalently as the map
\[
H^0(Y \times Y \times Y, \pi_3^*(L)) \otimes H^0(Y \times Y \times Y, \pi_1^*(L)) \otimes \pi_2^*(E \otimes L^a) \otimes \mathcal{O}_{Y \times Y \times Y}(-\Delta_{12})
\]
\[
\tilde{\gamma}_a \mapsto H^0(Y \times Y \times Y, \pi_1^*(L) \otimes \pi_2^*(E \otimes L^a) \otimes \pi_3^*(L)) \otimes \mathcal{O}_{Y \times Y \times Y}(-\Delta_{12}) \otimes \mathcal{O}_{\Delta_{23}}.
\] (2.54)

From tensoring the restriction sequence for $\Delta_{23}$ on $Y \times Y \times Y$ appropriately, we see that $\tilde{\gamma}_a$ is surjective if
\[
H^1(Y \times Y \times Y, \pi_1(L) \otimes \pi_2(E \otimes L^a) \otimes \pi_3(L)) \otimes \mathcal{O}_{Y \times Y \times Y}(-\Delta_{12} - \Delta_{23}) = 0.
\] (2.55)

For $L$ sufficiently ample, (2.55) holds for all $a \geq 1$. This proves Lemma 2.47.

References


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