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ABSOLUTELY EXTREMAL POINTS IN MINIMAL FLOWS

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Abstract

A point of a minimal flow is called absolutely extremal if it is an extreme point in every affine embedding. We show that distal points are absolutely extremal and give an example of a weakly mixing minimal flow all of whose points are absolutely extremal.

§1. Introduction

Let (X, T) be a metric flow i.e. X is compact metric and T a homeomorphism of X onto itself. A flow (Q, T) is called *affine* if Q is a convex compact subset of a locally convex linear space and T is an affine homeomorphism of Q. We say that a map $\varphi:(X, T) \to (Q, T)$ is an *affine embedding* if φ is continuous, one to one, equivariant and $\overline{co}(\varphi(X)) = Q$. Thus for any embedding, the set $ex(Q)$ of extreme points of Q is contained in $\varphi(X)$. We often identify X with $\varphi(X)$. If (X, T) is a minimal flow (every orbit is dense), then $(\varphi(X), T)$ is minimal and we have $\overline{ex}(Q) = \varphi(X)$. Call a point $x_0 \in X$ *absolutely extremal* if for every embedding $\varphi: X \to Q$, $\varphi(x_0)$ is an extreme point of Q. (X, T) is an absolutely extremal flow if every point of X is absolutely extremal.

Using this terminology a theorem of *I*. Namioka [3] asserts that a distal minimal flow is absolutely extremal. We simplify the proof of this theorem and generalize it in showing that every distal point of a minimal flow is absolutely extremal. We give examples of almost automorphic absolutely extremal flows (X, T) and (Y, T) such that in the product flow $(X \times Y, T)$ which is minimal and almost automorphic, there are points which are not absolutely extremal. Finally using a result of del Junco and Keane we demonstrate the existence of a weakly mixing minimal absolutely extremal flow.

§2. Distal points are absolutely extremal

For a flow (X, T) denote by $\mathscr{P}(X)$ the space of probability measures on X equipped with the weak* topology. The homeomorphism T of X

induces an affine homeomorphism of $\mathscr{P}(X)$ under which $(\mathscr{P}(X), T)$ is an affine flow. For $x \in X$, let δ_x be the point mass at x. If $\varphi: X \to Q$ is an affine embedding then the barycenter map $\beta: \mathscr{P}(X) \to Q$ is defined by

$$\beta(\lambda) = \int_X \varphi(x) d\lambda(x) \quad (\lambda \in \mathscr{P}(X)).$$

The map β is a continuous affine homomorphism (i.e. an equivariant map) of $\mathscr{P}(X)$ onto Q. A point $q \in Q$ is extremal iff $\beta^{-1}(q) = \{\delta_x\}$ for some $x \in X$ (with $\varphi(x) = q$).

LEMMA 2.1: *Let (X, T) be a metric flow, ν a probability measure on X and assume that in $\mathscr{P}(X)$,* $\lim T^{n_i}\nu = \delta_{x_0}$ *for some sequence $\{n_i\}$ and a point $x_0 \in X$. Then there exist an F_σ subset A of X and a subsequence $\{n'_i\}$ such that $\nu(A) = 1$ and for every $x \in A$,* $\lim T^{n'_i}x = x_0$. *In particular every two points of A are proximal.*

PROOF: Let $n'_i \in \{n_j\}$ satisfy $T^{n'_i}\nu(B_{1/i}(x_0)) > 1 - 2^{-(i+1)}$ $(i = 1, 2, \ldots)$, where $B_r(x_0)$ is the closed ball of radius r around x_0. Then

$$\nu\left(\bigcap_{i=k}^{\infty} T^{-n'_i} B_{1/i}(x_0)\right) > 1 - 2^{-k}$$

and thus

$$A = \bigcup_{k=1}^{\infty} \bigcap_{i=k}^{\infty} T^{-n'_i} B_{1/i}(x_0)$$

is the required set.

COROLLARY 2.2: *If (X, T) is a distal flow and* $\lim T^{n_i}\nu = \delta_{x_0}$ *for some $x_0 \in X$ and $\nu \in \mathscr{P}(X)$, then $\nu = \delta_x$ for some $x \in X$.*

THEOREM 2.3: *Let (X, T) be a metric minimal flow, $x_0 \in X$ a distal point then x_0 is an absolutely extremal point.*

PROOF: Let $\varphi: X \to Q$ be an affine embedding; we identify X with $\varphi(X)$. Let $\beta: \mathscr{P}(X) \to Q$ be the barycenter map. If x_0 is not an extreme point of Q then there exists a measure $\nu \in \mathscr{P}(X)$ with $\beta(\nu) = x_0$ and $\nu \neq \delta_{x_0}$. Taking $1/2(\delta_{x_0} + \nu)$ instead of ν we can assume ν has an atom at x_0. Choose $\bar{x} \in X$ which is extreme in Q; then by minimality there exists a sequence $\{n_i\}$ such that $\lim T^{n_i}x_0 = \bar{x}$. We can also assume that

$\lim T^{n_l}\nu = \bar{\nu}$ exists and then

$$\beta(\bar{\nu}) = \lim \beta(T^{n_l}\nu) = \lim T^{n_l}\beta(\nu) = \lim T^{n_l}x_0 = \bar{x}.$$

Thus $\bar{\nu} = \delta_{\bar{x}}$ and $\lim T^{n_l}\nu = \delta_{\bar{x}}$. By lemma 2.1. there exist an F_σ subset A of X with $\nu(A) = 1$ - hence $x_0 \in A$ - and a subsequence $\{n_l'\}$ such that $\lim T^{n_l'}x = x_0$ for every $x \in A$. In particular every point of A is proximal to x_0. Since x_0 is distal $A = \{x_0\}$ and $\nu = \delta_{x_0}$, a contradiction. Thus x_0 is extreme and the proof is completed.

§3. Almost antomorphic examples

Let $I = [0, 1)$, α an irrational number and (I, R_α) the rotation by α on I. Let $x_0(n) = \operatorname{sgn}\cos(2\pi n\alpha)$; consider x_0 as an element of $\{1, -1\}^{\mathbb{Z}} = \Omega$ and let T be the shift on Ω. Set $X = \bar{\mathcal{O}}(x_0)$, the orbit closure of x_0 in Ω; then (X, T) is an almost automorphic minimal flow. There exists a homomorphism $\pi: (X, T) \to (I, R_\alpha)$ such that for $\xi \in I \setminus E$, where $E = \{1/4 + n\alpha,\ 3/4 + n\alpha : n \in \mathbb{Z}\}$, $\pi^{-1}(\xi) = \{\operatorname{sgn}\cos(2\pi(n\alpha + \xi))\}$ and for $\xi = 1/4 + k\alpha$, $\pi^{-1}(\xi) = \{x^+, x^-\}$ where for $n \neq -k$, $x^+(n) = x^-(n) = \operatorname{sgn}\cos(2\pi(n + k)\alpha)$ and $x^+(-k) = 1$, $x^-(-k) = -1$. Similar situation exists for $\pi^{-1}(\xi)$ where $\xi = 3/4 + n\alpha$. Since every point of $\pi^{-1}(I \setminus E)$ is distal we conclude, by theorem 2.3 that these points are absolutely extremal. However it is clear that the points of the form $x^\pm$ are also absolutely extremal. For if, say $\beta(\nu) = x^+$ in some affine embedding then by the proof of theorem 2.3. ν is supported by the proximal cell of x^+ (i.e. the set $\{y \in X$: y is proximal to $x^+\}$) which in our case is the set $\{x^+, x^-\}$. Clearly than $\nu = \delta_{x+}$ and x is extremal. We have shown that *every point of X is* absolutely extremal. Next let Y be the orbit closure of y_0 in Ω where $y_0(n) = \operatorname{sgn}\cos(2\pi n\gamma)$ and γ is an irrational number independent of α. Put $z_0(n) = x_0(n) + \frac{1}{10}y_0(n)$ and set $Z = \bar{\mathcal{O}}(x_0) \subset \mathbb{R}^{\mathbb{Z}}$. Then (X, T) and (Y, T) are disjoint minimal flows and the minimal flow $(X \times Y, T)$ is isomorphic to (Z, T). Now $\mathbb{R}^{\mathbb{Z}}$ with its product topology is a locally convex linear space and, denoting $Q = \overline{co}(Z) \subset \mathbb{R}^{\mathbb{Z}}$, we see that the inclusion map of Z into Q is an embedding of (Z, T) into the affine flow (Q, T). Let $x^\pm$, $y^\pm$ be the points in X and Y respectively, which lie over $1/4 \in I$. Then e.g.

$$x^+ + \tfrac{1}{10}y^- = \tfrac{10}{11}\left(x^+ + \tfrac{1}{10}y^+\right) + \tfrac{1}{11}\left(x^- + \tfrac{1}{10}y^-\right)$$

and we conclude that although x^+ and y^- are absolutely extremal, the point (x^+, y^-) of the minimal flow $X \times Y$ is not absolutely extremal. Thus *the property of being an absolutely extremal point is not preserved under products.*

Taking $x_0(n) = f(\cos 2\pi n\alpha)$ where f is continuous on $[-1, 1]$ except for at zero where it has say, $[0, 1]$ as a limit set, the embedding

$X=\bar{\mathcal{O}}(x_0)\to\overline{co}(X)\subset\mathbb{R}^{\mathbb{Z}}$ will yield a continuum of non absolutely extremal points of X.

If (X,T) is an almost automorphic minimal flow, there exist an almost periodic flow (Y,T) and an almost one to one homomorphism $\pi:X\to Y$. Clearly every pair $(x,x')\in X\times X$ with $\pi(x)=\pi(x')$ has the property that the only minimal set in $\mathcal{O}(x,x')$ is the diagonal Δ. For a general minimal flow (X,T) put

$$L=\{(x,x')\in X\times X\colon \Delta \text{ is the unique minimal set in } \bar{\mathcal{O}}(x,x')\}.$$

L is an invariant equivalence relation but not necessarily closed. We shall use the following lemma in the next section.

LEMMA 3.1: *Let (X,T) be a minimal flow and $\varphi\colon X\to Q$ an affine embedding. Suppose $\beta(\nu)=\varphi(x)$ for some $\nu\in\mathcal{P}(X)$ and $x\in X$. If y is an atom of ν then $(x,y)\in L$.*

PROOF: Since $\beta(1/2(\nu+\delta_x))=\varphi(x)$ we can assume that x itself is an atom of ν. Write $\nu=a\delta_x+b\delta_y+(1-(a+b))\theta$ where $0<a,\ b<1$ and $\theta\in\mathcal{P}(X)$. If $M\subset\bar{\mathcal{O}}(x,y)$ is a minimal set then, since X is minimal, there is a point $(z,w)\in M$ with $\varphi(z)\in ex(Q)$. Let $\{n_i\}$ be a sequence with $\lim T^{n_i}(x,y)=(z,w)$ and we can assume that $\lim T^{n_i}\theta=\tilde{\theta}$ and $\lim T^{n_i}\nu=\tilde{\nu}$ exist. Then

$$\beta(\tilde{\nu})=a\beta(\delta_z)+b\beta(\delta_w)+(1-(a+b)\beta(\tilde{\theta})=\varphi(z),$$

whence $\delta_z=\delta_w=\tilde{\theta}$. Thus $M=\Delta$ and $(x,y)\in L$.

§4. A weakly mixing example

In this section we demonstrate the existence of a minimal absolutely extremal weakly mixing flow. Recall that a minimal flow (X,T) is P.O.D. if it is weakly mixing and for every $x,\ y\in X,\ x\neq y$ there exists some $n\neq 0$ with $T^n y$ and x proximal. Every P.O.D flow is prime [1]. Let (X,T) be P.O.D. and suppose $\varphi\colon X\to Q$ is an affine embedding of X and assume $X\subset Q$. Let $\bar{\nu}\in\mathcal{P}(X)$ and $x_0\in X$ with $\beta(\bar{\nu})=x_0$. If y is an atom of $\bar{\nu}$ then by lemma 3.1. $(x_0,y)\in L$. However it is easy to see that in a P.O.D. flow $L=\Delta$, so that $x_0=y$ and $\bar{\nu}$ can have an atom only at x_0. If $\bar{\nu}\neq\delta_{x_0}$ then there exists a measure $\nu\in\mathcal{P}(X)$ which is continuous (has no atoms) and for which $\beta(\nu)=x_0$.

Suppose further now that (X,T) is also strictly ergodic with an invariant measure μ and that for every $x\in X$ the set,

$$F_x=\{y\in X\colon (x,y) \text{ is not generic for } \mu\times\mu\},$$

is countable. We show that every point of X is absolutely extremal.

That such a flow exists is a result of A. del Junco and M. Keane [4]. They showed that the Chacon transformation which is P.O.D. [2] and strictly ergodic, has also the latter property. Set

$$F = \{(x, y) : (x, y) \text{ is not generic for } \mu \times \mu\},$$

then

$$\nu \times \nu(F) = \int 1_F \mathrm{d}\nu \times \nu = \iint 1_{F_x}(y)\mathrm{d}\nu(y)\mathrm{d}\nu(x) = 0,$$

since ν is continuous and for every x, F_x is countable.

If $f(x, y)$ is a continuous function on $X \times X$ then for $(x, y) \notin F$ we have

$$\frac{1}{2N+1} \sum_{j=-N}^{N} f(T^j x, T^j y) \to \int f \mathrm{d}\mu \times \mu.$$

In particular this convergence holds $\nu \times \nu$ a.e. and integrating we get

$$\frac{1}{2N+1} \sum_{j=-N}^{N} \iint f(T^j x, T^j y)\mathrm{d}\nu(x)\mathrm{d}\nu(y) \to \int f \mathrm{d}\mu \times \mu.$$

Now let g be a continuous affine function on X (i.e. g is the restriction of an affine function on Q). Then for $f(x, y) = g(x)g(y)$ we have

$$\left(\int g d\mu\right)^2 \leftarrow \frac{1}{2N+1} \sum_{j=-N}^{N} \iint g(T^j x) g(T^j y)\mathrm{d}\nu(x)\mathrm{d}\nu(y)$$

$$= \frac{1}{2N+1} \sum_{j=-N}^{N} \left(\int g(T^j x)\mathrm{d}\nu(x)\right)^2.$$

Since g is affine so is $g \circ T^j$ and recalling our assumption that $\beta(\nu) = x_0$ we see that $\int g(T^j x)\mathrm{d}\nu(x) = g(T^j x_0)$. Thus the right hand side of the above equation equals $1/(2N+1)\sum_{j=-N}^{N}(g(T^j x_0))^2$. By strict ergodicity this tends to $\int g^2 \mathrm{d}\mu$ so that

$$\int g^2 \mathrm{d}\mu = \left(\int g \mathrm{d}\mu\right)^2.$$

Choosing $g \neq 0$ with $\int g \mathrm{d}\mu = 0$ we get a contradiction. Thus x_0 is extremal in Q and the proof is completed.

PROBLEMS:

(1) Is there a minimal flow no point of which is absolutely extremal?
(2) Is every minimal flow with $L = \Delta$, absolutely extremal?
(3) Is the homomorphic image of an absolutely extremal point an absolutely extremal point?

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Added in proof: This problem has been solved by D. Maon and S. Glasner; see "On absolutely extremal points", to appear in this journal.